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ANALYTICAL AND NUMERICAL METHODS OF ANALYSIS OF STRUCTURES

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Torsion problem: stress statement and solution by the boundary element method

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Abstract. The formulation of the problem of torsion regarding stresses and its solution by the boundary elements method are described. The main advantage of the problem formulation in stresses is direct determination of stresses in the cross-section, unlike the classical formulation, when the result of the approximate solution is the Prandtl stress function values, and the determination of stresses is brought down to numerical differentiation. The boundary integral equation of the second kind is obtained to formulate the problem with respect to stresses. The procedure for solving the problem by the boundary elements method is described, the system of solving equations is compiled. Solutions of test problems on torsion of rods with rectangular and channel cross-sections are presented. Comparison of the calculation results with known analytical solutions illustrates the reliability and permissible engineering accuracy of the obtained solutions.

Keywords: Elastic Rod Torsion, Poisson's equation, Integral Representation of Stresses, Boundary Integral Equation, Integral Equation of the second kind

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Задача о кручении: постановка в напряжениях и решение методом граничных элементов

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Аннотация. Приводится постановка задачи о кручении относительно напряжений и ее решение методом граничных элементов. Основным достоинством данной постановки задачи является непосредственное определение напряжений в сечении, в отличие от классической постановки, где результатом приближенного решения являются значения функции напряжений Прандтля, а определение напряжений сводится к численному дифференцированию. Для постановки задачи относительно напряжений получено граничное интегральное уравнение второго рода. Описана процедура решения задачи методом граничных элементов, составлена система разрешающих уравнений. Представлены решения тестовых задач о кручении стержней прямоугольного и швеллерного сечений. Сопоставление результатов расчета с известными аналитическими решениями иллюстрирует достоверность и допустимую инженерную точность полученных решений.

Ключевые слова: кручение упругих стержней, уравнение Пуассона, интегральное представление напряжений, граничное интегральное уравнение, интегральное уравнение второго рода

1. Introduction

The torsion problem for elastic prismatic rods is one of the oldest problems in the theory of elasticity. It was mathematically formulated by Saint-Venant in the middle of the 19th century. Before the broad spread of ECM, many problems for bars with relatively simple shapes of cross-section were solved analytically. The obtained solutions were summarized in the monograph [1].

With the creation of ECM, it became possible to obtain numerical solutions to the problem of torsion for bars with an arbitrary cross-section. This led to the rapid development of numerical methods for solving torsion problems and problems of potential theory that are similar in their mathematical formulation. In spite of their considerable age, these problems are still the subject of research for many scientists and engineers. These works contain formulations and methods for solving problems for inhomogeneous bars [2; 3], nanosized bars [4], problems of dynamics [5] and others.

One of the widely used numerical methods for solving torsion problems is the boundary elements method (BEM) [7–13]. This method has been actively developed since the 70s of the 20th century, but up to now its new formulations continue to appear, including those for torsion problems of bars [14–18].

The traditional mathematical formulation of the torsion problem consists of finding the stress function[1] [1; 19], and the stresses themselves are subsequently found by differentiation of the stress function [20; 21]. From the point of view of determining the stresses in a cross-section in numerical solution, the formulation of the problem regarding the stress function has two significant disadvantages. Since the result of the approximate solution of the problem in such a formulation is actually the values of the stress function in the nodes of boundary elements, then the determination of stresses is reduced to numerical differentiation. This leads to an additional source of computational error. The second reason reducing the accuracy of the solution is the fact that the boundary integral equation regarding the stress function is a numerically unstable equation of the first kind[2].

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[1] Timoshenko S.P., Goodyear D. Theory of elasticity. Moscow: Nauka Publ.; 1975. (In Russ.)
[2] Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. Numerical methods. Moscow: Laboratoriya znanii Publ.; 2020. (In Russ.)

This paper presents a formulation of the torsion problem in stresses, derives the boundary integral equation regarding stresses, and describes the procedure for solving the problem by the boundary elements method. In difference from the classical formulation, the proposed formulation leads to an integral equation of the second kind, and its numerical algorithm is a stable one[2]. The verification of the methodology was carried out on the example of two cross-sections by means of comparing the numerical solution of the problem with the known analytical one.

2. Methods

Formulation of the torsion problem regarding stresses

Consider the problem of torsion for a prismatic bar of an arbitrary cross-section under the action of two external moments which lie in the plane of its outermost cross-sections. We consider the volumetric forces to be equal to zero and the lateral surface to be free of external loads.

The following coordinate system has been choosen: the z-axis coincides with the axis of torsion, i.e., the axis that remains unmoved when the bar is twisted; the *x*- and *y*-axes are mutually orthogonal and located randomly in the plane of the cross-section.

The problem of torsion of a bar with cross-section *S* and contour *Г*, is formulated in terms of the Prandtl stress function *F* in the following way[3] [24]:

$$S: \Delta F = -2, \tag{1}$$

$$\Gamma: F = \text{const}, \tag{2}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is Laplace operator on a plane; $\partial_x \equiv \frac{\partial}{\partial x}$, $\partial_y \equiv \frac{\partial}{\partial y}$.

If the area confined by the contour is single-linked, then the constant in equation (2) can be assumed to be an arbitrary one, since it does not affect the values of stresses. Hereinafter we take *F* at the boundary *Г* to be equal to zero.

Required tangential stresses τ_{zx} and τ_{zy} are expressed via the function *F* by the formulas:

$$\tau_{zx} = G\theta\partial_y F, \qquad \tau_{zy} = -G\theta\partial_x F, \tag{3}$$

where G is the shear modulus; θ is an angle of the bar twisting per the unit of length.

In the numerical solution of problem (1), the determination of stresses will require numerical differentiation according to formulas (3). The next step is to state the problem formulation, where the unknowns will be stresses directly.

Equation (1) can be rewritten using equation (3):

$$\partial_x\tau_{zy} - \partial_y\tau_{zx} = 2G\theta. \tag{4}$$

The functions τ_{zy} and τ_{zx} should satisfy the equation of equilibrium

$$\partial_x\tau_{zx} + \partial_y\tau_{zy} = 0. \tag{5}$$

Boundary condition on the surface *Г*:

$$n_x\tau_{zx} + n_y\tau_{zy} = 0, \tag{6}$$

where n_x and n_y are projections of the vector of unit external normal to the contour *Г*.

Finally, the following formulation of the torsion problem relative to stresses is obtained:

[3] Alexandrov A.V., Potapov V.D. Fundamentals of the theory of elasticity and plasticity. Moscow: Vysshaya shkola Publ.; 1990. (In Russ.)

$$S\colon \begin{cases} \partial_x \tau_{zx} + \partial_y \tau_{zy} = 0 \\ \partial_x \tau_{zy} - \partial_y \tau_{zx} = 2G\theta \end{cases} \tag{7}$$

$$\Gamma\colon n_x \tau_{zx} + n_y \tau_{zy} = 0\,. \tag{8}$$

Then it is possible to find the tangential stresses in the cross-section without finding the values of the stress function *F* by solving the system of equations (7)–(8).

The vector notation of the formulation (7)–(8) was given below with necessary explanations. It was supposed that vector $\boldsymbol{\tau}$ with components τ_{zx}, τ_{zy}, $\boldsymbol{\nabla}$ is a gradient operator in the plane, $\boldsymbol{n}$ is a vector of external unit normal to the boundary *Γ*, vector $\tilde{\boldsymbol{\tau}}$ is a vector $\boldsymbol{\tau}$. rotated by *π/2* clockwise when viewed from the *z*-axis. Thus, the formulation of the problem (7)–(8) acquires the following form:

$$S\colon \begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\tau} = 0 \\ \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\tau}} = 2G\theta \end{cases} \tag{9}$$

$$\Gamma\colon \ \boldsymbol{n} \cdot \boldsymbol{\tau} \equiv \tau_n = 0. \tag{10}$$

Boundary integral equations regarding the stress function

Let $\xi,\ \beta$ be two arbitrary points in the plane, $\boldsymbol{R_\xi}, \boldsymbol{R_\beta}$ are their radius vectors. Then the following notations can be introduced:

$$\boldsymbol{R} = \boldsymbol{R_\xi} - \boldsymbol{R_\beta}, \quad R = |\boldsymbol{R}| = \sqrt{\boldsymbol{R} \cdot \boldsymbol{R}}. \tag{11}$$

Also, the function $v(x,y)$ were introduced, which is the fundamental solution of the Laplace equation in the plane[4]:

$$v = lnR. \tag{12}$$

By definition of a fundamental solution:

$$\Delta_\xi v = -2\pi\delta(\xi,\beta), \tag{13}$$

where Δ_ξ is the Laplace operator, with differentiation on the coordinates of the point ξ; $\delta(\xi,\beta)$ is a Dirac delta function with the following basic property [7]:

$$\int\limits_S g(\beta)\delta(\xi,\beta)dS(\beta) = g(\xi)c(\xi), \tag{14}$$

Here $g(\beta)$ is an arbitrary continuous function; the notation $dS(\beta)$ underlines that integration was performed on the coordinates of the point *β*:

$$c(\xi) = \begin{cases} 1, & \xi \in S, \\ 0{,}5, & \xi \in \Gamma, \\ 0, & \xi \notin S + \Gamma. \end{cases} \tag{15}$$

The next step was to write the second Green's formula for the functions $v(x,y)$ and $F(x,y)$ [25]:

$$\int_S \ [-v(\Delta F) + F\Delta v]\, dS = \int_\Gamma \ \left(F\frac{\partial v}{\partial n} - v\frac{\partial F}{\partial n}\right) d\Gamma. \tag{16}$$

The following form was obtained using equations (13)–(14):

[4] Tikhonov A.N., Samarsky A.A. Equations of mathematical physics: 6th ed. Moscow: House of Moscow State University Publ.; 1999. (In Russ.)

$$2\pi c(\xi)F(\xi) = \int_S -v(\Delta F)\, dS + \int_\Gamma \left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n}\right) d\Gamma.$$

Let us place the point ξ on the boundary of Γ, and take the value of the function $c(\xi)$ according to (15):

$$\pi F(\xi) = \int_S -v(\Delta F)\, dS + \int_\Gamma \left(v\frac{\partial F}{\partial n} - F\frac{\partial v}{\partial n}\right) d\Gamma. \quad (17)$$

The sought function in equation (17) in solving the torsion problem is the normal derivative of the stress function F at the boundary Γ. Thus, equation (17) with respect to $\frac{\partial F}{\partial n}$ is a Fredholm equation of the first kind. The numerical solution of the equation of the first kind can lead to highly inaccurate results, since equations of this type are predominantly poorly specified and numerically unstable[5].

Boundary integral equations regarding stresses

To obtain the boundary integral equations regarding stresses, it is necessary to have an integral representation on the plane for the stress vector $\boldsymbol{\tau}$.

Suppose $\boldsymbol{a}$ is a unit tensor in the plane, $\boldsymbol{c}$ is a basis co-symmetric tensor in the plane. If $\boldsymbol{e_x}$, $\boldsymbol{e_y}$ are orthants of the Cartesian coordinate system, then the introduced tensors have the following dyadic representation:

$$\boldsymbol{a} = \boldsymbol{e_x e_x} + \boldsymbol{e_y e_y}, \quad \boldsymbol{c} = \boldsymbol{e_x e_y} - \boldsymbol{e_y e_x}.$$

The main property of the introduced tensors:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}, \quad \boldsymbol{c}\cdot\boldsymbol{b} = \widetilde{\boldsymbol{b}},$$

where $\boldsymbol{b}$ is an arbitrary vector.

Let us write the following representation of the Laplacian from an arbitrary vector $\boldsymbol{\varphi}$:

$$\Delta\boldsymbol{\varphi} = \boldsymbol{\nabla}\cdot(\boldsymbol{a}\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + \boldsymbol{c}\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\varphi}}\,). \quad (18)$$

Equation (18) was multiplied scalarly by the arbitrary vector $\boldsymbol{f}$, transforming the right part using equality [19]:

$$\boldsymbol{f}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{Q} = \boldsymbol{\nabla}\cdot(\boldsymbol{Q}\cdot\boldsymbol{f}) - \boldsymbol{Q^T}\cdot\cdot\,\boldsymbol{\nabla f}, \quad (19)$$

where $\boldsymbol{Q}$ is an arbitrary tensor of the second rank; symbol "··" means the double scalar product of two tensors [19].

The obtained equation has the form:

$$\boldsymbol{f}\cdot\Delta\boldsymbol{\varphi} = \boldsymbol{\nabla}\cdot\left(\boldsymbol{f}\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + \tilde{\boldsymbol{f}}\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\varphi}}\right) - \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}\,\boldsymbol{\nabla}\cdot\boldsymbol{f} - \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\varphi}}\,\boldsymbol{\nabla}\cdot\tilde{\boldsymbol{f}}. \quad (20)$$

By integrating (20) over area S and using the formula of Ostrogradsky-Gauss [19]

$$\int_S \boldsymbol{\nabla}\cdot\boldsymbol{b}\, dS = \int_\Gamma \boldsymbol{n}\cdot\boldsymbol{b}\, d\Gamma, \quad (21)$$

which is valid for any vector $\boldsymbol{b}$, the following formula was obtained:

$$\int_S \boldsymbol{f}\cdot\Delta\boldsymbol{\varphi}\, dS = \int_\Gamma (f_n\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + f_t\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\varphi}})d\Gamma - \int_S \left(\boldsymbol{\nabla}\cdot\boldsymbol{f}\,\boldsymbol{\nabla}\cdot\boldsymbol{\varphi} + \boldsymbol{\nabla}\cdot\tilde{\boldsymbol{f}}\,\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{\varphi}}\right)dS\,, \quad (22)$$

where $f_n \equiv \boldsymbol{n}\cdot\boldsymbol{f}$, $f_t \equiv \boldsymbol{t}\cdot\boldsymbol{f}$ are projections of the vector f on the normal and tangent to the contour Γ.

[5] Bakhvalov N.S., Zhidkov N.P., Kobelkov G.M. Numerical methods. Moscow: Laboratoriya znanij Publ.; 2020. (In Russ.)

If using equations (11)–(13) and assuming that $\boldsymbol{\varphi} = \boldsymbol{b} lnR$ in equality (22), where $\boldsymbol{b}$ is an arbitrary constant vector, then the following formula for vector $\boldsymbol{\varphi}$ is got:

$$\Delta_\beta \boldsymbol{\varphi} = \boldsymbol{b} 2\pi\delta(\xi,\beta), \qquad \boldsymbol{\nabla}_\beta \cdot \boldsymbol{\varphi} = \boldsymbol{b} \cdot \frac{\boldsymbol{R}}{R^2}, \qquad \boldsymbol{\nabla}_\beta \cdot \widetilde{\boldsymbol{\varphi}} = -\boldsymbol{b} \cdot \frac{\widetilde{\boldsymbol{R}}}{R^2}. \tag{23}$$

Substituting this formula into (22) and using property (14), the next equation was obtained (if discarding an arbitrary vector $\boldsymbol{b}$)):

$$2\pi c(\xi)\boldsymbol{f}(\xi) = \int_\Gamma \left(f_n \frac{\boldsymbol{R}}{R^2} - f_t \frac{\widetilde{\boldsymbol{R}}}{R^2} \right) d\Gamma(\beta) + \int_S \left(\frac{\widetilde{\boldsymbol{R}}}{R^2} \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{f}} - \frac{\boldsymbol{R}}{R^2} \boldsymbol{\nabla} \cdot \boldsymbol{f} \right) dS(\beta). \tag{24}$$

Equality (24) is the required integral representation of an arbitrary vector $\boldsymbol{f}$ in the plane. After replacing the arbitrary vector $\boldsymbol{f}$ by the vector of tangential stresses $\boldsymbol{\tau}$ and using equations (9), (10), formula (24) can be written in the form:

$$2\pi c(\xi)\boldsymbol{\tau}(\xi) = -\int_\Gamma \tau_t \frac{\widetilde{\boldsymbol{R}}}{R^2} d\Gamma(\beta) + \int_S \frac{\widetilde{\boldsymbol{R}}}{R^2} 2G\theta \, dS(\beta). \tag{25}$$

Using formula (25), it is possible to determine the value of vector $\boldsymbol{\tau}$ at any point of cross-section S if its tangent projection τ_t is known at the boundary Γ. To find the function τ_t, it is necessary to multiply (25) scalarly by the vector $\boldsymbol{t}_\xi$ (that is a unit vector tangent to the contour Γ at the point ξ) and impose the condition $\xi \in \Gamma$. Thus, the following integral equation with respect to the function τ_t ($\boldsymbol{t}_\xi \cdot \widetilde{\boldsymbol{R}} = -\boldsymbol{n}_\xi \cdot \boldsymbol{R}$) can be obtained:

$$\tau_t(\xi) - \frac{1}{\pi} \int_\Gamma \tau_t \frac{\boldsymbol{n}_\xi \cdot \boldsymbol{R}}{R^2} d\Gamma(\beta) = -\frac{1}{\pi} \int_S \frac{\boldsymbol{n}_\xi \cdot \boldsymbol{R}}{R^2} 2G\theta \, dS(\beta). \tag{26}$$

Equation (26) is an integral equation of the second kind, unlike the formulation with the use of the stress function F (17).

The next step is to convert the integral in the right part of equation (26) into an integral over the contour:

$$\int_S \frac{\boldsymbol{n}_\xi \cdot \boldsymbol{R}}{R^2} dS(\beta) = \int_S \boldsymbol{n}_\xi \cdot \boldsymbol{\nabla}_{\boldsymbol{\beta}} ln\, R \, dS(\beta) = \int_\Gamma \boldsymbol{n}_\xi \cdot \boldsymbol{n}_{\boldsymbol{\beta}} ln\, R \, d\Gamma(\beta).$$

As a result, equation (26) takes the form:

$$\tau_t(\xi) - \frac{1}{\pi} \int_\Gamma \tau_t \frac{\boldsymbol{n}_\xi \cdot \boldsymbol{R}}{R^2} d\Gamma(\beta) = -\frac{2G\theta}{\pi} \int_\Gamma \boldsymbol{n}_\xi \cdot \boldsymbol{n}_{\boldsymbol{\beta}} ln\, R \, d\Gamma(\beta). \tag{27}$$

The solution of equation (27) is the tangential stresses on the contour Γ, and one of them is the maximum tangential stress in the section [19].

Boundary elements method

After that the discrete form of the integral equation (27) is written, using the so-called “constant” boundary elements. The nodes, where the values of the unknown function are calculated, are in the middle of each element. The value of the sought function within one boundary element is assumed to be constant. The boundary was divided into N elements, and equation (27) was transformed into the form:

$$\tau_{ti} - \frac{1}{\pi} \sum_{j=1}^{N} \int_{\Gamma_j} \tau_{tj} \frac{\boldsymbol{n}_{\boldsymbol{i}} \cdot \boldsymbol{R}_{\boldsymbol{ij}}}{R_{ij}^2} d\Gamma(j) = -\frac{2G\theta}{\pi} \sum_{j=1}^{N} \int_{\Gamma_j} \boldsymbol{n}_{\boldsymbol{i}} \cdot \boldsymbol{n}_{\boldsymbol{j}} \cdot lnR_{ij} d\Gamma(j), \tag{28}$$

where i is the number of the boundary element.

The function τ_{tj} in equation (28) under the sign of the integral has the constant values in the area of each element and, therefore, can be taken out of the sign of integral:

$$\tau_{ti} - \frac{1}{\pi}\sum_{j=1}^{N} \tau_{tj} \int_{\Gamma_j} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R_{ij}}}{R_{ij}^2} d\Gamma(j) = -\frac{2G\theta}{\pi}\sum_{j=1}^{N} \int_{\Gamma_j} \boldsymbol{n_i} \cdot \boldsymbol{n_j} \cdot lnR_{ij} d\Gamma(j).$$

If points *i* and *j* are located on the same element, then the integral

$$\int_{\Gamma_j} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R_{ij}}}{R_{ij}^2} d\Gamma(j)$$

is identically equal to zero due to orthogonality of vectors $\boldsymbol{n_i}, \boldsymbol{R_{ij}}$, and the integral

$$\int_{\Gamma_j} \boldsymbol{n_i} \cdot \boldsymbol{n_j} \cdot lnR_{ij} d\Gamma(j)$$

can be calculated analytically:

$$\boldsymbol{n_i} \int_{l} \boldsymbol{n_j} lnR_{ij} d\Gamma(j) = \frac{1}{2} l \int_{-1}^{1} ln\frac{l|\xi|}{2} d\xi = \frac{1}{2} l \left(\int_{-1}^{1} ln\frac{l}{2} d\xi + \int_{-1}^{1} ln|\xi|\, d\xi \right) =$$

$$= \frac{1}{2} l \left(2ln\frac{l}{2} + \lim_{\varepsilon \to 0} \left(\int_{-1}^{-\varepsilon} ln|\xi| d\xi + \int_{\varepsilon}^{1} ln\, \xi\, d\xi \right) \right) = \frac{1}{2} l \left(2ln\frac{l}{2} + 2\lim_{\varepsilon \to 0} (\xi ln\, \xi - \xi)|_{\varepsilon}^{1} \right) = l\left(ln\frac{l}{2} - 1 \right).$$

The following notations were introduced:

$$H_{ij} = \delta_{ij} - \frac{1}{\pi} \int_{\Gamma_j} \frac{\boldsymbol{n_i} \cdot \boldsymbol{R_{ij}}}{R_{ij}^2} d\Gamma(j),$$

$$B_i = -\frac{2G\theta}{\pi} \left[\sum_{\substack{j=1 \\ i \neq j}}^{N} \int_{\Gamma_j} \boldsymbol{n_i} \cdot \boldsymbol{n_j} \cdot lnR_{ij} d\Gamma(j) + l\left(ln\frac{l}{2} - 1 \right) \delta_{ij} \right],$$

where δ_{ij} is the Kronecker delta.

The obtained system of equations can be represented in a matrix form:

$$\underbrace{\boldsymbol{H}}_{N \times N}\ \underbrace{\boldsymbol{T}}_{N \times 1} = \underbrace{\boldsymbol{B}}_{N \times 1},$$

where $\boldsymbol{T}$ is a vector, whose components are unknown values of tangential stresses at the boundary.

3. Results and discussion

The calculations for two cross-sections (a rectangular and channel ones) were performed in order to verify the described formulation of the problem. The dimensions of these cross-sections are shown in Figure 1.

To simplify the interpretation of the results, it was assumed that the values of the angle of section's torsion θ (rad/cm) and shear modulus G (N/cm^2) are equal to one, since their values do not affect the distribution of stresses in the cross-section. The distribution of maximum tangential stresses along the boundaries of the cross-sections was studied next. Only a part of the boundaries was considered due to the symmetrical distribution of stresses along the cross-section: there are two edges for a rectangular cross-section (from point 1 to point 3, according to Figure 1), and four edges for a channel section (from point 1 to point 5, according to Figure 1).

The results of the calculation were compared with the known analytical solution of the Poisson equation describing the torsion of bars with constant profile[6] [1]. The calculations were performed with consecutive reduction of the boundary element size: 1, 0.5, 0.25 cm.

Figure 2 shows the results of the calculation of the rectangular cross-section.

Figure 3 presents the results of the calculation for the channel cross-section.

The table contains the comparative values of stresses for several nodes of the calculated cross-sections.

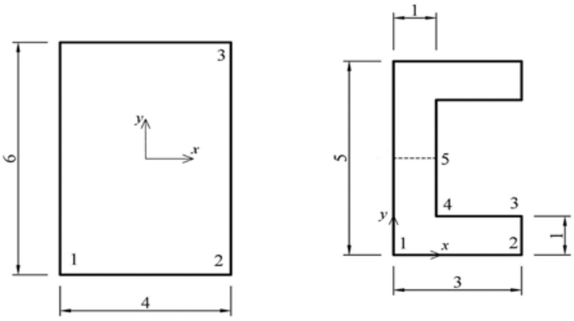

Figure 1. Dimensions of design sections, cm

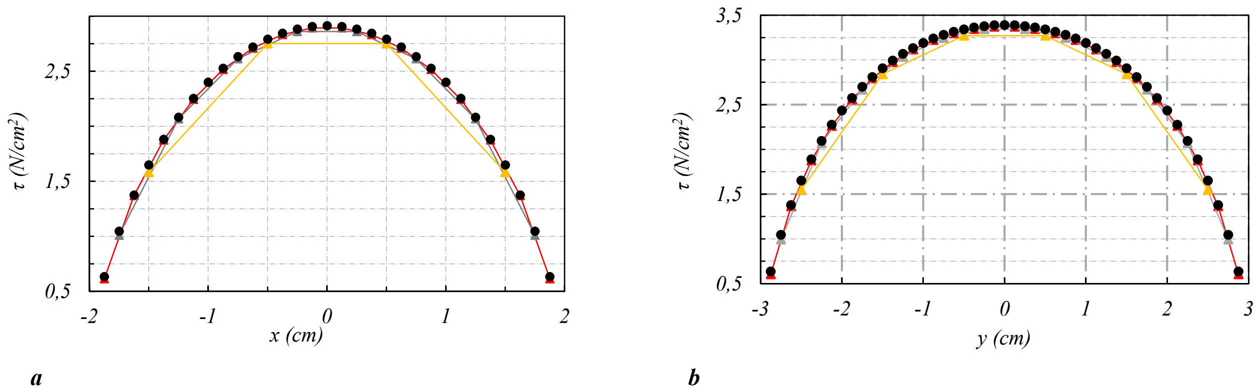

Figure 2. Distribution of tangential stresses along the boundary of a rectangular cross-section:
a — edge between points 1 and 2; *b* — edge between points 2 and 3
● — analytical solution; — 0.25 cm mesh; — 0.5 cm mesh; — 1 cm mesh

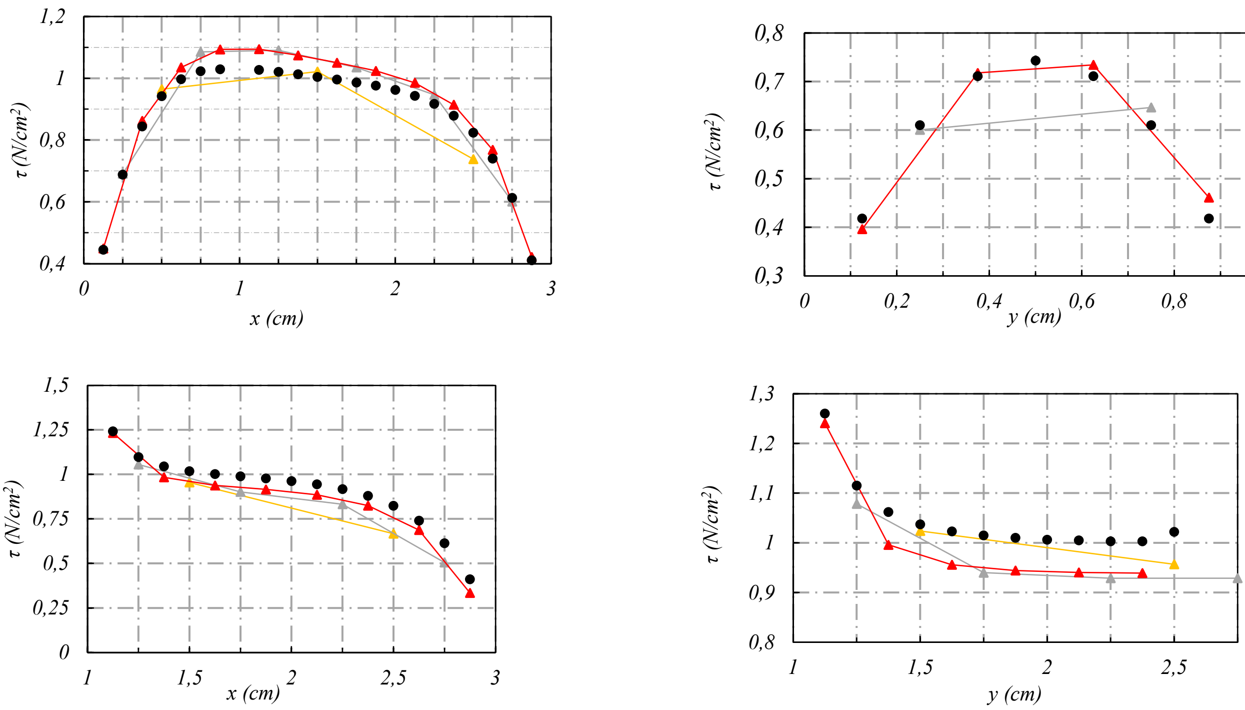

Figure 3. Distribution of tangential stresses along the boundary of a channel cross-section:
a — edge between points 1 and 2; *b* — edge between points 2 and 3;
c — edge between points 3 and 4; *d* — edge between points 4 and 5;
● — analytical solution; — 0.25 cm mesh; — 0.5 cm mesh; — 1 cm mesh

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Values of tangential stresses at the boundaries of design cross-sections

Cross-section	Coordinate of node		Size of boundary element			Analytical solution of the Poisson equation	Relative inaccuracy of the numerical method, %
	x, cm	*y*, cm	1 cm	0.5 cm	0.25 cm		
Rectangle	–1.5	–3	1.582	–	–	1.648	4
	2	0.25	–	3.345	–	3.378	1
	2	0.125	–	–	3.371	3.387	0.5
Channel	1.5	0	1.022	–	–	1.004	1.8
	3	0.75	–	0.647	–	0.610	1.5
	1.125	1	–	–	1.232	1.242	0.8
	1	1.125	–	–	1.241	1.260	1.5

The analysis of the obtained results demonstrates the acceptable inaccuracy for solving the problem by the method of boundary elements in stresses. On the basis of comparison of the calculation results with the analytical solution, it has been shown that satisfactory accuracy was achieved on the coarsest mesh of boundary elements. It has also been shown that acceptable accuracy was achieved in the nodes that are most important from the point of view of evaluating the strength of the cross-section, which are located in the zones of extreme stresses along the cross-section.

4. Conclusion

In this paper we have presented a formulation of the torsion problem in stresses, have derived the boundary integral equation regarding stresses, and have described the procedure for solving the problem by the boundary elements method. The main results are:

1. The integral representation of an arbitrary vector $\boldsymbol{f}$ in the plane.
2. The integral representation of a stress vector $\boldsymbol{\tau}$ in the rod torsion problem.
3. The boundary integral equation of the second kind for the rod torsion problem.
4. The procedure for solving the proposed formulation using the boundary elements method.
5. The verification of the method on the example of two cross-sections.

It has been shown that the numerical solution satisfactory accuracy was achieved in comparison with the analytical solution.

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