Surfaces of congruent sections of pendulum type on cylinders with generatrix superellipses

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Abstract. In 1972, I.I. Kotov proposed to separate the surfaces of congruent sections into a separate class and to include the surfaces of plane-and-parallel translation, surfaces of revolution, carved surfaces of Monge, cyclic surfaces with a generatrix circle of constant radius, rotative, spiroidal, and helical surfaces in it. The aim of the research is to obtain generalized parametric equations of surfaces of congruent sections of the pendulum type on right cylinders with plane-and-parallel translation of movable rigid superellipses. Analytical geometry methods are used. Computer systems MathCad and AutoCAD are applied to visualize surfaces. The results consist in the derivation of parametric equations of the studied surfaces in a general form convenient for the use of computer modeling methods. The technique is demonstrated on five examples with congruent mobile superellipses. The possibility of using obtained surface shapes in parametric architecture, free-form architecture, and in shaping of the surfaces of some technical products is noted.

Keywords: surfaces, congruent sections, superellipse, plane-parallel transfer, curves, asteroid, surface assignment
Для цитирования

Introduction

Recently, several papers have been published [1–4] devoted to the formation of surfaces of congruent sections of the pendulum type on arbitrary cylinders with forming plane curves in the form of circles [1; 2], parabolas [2; 3] and ellipses [4]. A surface of congruent sections is a surface bearing a continuous one-parameter family of plane lines. Such a surface is obtained as a result of moving some flat line (generatrix). The simplest types of surfaces of congruent sections are plane-and-parallel translation surfaces relative to the projection plane [5]. A plane-and-parallel transfer of a figure relative to the plane of projection is its movement in space, in which each of its points moves in its plane of level. Varieties of plane-and-parallel translation surfaces are right translation surfaces [6] (Figure 1).

Figure 1. The circular translation surface
(Cheremushkinsky Market, Moscow, photo by I.A. Mamieva)

The number of surfaces under consideration can be significantly expanded if we accept congruent plane curves, given in the form

\[ |z|^n = T^n \left(1 - \frac{|y|^m}{W^m}\right), \]

where \( n \) and \( m \) are constant non-negative numbers.

Ключевые слова: поверхность конгруэнтных сечений, суперэллипс, плоскопараллельный перенос кривых, астроида, задание поверхности
By giving different values to the parameters $n$ and $m$, it is possible to obtain various closed and open plane curves. For $n = m$, closed curves called superellipses are obtained [7]. Superellipses with $T = W$ are called Lame curves$^1$, for $n = m = 2$ and $T = W$, a circle is obtained, and for $n = m = 2$ and $T \neq W$, an ellipse. The more the value of the parameter $n = m$, the more precisely the shape of the superellipse approaches a rectangular contour.

Taking into account the method of forming of the surfaces under consideration, one can rank them among kinematic surfaces [8].

So far, superellipses and Lame curves have made possible to expand the range of solved geometric problems only in shipbuilding [9]. In architecture and construction, surfaces of congruent sections of the pendulum type with simple generating Lame curves in the form of a circle and an ellipse have been used [2; 3]. A paper [10] provides an example of using surface of congruent sections for cover of a bridge over the Kura River (Figure 2).

![Figure 2. The glass Bridge of Peace, Tbilisi, Georgia (photo by I.A. Mamieva)](image1)

There is also an example of a surface of congruent sections (Figure 3) in the city of Khimki (Moscow region). The need to construct an envelope of a family of congruent curves arises when surfaces of some technical products are formed [11].

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Problem statement

Consider a right cylinder with a guiding superellipse, given in the form

$$z_0 = \pm T \left(1 - \frac{|y_0|^{m}}{W^{m}}\right)^{\frac{1}{n}},$$

and a movable generatrix superellipse, given in the local coordinate system as

$$Z = \pm t \left(1 - \frac{|y|^k}{\omega^k}\right)^{\frac{1}{s}},$$

where \(n, m, k, s\) are constant non-negative numbers; the geometric parameters \(T, W, t, \omega\) are shown in Figure 4.

In this case, the area, covered by the movement of the center of the movable superellipse (2) along the contour of the stationary superellipse (1), can be set according to Figure 4 by the equations:

$$y = y(y_0, Y) = y_0 + Y, \quad z = z(y_0, Y) = |z_0| + Z = \left[T \left(1 - \frac{|y_0|^{m}}{W^{m}}\right)^{\frac{1}{n}} \pm t \left(1 - \frac{|y|^k}{\omega^k}\right)^{\frac{1}{s}}\right];$$

$$y = y(y_0, Y) = y_0 + Y, \quad z = z(y_0, Y) = -|z_0| + Z = \left[-T \left(1 - \frac{|y_0|^{m}}{W^{m}}\right)^{\frac{1}{n}} \pm t \left(1 - \frac{|y|^k}{\omega^k}\right)^{\frac{1}{s}}\right],$$

where \(-W \leq y_0 \leq W, -\omega \leq Y \leq \omega\).

In formulas (3) and (4), \(y_0\) and \(Y\) are independent variable parameters.

![Figure 4. Scheme of formation of the surface of congruent sections](image)

Considering that the movable superellipse performs oscillatory movements of the pendulum type and simultaneously moves uniformly along the \(x_0\) axis (Figure 4), we can write:

$$y_0 = A \sin \frac{\pi x}{l},$$

where \(A\) is the maximum deviation of the center of the moving superellipse from the \(Oz\) axis, that is, the amplitude of the sine wave in the horizontal plane \(xOy\); \(l\) is the step of the half-wave of the sine wave.

In this case, the parametric equations of the surface of congruent sections of the pendulum type will have the form

$$x = x(x); \quad y = y(x, Y) = y_0 + Y;$$
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Moreover, a formula (7) is used when constructing surface with a line of centers with $z_0 > 0$, and a formula (8) with $z_0 < 0$. The limits of the change in the parameter $x$ are chosen arbitrarily, if necessary.

Example 1. Let formulas (1) and (2) define circles, that is, $m = n = k = s = 2$, and $T = W = 3 \text{ m}$, $t = \omega = 1 \text{ m}$, $A = 2 \text{ m}$, $l = 2 \text{ m}$, $-t \leq Y \leq t$, $0 \leq x \leq 4l$.

In this case, formulas (5)–(8) will take the form

$$
x = x(x) = x; \quad y = y(x, Y) = y_0 + Y;
$$

$$
z = z(x, Y) = |z_0| + Z = (T^2 - y_0^2)^{1/2} \pm (t^2 - Y^2)^{1/2},
$$

where

$$
y_0 = A\sin(\pi x/l).
$$

The surface is shown in Figure 5. This surface can be attributed to the subgroup of cyclic surfaces with a plane of parallelism from the class “Cyclic Surfaces.” Some varieties of these surfaces are presented in [12].

![Figure 5. A cyclic surface on a circular cylinder and a line of centers of a movable circle on the cylinder](image1)

![Figure 6. A congruent surface with a generative astroid on an oval cylinder and a line of centers of a movable astroid on the cylinder](image2)
Example 2. Let the cross section of a right cylinder has the form of a superellipse given by the formula (1), where \( T = 1 \text{ m}, \ W = 1.5 \text{ m}; \ m = n = 10/7 \), and the mobile superellipse has \( k = s = 2/3, \ t = \omega = 0.5 \text{ m} \), that is, the mobile superellipse is an astroid [8]. In addition, \(-\omega \leq Y \leq \omega; 0 \leq x \leq 4l, \ l = 2 \text{ m}, \ A = 1 \text{ m}\). Substituting the given values into formulas (5)–(8), we obtain parametric equations of the desired surface. The surface itself is shown in Figure 6.

Example 3. Let formulas (1), (2) have the form

\[
Z_0 = \pm T \left(1 - \frac{|Y_0|}{W}\right); \quad Z = \pm t \left(1 - \frac{|Y|}{\omega}\right),
\]

that is \( m = n = k = s = 1 \), and \( T = 1 \text{ m}, \ W = 1.5 \text{ m}; \ t = \omega = 0.5 \text{ m}; \ A = 1 \text{ m}; \ -\omega \leq Y \leq \omega; 0 \leq x \leq 4l, \ l = 1 \text{ m}\).

In this case, using formulas (5)–(8), it is possible to construct a box-shaped surface, shown in Figure 7. Box-shaped surfaces can be used in some sectors of the national economy. Various box-shaped surfaces with curved lines of centers are studied in [13].

![Figure 7](image1.png)

**Figure 7.** A congruent box-shaped surface on a box-shaped cylinder and a line of centers of a movable quadrilateral on a box-shaped cylinder

![Figure 8](image2.png)

**Figure 8.** A congruent surface with generatrix ovals on an oval cylinder and a line of centers of a movable oval

Example 4. Let the center of the movable oval (2) with \( k = s = 1.5; \ t = 0.5 \text{ m}, \ \omega = 0.8 \text{ m} \) moves along a fixed oval (superellipse) (1) with \( m = n = 1.5; \ T = 1.5 \text{ m}; \ W = 2.5 \text{ m} \) and besides \( A = 2 \text{ m}, \ l = 2 \text{ m}, \ -\omega \leq Y \leq \omega; 0 \leq x \leq 3l\).
In this case, a formula (5) and parametric equations of the projected surface will take the form

\[
x = x(x); \quad y = y(x, Y) = y_0 + Y;
\]

\[
z = z(x, Y) = |z_0| + Z = T \left( 1 - \frac{|y_0|^{1.5}}{W^{1.5}} \right)^{1/1.5} + t \left( 1 - \frac{|Y|^{1.5}}{a^{1.5}} \right)^{1/1.5};
\]

\[
y_0 = 2\sin(\pi x/2).
\]

The surface is shown in Figure 8.

**New problem statement**

Superellipses (1), (2), taking into account that \(m = n\) and \(k = s\), can be represented as

\[
y_0 = y_0(\beta) = W\cos^{2/m}\beta; \quad z_0 = z_0(\beta) = T\sin^{2/m}\beta; \quad (9)
\]

\[
Y = Y(\gamma) = W\cos^{2/k}\gamma; \quad Z = Z(\gamma) = T\sin^{2/k}\gamma, \quad (10)
\]

then the equation of the surface of congruent sections of the pendulum type can be represented as

\[
x = x(x); \quad y = y(x, \gamma) = y_0 + Y = A\sin(\pi x/l) + W\cos^{2/k}\gamma;
\]

\[
z = z(x, \gamma) = z_0 + Z = T\sin^{2/m}\beta + T\sin^{2/k}\gamma = T \left( \frac{1 - \cos^2(2\beta)}{1} + \frac{1 - \cos^2(2\gamma)}{1/m} \right) + t\sin^{2/k}\gamma. \quad (11)
\]

It should be borne in mind that \(0 \leq x \leq C; \beta, \gamma\) are the angles measured from the horizontal axis \(x\) or \(X\) (Figure 4), \(C\) is the required surface length,

\[
tg \alpha = \frac{y_0}{z_0} = \frac{W}{T} \frac{z}{\sqrt{m}}\beta,
\]

where \(\alpha\) is the angle measured from the vertical axis \(Oz\) clockwise (Figure 4).

**Example 5.** Let the center of the movable shaft (10) with \(k = 1.5; t = 0.5 \text{ m}, \omega = 0.8 \text{ m}\) moves along the stationary shaft (superellipse) (9) with \(m = 1.5; T = 1.5 \text{ m}; W = 2.5 \text{ m}\) and in addition \(A = 2.5 \text{ m}, l = 2 \text{ m}, -\omega \leq Y \leq \omega; 0 \leq x \leq 4l, 0 \leq \gamma \leq 2\pi.\)

**Figure 9.** A congruent surface with generatrix ovals on an oval cylinder and a line of centers of a movable oval (10)

Substituting the above geometric parameters into the parametric equations of the surface (11), we obtain a pendulum-type surface with congruent curves, shown in Figure 9.
Results

Parametric equations of surfaces of congruent sections in the form of superellipses on right cylinders with guiding superellipses are obtained. The given method of constructing considered plane-and-parallel translation surfaces is illustrated by 5 examples. The four obtained surfaces are presented for the first time in Figures 5–8. With the help of parametric equations of general form obtained in this article, a large number of new surfaces of congruent sections of the pendulum type, as well as helical surfaces, can be constructed. Apparently, the surfaces of congruent sections of the pendulum type can be distinguished into a separate subgroup of the class “Surfaces of Congruent Sections.”

Conclusion

The article considers surfaces, formed by superellipses, that is, with $n = m$ and $s = k$. But the obtained parametric equations of the surface of the general form make it possible to consider the cases when $n \neq m$ and $s \neq k$. This will further expand the range of surfaces of congruent sections under consideration, since formulas (1), (2) can describe parabolas, hyperbolas and other open plane curves.

References


Список литературы


