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Is It Possible to Determine the Whole Crack Path at Once?

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Abstract. A brief review of crack path calculation methods using integral principles of mechanics is presented. In two-dimensional setting, a crack is considered as a geodesic line on the surface of a body with a metric that depends on the initial stress state. The possibility of approximate determination of crack path on the basis of integral principles is illustrated on a number of problems. In particular, crack paths in a half-plane under uniformly distributed load applied on its edge are determined. The calculations include the stress state of the half-plane taken from the solution for a body without a crack. The fruitfulness of the representation of displacements of crack edges using the Winkler's hypothesis is shown. To study the subcritical behavior of the crack, the concept of cracon, a quasi-particle simulating the motion of the crack tip, can be introduced. The problem of determining the crack path on the basis of integral principles of mechanics is insufficiently investigated and requires further research.

Keywords: fracture mechanics, solid mechanics, cracks path, quasi-brittle fracture, fracture stress, composite material

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Authors' contribution. *Morozov E.M.* — scientific guidance, research concept, development of methodology, final conclusions. *Kurbanmagomedov A.K.* — numerical analysis, evaluation of research results, preparation of text and infographics, final conclusions.

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Возможно ли определение траектории трещины сразу и в целом?

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Аннотация. Представлен краткий обзор методов расчета траектории трещины с использованием интегральных принципов механики. В двумерной постановке трещина рассматривается как геодезическая линия на поверхности тела с метрикой, которая зависит от начального напряженного состояния. Возможность приближенного определения траектории трещины на основе интегральных принципов проиллюстрирована на ряде задач. В частности, определены траектории трещины в полуплоскости под действием равномерно распределенной нагрузки на ее кромку. Расчеты включают напряженное состояние полуплоскости, взятое из решения для тела без трещины. Показана плодотворность представления смещений краев трещины с помощью гипотезы Винклера. Для изучения докритического поведения трещины может быть введено понятие cracon — квазичастицы, имитирующей движение вершины трещины. Проблема определения траектории трещины на основе интегральных принципов механики изучена недостаточно и требует дальнейших исследований.

Ключевые слова: фрактальная механика, механика твердого тела, траектория трещины, квазихрупкий фрактал, фрактальное напряжение, композиционный материал

Заявление о конфликте интересов. Авторы заявляют об отсутствии конфликта интересов.

Вклад авторов. *Морозов Е.М.* — научное руководство, концепция исследования, развитие методологии, итоговые выводы. *Курбанмагомедов А.К* — проведение численных исследований, анализ результатов исследования, подготовка исходного текста, подготовка инфографиков, итоговые выводы.

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1. Introduction

In a three-dimensional setting, the path along which a crack propagates is the surface on which the front of the crack is located, limiting its area; in a two-dimensional setting, the line on which the tip (apex) of the crack is located is the point limiting the extent of the crack. The problem of calculating the crack path is not new. Moreover, it is completely solvable using numerical, step-by-step methods (at least in plane formulation). Finally, it is not among the primary ones, although it is possible to think of computational methods for determining a stronger or more durable body configuration based on the analysis of shape and length of cracks. Algorithms for finding the crack propagation path using the step-by-step method are clear because they rely on familiar mathematical operations. The specificity of fracture mechanics arises when choosing a criterion on the basis of which, at each step of the increase in the length of a crack (hereinafter, cracks in the form of a line are implied), the direction of this increase is established and, if the crack growth process is stable, then the load step or, conversely, the length of this increase at the selected load step as well. Each increment in crack length is accompanied by a final increment in load or stress intensity factor. Several such criteria are known [1–3]. For example in [4], a relationship for the crack growth rate under cyclic loading (such as the Paris formula) is used to calculate the increment in the length of subcritical crack

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growth, implemented in [5; 6]. This addition to the step-by-step method, in contrast to the option with a constant crack increment, takes into account the "inertia" of crack propagation, determined by the crack speed (and therefore a crack growing at a higher speed is "straighter" than one growing at a low speed). In this regard, it would be interesting to compare the paths for materials with different mechanical properties of cyclic crack resistance. What follows is a plane problem (or a two-dimensional one on a curved surface of a body) for an isotropic material and, in addition, the slow subcritical growth of a crack due to cyclic loading, creep, etc. is not considered.

2. Methods

It is important that each state before the subsequent small increment in the crack length is a critical equilibrium state, in which the accepted failure criterion is met and this state is stable ($W\frac{\partial p}{\partial l} > 0$, p is the load parameter, l is the crack length). In this case, there is no avalanche-like growth of a crack (or its jumps from one stable equilibrium state to another), since the step-by-step method is not able to describe the unstable, avalanche-like growth of a crack, which is dynamic in nature. At the same time, one can imagine a calculation algorithm with the necessary load drop per crack step, in order to try to take into account the unstable domain $W\frac{\partial p}{\partial l} < 0$ of the critical equilibrium state. At the same time, however, the question remains open about the adequacy of the calculated path compared to the observation. A brief overview of some techniques for the numerical implementation of path calculations using finite element method can be found in [7]. However, let us discuss the question of whether there are any possibilities for determining crack paths in general. The well-known example of determining the entire crack path at once is based on the intuitive assumption that the crack coincides with the path of the smallest principal stress of the initial (without crack) stress field. This approach is confirmed by many particular cases of fracture, and is implemented in design, specifically, for diagnostic purposes using the "brittle coating method". And yet there is no "theoretical" evidence of the validity of this approach. In addition to discrete (differential) methods for finding the path of a process, theoretical mechanics also considers integral methods that are interconnected with differential ones and describe the same process, but in a different mathematical implementation. Assuming that the propagation of a crack is a process of mechanical movement of its tip (as a material point) along an optimal path, in some sense in comparison with other possible paths, then one arrives at the variational problem

$$\delta L = 0,\ L = \int_A^B \Phi\left(x, y, y'\right) \mathrm{d}s, \tag{1}$$

in which $y = y(x)$ is the crack path equation, and the conditions at ends A and B may be different depending on the formulation of the problem. Condition (1) allows to interpret the crack propagation line as a generalized geodesic line with a metric depending on the stress state, namely $ds^* = \Phi ds$. The art of choosing the integral function Φ, as well as the general choice of the Lagrange function in the integral variational principles of physics and mechanics [8], completely determines the success of the solution. Some simple specified relationships for function Φ in the form of its proportionality to the greatest principal stress, deformation, and the product of stress and displacement, used in [9–11], gave acceptable forms of crack paths for the cases studied. The optimization condition can be the condition of the least cost of fracture work in comparison with the energy supplied to this process. Based on this, the shape of crack cells on the plane surface of a uniformly stressed body in the form of hexahedrons and rectangles was obtained [12]. Parts of these hexagonal cells in the form of three cracks converging at an angle of 120° can be observed on the surface of dried ground. The natural assumption of stress relaxation around the first crack formed and the sequential formation and growth of subsequent cracks, together with the analysis of absorbed and available energies, made it possible to explain [13] various patterns of cracks in the Earth's crust on river floodplains, dried silt, on enamel dishes, on the surface of oil paintings etc. By the way, the idea of two types of energy required for the formation of cracks, allowed the authors together with Ya.B. Friedman, using the well-known analysis of fractures, introduce crack analysis into the mechanics of materials [14], which helps to draw useful conclusions about the mechanical behavior of a collapsing body. These representations, in accordance with the law of conservation of energy when varying the crack path, made it possible to write function Φ in the form

$$\Phi = 2\gamma - \frac{1}{2}\left(p_i^+ u_i^+ + p_i^- u_i^-\right). \tag{2}$$

Here, $2\gamma = \gamma^+ + \gamma^-$ is the specific work of fracture, p_i is the load acting on the surface of the crack $p_i = -\sigma_{ji} n_j$, σ_{ji} is the stress in the solid body at the points where a crack is assumed (the principle of superposition is used to transform the external load to the load distributed over the surface of the crack), n_j is the normal line to the surface of the crack, u_i is the displacement on the surface of the crack due to load p_i. The superscripts "plus and minus" refer to the opposite edges of the crack along which integration is carried out.

From the definition of the intensity of the distributed load p_i, from the very meaning of the integral principles, it is clear that the crack path is determined by the initial stressed state of the body, i.e. without taking into account its distortion by the presence of a crack (which manifests itself and, therefore, can only be taken into account when analyzing the process of crack growth).

The use of expression (2) is associated with the difficulty of determining displacement u_i included in this expression. It is proposed to circumvent this complication by applying the Winkler hypothesis, used in foundation theory. According to this hypothesis, the movement of a point on the boundary of a half-plane is proportional to the load on this point. The Winkler hypothesis $u_i = \beta p_i$ gives the greatest deviations from the solution of the theory of elasticity for the case of concentrated forces. Since in the present case a distributed load of intensity p_i acts on the edges of the crack, Winkler's hypothesis should give acceptable results. In addition, not a constant, but a variable bedding coefficient is proposed in the form $\beta = \beta\sqrt{l^2 - s^2}$, based on the elastic solution for a straight cut (in the Griffiths problem $\nu = \left(\frac{2p}{E}\sqrt{l^2 - s^2}\right)$). In this case, at the end of the crack (at $s = l$) the displacement is always zero, according to the physical meaning in the elastic problem. The introduction of such assumption made it possible to obtain reasonable solutions to a number of problems with crack paths, for example, a spiral for a separation crack during torsion of a circular cylinder or gouging out an area around the point of application of a concentrated force in a plane [9].

In [15], the inhomogeneous distribution of the fracture work density was considered in the form $\gamma = \gamma_0 + \gamma_1 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$, the accepted Winkler hypothesis for displacements on the desired path. The tensile stress on the plane is constant and satisfies the condition $2\gamma_0 + p_y u_y = 0$, which is obtained from (1) and (2) at $l \to 0$. As a result of the numerical calculation, a wavy crack path (bypassing the area of increased fracture resistance) across the tensile direction was obtained.

To be fair, there are negative opinions concerning the possibility of using integral principles to find crack paths [2; 16]. At the report of E.M. Morozov at the seminar of the Department of Mechanics of the Steklov Mathematical Institute, Professor G.P. Cherepanov stated that the crack path cannot be immediately determined as a whole, to which L.I. Sedov [2] remarked — "well, why not, it is an adequate formulation of the problem — among the multitude of lines passing through the predetermined points A and B, to find one that satisfies the condition of optimality in energies". It was also pointed out in [16] that the path obtained from the variational principle [17; 18] will depend on the external load as a parameter and will not "continue itself". However, in the authors' opinion, an additional condition (in the form of a fracture criterion) that crack growth along the desired path occurs at a critical load is omitted here, thereby eliminating the load parameter from the path equation.

The possibility of finding the path as a whole is also questionable in the case of combined loading, i.e. when in an unpredictable way, somewhere in the middle of the loading path, another load system is wedged in. In this case, the loading path must be foreseen in advance, divided into known simple ones, and the solution must be framed separately at each stage within which the loading is simple. This question is partly reminiscent of Poincaré's judgment about the existence of non-analytic solutions to Lagrange's equations of motion.

Let us consider the calculation of the path of crack motion under the action of a uniformly distributed load on a part of the half-plane boundary in more detail [18; 19]. The cracks enter from the ends of the loaded sections and have a form corresponding to the solution from the orthogonality of the crack to the lines of identical principal stresses [11; 18]. The determining equation is not the variational condition (1), but the assumption made that the stationary value of the functional L itself is equal to zero. This assumption was justified by the fact that the required and released energies are equal during the growth of a static stable crack, i.e. $\left(p_i = pf_i\left(s\right)\right)$.

$$\int_0^l \left(2\gamma - \beta p^2 f_i^2\left(s\right)\sqrt{l^2 - s^2}\right) ds = 0.$$

The modified Winkler hypothesis was used for displacements. The load parameter p, as well as the specific work of fracture 2γ, are excluded from the expression for L by the usual condition $\frac{\partial L}{\partial l} = 0,$ which gives a connection between the load parameter and the crack length. The desired crack path $y = y(x)$ is approximated by the expression $x = a_o + a_1 \exp(b_1 y) + a_2 \exp(b_2 y)$, constants a_0, a_1, a_2, b_1, b_2 in which are determined by the equation

$$\int_0^l \left[1 - \frac{f_i^2\left(s\right)\sqrt{l^2 - s^2}}{\int_0^l f_i^2\left(s\right) l \left(l^2 - s^2\right)^{\left(-\frac{1}{2}\right)} ds}\right] ds = 0.$$

The length of the crack along its found path depends on the load parameter according to the formula

$$p^2 = \frac{2\gamma}{\int_0^l \beta f_i^2\left(s\right) l \left(l^2 - s^2\right)^{-\frac{1}{2}} ds}.$$

The load parameter increases monotonically with increasing crack length, which indicates its stable growth.

In [20; 21], the same problem was solved for the crack equation in the form of polynomials of the fourth and fifth orders, obtaining approximately the same results. The calculated paths originated from the ends of the loaded area, going deeper and moving away from it in qualitative agreement with the experiments [22].

If in a material that is homogeneous and isotropic in terms of crack resistance properties, it is assumed that when the path is varied at a fixed length, the variation in the work of fracture is equal to zero and taking into account that,

$$\delta W + \delta A = \frac{1}{2}\delta\int_A^B \left[\left(\sigma_{ij} n_j u_i\right)^+ + \left(\sigma_{ij} n_j u_i\right)^- \right] ds \ ,$$

then this expression can be used to formulate a variational condition in which the variation of the functional is caused by the variation of the crack path. Then condition (1) will be rewritten in the form

$$\delta\int_A^B \left(p_i^+ u_i^+ + p_i^- u_i^- \right) ds = 0 \,, \tag{3}$$

Considering the problem in [23]

$$\delta I = \delta\int_A^B \frac{1}{2}\left(p_i^+ u_i^+ + p_i^- u_i^- \right) ds = 0 \tag{4}$$

with isoparametric condition $\int_0^l ds = \int_0^l \sqrt{1+y^2}\,dx = l - \text{const}$, one obtains a problem for the unconditional extremum of the functional $L = \int_0^{x_B} \left[\frac{1}{2}\left(p_i^+ u_i^+ + p_i^- u_i^- \right) + \Lambda \right] \sqrt{1+y^2}\,dx$ with boundary condition $\frac{\partial M}{\partial y} = 0$ for $x = x_B$ (since $\delta y_B \neq 0$, M is the integrand of L). Lagrange multiplier Λ in the isoparametric problem is the derivative of the extreme value of the original functional along the length of the crack, $\Lambda = \frac{dI}{dl} = \frac{\partial I}{\partial l} + \frac{\partial I}{\partial p}\frac{dp}{dl}$. For a crack growing in a subcritical state according to the law $p = p(l)$ at the critical moment one obtains $\frac{dp}{dl} = 0$ and then from $\Lambda = \frac{dI}{dl}$ we can find

$$I = \Lambda l - C \,. \tag{5}$$

At the same time, on a known path (in this case on the extremal), the load parameter p is related to the length of the crack by the relation $\frac{\partial}{\partial l}\left(2\gamma l - I\right) = 0$, from which $2\gamma - \Lambda = 0$. Now comparing (4) and (5), it can be concluded that at the extremal the energy functional takes on a constant value equal to zero.

$$\int_0^l \left[2\gamma - \frac{1}{2}\left(p_i^+ u_i^+ + p_i^- u_i^- \right) \right] ds = C. \tag{6}$$

It becomes possible to find the crack path based on the equality of functional (6) to zero. On the other hand, by immediately writing down the stationary condition of the functional in the form

$$\delta L = \delta\int_0^l \left[2\gamma - \frac{1}{2}\left(p_i^+ u_i^+ + p_i^- u_i^- \right) \right] ds = 0, \tag{7}$$

with the same condition $l = const$, then assuming that the crack develops stationary, all the supplied energy is spent on fracture, and then with a slow increase or decrease in the external load, the crack also slowly and steadily spreads along the desired path. This means that the total derivative of functional *L* (but not *I*) with respect to the length of the crack must be equal to zero. Consequently, in essence, both formulations of the problem gave the same result. An example of a combined solution method – step-by-step and integral ones — is given in the book [24], where the problem of stretching a plane with a crack, the line of which makes a specified angle with the direction of stretching, was solved. The displacements u_i were determined by mapping a plane with a broken crack onto a straight segment and then onto the exterior of a unit circle, and the increment in length at each step was fixed. The use of a step-by-step method made it possible to take into account the change in stress state during crack growth. It was found that a growing crack, starting at the end of the initial crack, is curved and oriented along the direction of tension. This result is consistent with experiments and other calculations [1; 25].

It is known that catastrophic fracture of main gas pipelines is characterized by the propagation of a crack along a sawtooth path with rounded teeth. Between the teeth of the saw, the crack path is close to a spiral one. The sharp change in the direction of the crack in the tooth area is apparently explained by the fact that the asymmetry of the soil backpressure on the left and right wings of the crack becomes significant as the crack tip approaches the ends of the horizontal (parallel to the day surface) diameter of the pipe. In this regard, the problem was posed about the spiral propagation of a crack with a constant pre-Rayleigh velocity in a pipe under the influence of internal pressure and with a given axial stress [26]. The problem was solved as a plane one, to develop a cylindrical surface onto an infinite plane containing a number of parallel semi-infinite cracks. Using the integral [2], it was concluded that the spiral mode of crack propagation is possible only at negative values of the axial stress. The crack growth rate and the angle between its vector and the pipe axis are determined. The beginning of the discussion about the ability of integral variational principles to describe the whole mechanical or physical process at once for a certain period of time goes back to the times of formation of its formulations, which involved such names as Huygens, Maupertuis, Euler, Fermat, Hertz, Helmholtz, Ostrogradsky and others. The history of the creation of these principles can be found in [8]. It seemed impossible to find a process flowing over time, all at once, with a seeming violation of causality, bearing the character of foresight. For example, a ray of light according to Fermat's principle in an inhomogeneously refracting medium, leaving a point, follows the path of the shortest time; a material point (in the absence of external forces) moves along a surface along a geodesic line according to Hertz's principle; even Bernoulli's classical problem of the fastest movement of a material point from position A to position B is solved by the integral principle.

In mathematical terms, confusion is dispelled due to the equivalence of the variational formulation of the integral criteria to the corresponding Euler — Lagrange differential equations. However, philosophically, confusion remains. For example, Planck tried to explain the process (preliminary) of a ray of light finding its future path — at first (after switching on) the light goes arbitrarily, but on all other paths except the suitable one, it stalls, i.e. photons are scattered and their main flux, although not immediately (on the appropriate time scales), begins to follow the path predicted by the integral principle. Generally speaking, considering the growth of a crack as a process that occurs in time along the desired path, it should be recognized that the crack path satisfies variational principles (in the form of the variation of the functional being equal to zero) only in special cases. In general, one should require that the integral over time of the variation of the Lagrange function be equal to zero, as required by the Hamilton — Ostrogradsky principle for non-conservative systems (in which the operations of integration and variation are non-permutable), i.e.

$$\int_{l_A}^{l_B} \delta L dt = 0, \delta L = \delta \int_{x_A}^{x_B} \Phi ds, \tag{8}$$

where variations of the function $y(x)$ describing the crack path are isochronous and not only the crack path must be found, but also the law of motion in time of its end $x(t)$, $y(t)$ (the position of which also varies) within a given time interval $t_B - t_A$. Here, the whole path is no longer determined at once, since it does not provide stationarity to any integral. Only in the case of explicit independence of L from time, $\delta L = 0$. This results in an equation in variations, to obtain which one can use the variational equation of L.I. Sedov [2], taking into account all possible associated effects.

Let us consider the case when the crack path is known. Then, when the position of the crack end is varied along path $\delta L < 0$, then δL is essentially dissipation (L is the difference between the required and available energy) and the crack grows spontaneously; if $\delta L > 0$, then δL is the increment of free energy and the crack does not grow spontaneously. Therefore, the condition $\delta L = 0$ determines the critical state corresponding to the beginning of crack propagation.

Apparently, one can also imagine (by analogy with other quasiparticles introduced in physics, for example, phonon, hydron, etc.) the tip of a crack in the form of a material quasiparticle — cracon — a point with mass and moving under the action of the corresponding crack-driving forces according to the laws of mechanics [27]. The concept of cracon in combination with an additional optimality condition involving the Pontryagin maximum was quite successfully used to solve the problem of how the stress applied to a plane should change, so that the ends of a single rectilinear symmetric crack, starting to move at a critical moment, stop at specified points (the problem of speed). It turned out that for this purpose the initial tensile stress in the middle of the cracon path, i.e. the end of the crack, should change sign.

In a plane stress field, having written down and solved Newton's laws of motion for cracon, one can establish its law of motion and thereby find the crack path.

In addition, in the supercritical stage, the laws of cracon motion for a straight crack were obtained [28], which made it possible to establish the average growth rate in connection with the critical load. The result agrees satisfactorily with experiment. It has been shown that in the supercritical stage (in the Griffiths problem), the crack growth rate practically corresponds to the well-known Mott formula [29].

For a more adequate formulation of the problem, it is desirable to clarify the state and the following process not only from a mechanical, but also from a thermodynamic point of view, since fracture is the process of transition of one (initial) state of equilibrium to another (final), one type of energy to another (taking into account the accompanying conditions). An attempt at such a classification is presented in [10; 30], where, in particular, on this basis it is shown that for cases with irreversible processes occurring inside the volume of a body with a crack, on the actual path of fracture the amount of heat acquired by the body (ultimately dissipation) is stationary. To highlight the methodologically general cognitive usefulness of thermodynamic signs, let us consider the process of transition of a body from one mechanical and thermodynamic equilibrium to another using the example of a sample stretched by force. In the case of a thermodynamically closed system, the increment in internal energy consists of increments in free energy δF and unavailable energy $T\delta S$ (at a constant temperature T, it cannot be separated). A decrease in free energy F is accompanied by a corresponding increase in entropy *S*. The sign of the increment δF indicates the possibility of a "spontaneous" process (fracture under constant external forces), and the absolute value of δF can serve as an indication of the rate of this process. Based on the second law of thermodynamics, the process cannot occur spontaneously without reducing the free energy of the system. In this case, since $\delta F < 0$, the equilibrium state of the system is unstable, but for instability to be realized, initial energy is required to activate this process.

Let us consider the process of deformation over time. First, while there is no external force, the body is in equilibrium and $F = 0$. Now (in a closed system that allows the exchange of energy, but not mass), an external force is applied, the mechanical work of which leads to an increase in internal energy, which in turn increases free energy (due to an increase in potential deformation energy). With increasing force, the excess free energy is supported by the work of this force, and the unavailable energy TS still remains approximately constant (although it may be due to elastic imperfections). If now the system at a certain achieved level of *F* (and force) is considered, then the energies can be redistributed towards an increase in entropy and a decrease in free energy. Then $\delta F < 0$, and all states at $F > 0$ are unstable, but this is instability at rest, such as a supersaturated solution or a superheated liquid. In this case, the degree of nonequilibrium is proportional to the absolute value of the increase in free energy when the body is loaded with an external force. The system seems to be stable in small things: the higher *F*, the smaller "the hole in which the ball is located, and the steeper the hill on top of which this hole is located." The system is unstable at large; but the initial energy is not yet enough to unbalance it. Finally, while *F* increases, an inevitable accident occurs – temperature fluctuations (apparently) provided the initial activation energy, which was sufficient to remove the system from unstable equilibrium (the system becomes unstable in small quantities) and fracture is a spontaneous irreversible process of transition to a new state of equilibrium — continues, $\delta F < 0$, $T\delta S > 0$, and based on Prigogine's principle [18; 31] $\frac{\delta S}{\delta t}$ is minimal and therefore

$\frac{F}{t}$ is maximum. From here there are two conclusions — with increasing temperature of the environment, the strength of the body decreases, and fracture occurs at the maximum possible (for given conditions) speed. In this case, the free energy comes down, but after complete fracture, when a new equilibrium state is reached, the value $F > 0$ (i.e., is established at a new, higher level), since the new surface of the body has surface energy, which is part of the free energy.

3. Results

There are several ways to calculate the crack propagation path on the surface of a stressed body — the crack trajectory — at which the crack equation is immediately found. Eventually, it turned out that it will be possible using the mathematical apparatus of calculus of variations and the variational principles of mechanics. At the same time, in each specific case, among those considered, there are complicating circumstances, the overcoming of which should be sought in accordance with the formulation of this task. Several successfully solved tasks can be found in [33].

4. Conclusions

1. An analysis of possible options for determining the crack growth trajectory as a whole is given.
2. It is concluded that the equation of the crack trajectory can be obtained using the basis of the calculus of variations of mathematics and the variational principles of mechanics.

References / Список литературы

1. Xu S., Reinhardt H.W. Determination of double-K criterion for crack propagation in quasi-brittle fracture, Part I: Experimental investigation of crack propagation. *International Journal of Fracture*. 1999;98(2):111–149. https://doi.org/10.1023/A:1018668929989

2. Sedov L.I. *Continuum mechanics.* St. Petersburg: Lan Publ.; 2004. (In Russ.) EDN: QJMTCB

Седов Л.И. Механика сплошной среды. СПб.: Лань, 2004. EDN: QJMTCB

3. Elices M., Guinea G.V., Gomez J., Planas J. The cohesive zone model: advantages, limitations and challenges. *Engineering Fracture Mechanics*. 2002;69(2):137–163. EDN: ASFTOD

4. Schapery R.A. A theory of viscoelastic crack growth: revisited. *International Journal of Fracture.* 2022;233:1–16. https://doi.org/10.1007/s10704-021-00605-z

5. Dombrovskii Y.M., Stepanov M.S. Mechanisms of Intragrain Plastic Deformation in Steel Heating Process. *Metal Science and Heat Treatment.* 2024;65:747–750. https://doi.org/10.1007/s11041-024-01000-w

6. Komarov O.N., Sevastyanov G.M., Abashkin E.E., Khudyakova V.A. Shift of a Spherical Layer Under High Pressures. *Metallurgist.*2023;67:801–813. https://doi.org/10.1007/s11015-023-01568-3

7. Lepikhin A.M., Morozov E.M., Makhutov N.A., Leschenko V.V. Possibilities of Estimation of Fracture Probabilities and Allowable Sizes of Defects of Structural Elements According to the Criteria of Fracture Mechanics. *Inorganic Materials*. 2023;59(15);1524–1531. https://doi.org/10.1134/S0020168523150074

8. Mahutov N.A., Morozov E.M., Gadenin M.M., Reznikov D.O., Yudina O.N. Coupled thermo-mechanical analysis of stress–strain response and limit states of structural materials taking into account the cyclic properties of steel and stress concentration. *Continuum Mechanics and Thermodynamics.* 2023;35:1535–1545. https://doi.org/10.1007/s00161-022-01160-1

9. Morozov E.M., Alymov M.I. Fracture Pressure in Microdefects of Consolidated Materials. *Doklady Physical Chemistry.* 2021;501:111–113. https://doi.org/10.1134/S0012501621110026

10. Matvienko Y.G., Morozov E.M.Two basic approaches in a search of the crack propagation angle. *Fatigue and Fracture of Engineering Materials and Structures*. 2017;40(8):1191–1200. https://doi.org/10.1111/ffe.12583

11. Pook L.P. The linear elastic analysis of cracked bodies, crack paths and some practical crack path examples. *Engineering Fracture Mechanics*. 2016;167:2–19. https://doi.org/10.1016/j.engfracmech.2016.02.055

12. Morozov E.M., Alymov M.I. Fracture Pressure in Microdefects of Consolidated Materials. *Doklady Physical Chemistry.* 2021;501(1):111–113. https://doi.org/10.1134/S0012501621110026

13. Kurbanmagomedov A.K. Crack of the normal gap in the elastic layer. *Bulletin of the Yakovlev chuvash state pedagogical university series: Mechanics of limit state*. 2017;(1):96–104. (In Russ.) EDN: ZGIEKB

Курбанмагомедов А.К. Трещина нормального разрыва в упругом слое // Вестник ЧГПУ им. И.Я. Яковлева. Серия: Механика предельного состояния. 2017. № 1. С. 96–104. EDN: ZGIEKB

14. Kolesnikov Yu.V., Morozov E.M. *Mechanics of contact failure.* Moscow: Nauka Publ.; 2012. (In Russ.)

Колесников Ю.В., Морозов Е.М. Механика контактного разрушения. М.: Наука, 2012. 224 с.

15. Gordeeva G.V., Kurbanmagomedov A.K., Spitsov D.V. Strength control of concrete structures during the assessment of the residual life of buildings and structures of a hazardous production facility in the field of thermal power engineering. *The System technologies.* 2022;(4):73–86. (In Russ.) https://doi.org/10.55287/22275398_2022_4_73

Гордеева Г.В., Курбанмагомедов А.К., Спицов Д.В. Контроль прочности бетонных конструкций при проведении оценки остаточного ресурса зданий и сооружений опасного производственного объекта в сфере теплоэнергетики // Системные технологии. 2022. № 4. С. 73–86. https://doi.org/10.55287/22275398_2022_4_73

16. Kurbanmagomedov A., Radzhabov Z., Okolnikova G. Investigation of Normal Fracture Cracks in an Infinite Elastic Medium. In: Guda, A. (eds) *Networked Control Systems for Connected and Automated Vehicles. NN 2022. Lecture Notes in Networks and Systems*. Springer, Cham. 2022;509. https://doi.org/10.1007/978-3-031-11058-0_142

17. Nikhamkin M., Ilinykh A. Low cycle fatigue and crack grow in powder nickel alloy under turbine disk wave form loading: Validation of damage accumulation model. *Applied Mechanics and Materials*. 2014;467:312–316. https://doi.org/10.4028/www.scientific.net/AMM.467.312

18. Yablonsky A.A., Nikiforova V.M. Course of theoretical mechanics. Moscow: KnoRus Publ.; 2011. (In Russ.) EDN: QJYXKR

Яблонский А.А., Никифорова В.М. Курс теоретической механики. М.: Изд-во КноРус, 2011. 603 с. EDN: QJYXKR

19. Hou J.-P., Wang Q., Yang H.-J., Wu X.-M., Li Ch.H., Zhang Zh.-F., Li X.W. Fatigue and Fracture behavior of a Cold-Drawn Commercially pure aluminum wire. *Materials.* 2016;9(9):764. https://doi.org/10.3390/ma9090764

20. Wang Q., Ren J.Q., Wu Y.K., Jiang P., Sun Z.J., Liu X.T. Comparative study of crack growth behaviors of fully-lamellar and bi-lamellar Ti-6Al-3Nb-2Zr-1Mo alloy. *Journal of Alloys and Compounds.* 2019;789:249–255. https://doi.org/10.1016/j.jallcom.2019.02.302

21. Sadananda K., Babu M.N., Vasudevan A.K. A review of fatigue crack growth resistance in the short crack growth regime. *Materials Science and Engineering: A.* 2019;754:674–701. https://doi.org/10.1016/j.msea.2019.03.102

22. Mall S., Perel V.Y. Crack growth behavior under biaxial fatigue with phase difference. *International Journal of Fatigue.* 2015;74:166–172. https://doi.org/10.1016/j.ijfatigue.2015.01.005

23. Berto F., Ayatollahi M.R., Borsato T., Ferro P. Local strain energy density to predict size-dependent brittle fracture of cracked specimens under mixed mode loading. *Theoretical and Applied Fracture Mechanics*. 2016;86(Part B): 217–224. https://doi.org/10.1016/j.tafmec.2016.07.004

24. Kaminsky A.A., Kurchakov E.E. Fracture Process Zone at the Tip of a Mode I Crack in a Nonlinear Elastic Orthotropic Material. *International Applied Mechanics.* 2019;55(1):23–40. https://doi.org/10.1007/s10778-019-00931-9

25. Tumanov A.V., Shlyannikov V.N., Zakharov A.P. Crack growth rate prediction based on damage accumulation functions for creep-fatigue interaction. *Fracture and Structural Integrity*. 2020;14(52):299–309. https://doi.org/10.3221/IGF-ESIS.52.23

26. Ghelichi R., Kamrin K. Modeling growth paths of interacting crack pairs in elastic media. *Soft Matter*. 2015;(11): 7995–8012. https://doi.org/10.1039/c5sm01376c

27. Musayev V.K. Mathematical Modeling of Stresses Under Unsteady Wave Action in Geo-Objects. *Power Technology and Engineering.* 2023;57(3):351–364. https://doi.org/10.1007/s10749-023-01668-9

28. Stepanova L.V., Roslyakov P.S. Multi-parameter description of the crack-tip stress field: analytic determination of coefficients of crack-tip stress expansions in the vicinity of the crack tips of two finite cracks in an infinite plane medium. *International Journal of Solids and Structures.* 2016;100–101:11–28. https://doi.org/10.1016/j.ijsolstr.2016.06.032

29. Xu S., Xiong L., Deng Q., McDowell D.L. Mesh refinement schemes for the concurrent atomistic-continuum method. *International Journal of Solids and Structure.* 2016;90:144–152. https://doi.org/10.1016/j.ijsolstr.2016.03.030

30. Carpinteri A., Brighenti R., Spagnoli A. Part-through cracks in pipes under cyclic bending. *Nuclear Engineering and Design.* 1998;185(1):1–14. https://doi.org/10.1016/S0029-5493(98)00189-7

31. Carpinteri A., Brighenti R., Spagnoli A. Fatigue growth simulation of part-through flaws in thick-walled pipes under rotary bending. *International Journal of Fatigue.* 2000:22(1):1–9. https://doi.org/10.1016/S0142-1123(99)00115-2

32. Musayev V.K. Mathematical Modeling of Explosive and Seismic Impacts on an Underground Structure. *Power Technology and Engineering*. 2024;57(6):875–881. https://doi.org/10.1007/s10749-024-01751-9

33. Astafyev V.I., Radaev Yu.N., Stepanova L.V. *Nonlinear mechanics of destruction*. Samara: Samara National Research University named after Academician S.P. Krolev. 2001. (In Russ.) EDN: XDVQPF

Астафьев В.И., Радаев Ю.Н., Степанова Л.В. Нелинейная механика разрушения Сам. гос. ун-т. Самара: Сам. ун-т, 2001. 630 с. EDN: XDVQPF