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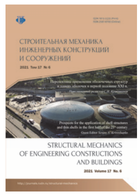

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Geometry of the normal ruled surfaces

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Abstract. The wide circle of the surfaces formed by the motion of the right line in the normal plain of some base directrix curve is regarded. The generate right line may rotate at some low at the normal plane of the base curve. The vector equation of the surface with any plane or space base curve is received. There are given the formulas of the geometry characteristics of the surfaces, on the base of them there is shown that the coordinate system of the normal ruled surfaces is orthogonal but there is not conjugated in common, that is that the normal ruled surfaces there are not developable surfaces in common way. The condition of the rotation of directrix plane line when the coordinate system of the normal ruled surfaces will be conjugated and the normal ruled surface will be developable is received. The condition that the normal ruled surface with space base curve will be the developable surface there is connected with its curvature of base curve. The developable normal ruled surface with plane base curve is formed by motion of right line at the normal plane of the base curve with the constant angle to the plane of the base curve; the received surface is a surface of constant slope. On the base of the vector equation of the surfaces there are made the figures of the normal ruled surfaces with the help of program complex MathCAD.

Keywords: geometry of the curves, geometry of the surfaces, normal surfaces, line surfaces, geometrical characteristics, surface

Геометрия нормальных линейчатых поверхностей

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Аннотация. Рассматривается формирование широкого круга поверхностей на основе нормальных линейчатых поверхностей, образуемых движением прямой линии в нормальной плоскости базовой направляющей кривой. Образующая прямая может вращаться по заданному закону в нормальной

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плоскости базовой кривой. Приводится векторное уравнение поверхностей, с произвольной пространственной и плоской направляющей кривой. Получены формулы геометрических характеристик поверхности. На основе полученных формул показано, что координатная система нормальной линейчатой поверхности является ортогональной, но в общем случае не сопряженной. Прямые линии не являются линиями главных кривизн поверхности и нормальные линейчатые поверхности в общем случае не являются торсовыми, развертывающимися поверхностями. Получено условие вращения образующей прямой в нормальной плоскости базовой кривой, при выполнении которого координатная сеть будет сопряженной – нормальная линейчатая поверхность развертывающейся. Для пространственной базовой кривой это условие связано с кривизной базовой кривой, для плоской кривой образующая прямая движется в нормальной плоскости направляющей плоской кривой с постоянным наклоном к плоскости базовой плоской кривой – поверхность одинакового ската. На основе векторного уравнения построены рисунки нормальных линейчатых поверхностей с использованием программного комплекса MathCAD.

Ключевые слова: геометрия кривых, геометрия поверхностей, нормальные поверхности, линейчатые поверхности, геометрические характеристики поверхности

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Introduction

The thin state constructions in the form of ruled surfaces are used wildly in building, machine-building, air and rocket production. It's based on the most simple way to realize the space constructions formed by the straight lines. It's comfortable to made the line straight timbering on the building site or made the timbering forms for concrete blocks. The thin-walled metal ruled forms (developable surfaces) are made by bending of the metal list plate.

The geometry of the ruled surfaces are considered in many monographs and science articles. Usually at all classic monographs [1–7] and at the present monographs and textbooks on differential geometry and at the most monographs and textbooks on the theory and methods of analyses of the thin-walled state structures, one can find a section considered to the ruled surfaces [8–16]. In a monography of S.N. Krivoshapko [17], the most full classification of the ruled surfaces is given and the variety of the subclasses of the ruled surfaces is shown. The normal ruled surfaces form a subclass of the ruled surfaces. The cylindrical and conical surfaces, the constant slope surfaces, and normal helicoids enter at this subclass. Shaping of the torus surfaces are based on the construction of surfaces by the system of the tangents to the given space base directrix curve which is the cuspidal edge of the torus surface, the exception is a forming of torus surfaces with two plane base curves. The coordinate system of these torus surfaces isn't orthogonal [9–11; 13]. The analyses of these shells by the analytical methods aren't possible usually. Numerical methods are used, for example the finite elements method [18; 19]. As it will be shown further the coordinate system of the normal ruled surfaces is orthogonal but not conjugated when the straight line rotates at the normal plane of the base directrix. But at any case it is possible to use analytical methods for some shells or to use a variation difference method [13; 20; 21] which gives more exact results in comparison with finite elements method.

The vector equation and the geometric characteristics of the normal ruled surfaces

The normal ruled surfaces are formed by the moving generating straight at the normal plane of the directrix curve [16; 22]. The generating curve may transform and rotate at the normal plane of the base directrix. The normal ruled surfaces with straight generatrixes at the normal planes of the base directrix constitute one of subclasses of the normal surfaces (Figure 1).

The vector equation of a normal ruled surface can be written as

$$\boldsymbol{\rho}(u,v)=\boldsymbol{r}(u)+v\boldsymbol{e}(u), \tag{1}$$

where $\boldsymbol{r}(u)=x(u)\boldsymbol{i}+y(u)\boldsymbol{j}+z(u)\boldsymbol{k}$ is a radius-vector of the base directrix curve; *i, j, k* are the orts of the Cartesian coordinate system; $\boldsymbol{e}(u)=\boldsymbol{\nu}\cos\theta(u)+\boldsymbol{\beta}\sin\theta(u)$ is a unit vector in the direction of the generating straight line; **τ**, **ν**, **β** are the unit vectors of tangent, normal and binormal of the base directrix curve; $\theta(u)$ is a function of rotation of the generating right line at the normal plane of the base directrix curve; $\boldsymbol{\rho}(u,v)$ is a radius-vector of the surface.

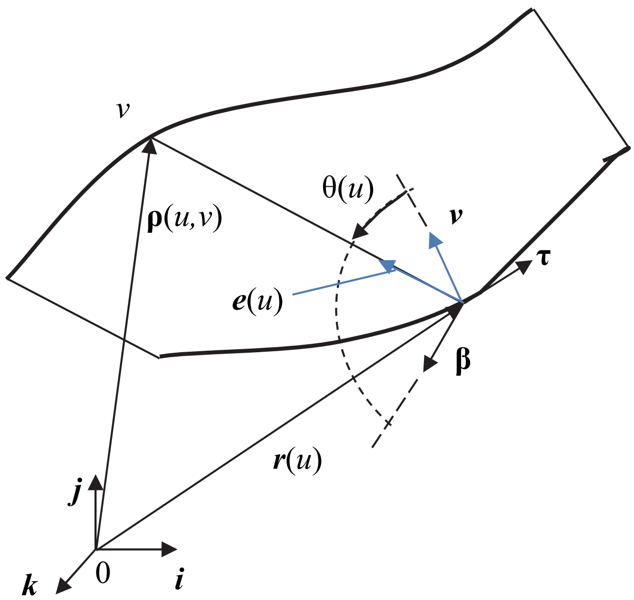

Figure 1. Normal ruled surface

The derivative of the vector $\boldsymbol{e}(u)$ is

$$\boldsymbol{e}'(u) = -k_s\cos\theta(u)\boldsymbol{\tau} + \left[-\boldsymbol{\nu}\sin\theta(u) + \boldsymbol{\beta}\cos\theta(u)\right](\chi_s + \theta') = -k_s\cos\theta\boldsymbol{\tau} + (\chi_s + \theta')\boldsymbol{g}(u), \quad (2)$$

$k_s = s'k$, $\chi_s = s'\chi$, $s' = |r'|$, k, χ are curvature and torsion of the base curve, $\boldsymbol{g}(u) = -\boldsymbol{\nu}\sin\theta(u) + \boldsymbol{\beta}\cos\theta(u)$ is the unit vector in the normal plane of the base curve orthogonal to the vector $\boldsymbol{e}(u)$,

$$\boldsymbol{g}'(u) = -\boldsymbol{\nu}\sin\theta(u) + \boldsymbol{\beta}\cos\theta(u) = k_s\sin\theta(u)\boldsymbol{\tau} - (\chi_s + \theta')\boldsymbol{e}(u). \quad (3)$$

The derivatives of the equation of the normal ruled surface give

$$\boldsymbol{\rho}_u = (s' - vk_s\cos\theta)\boldsymbol{\tau} + v(\chi_s + \theta')\boldsymbol{g};\ \boldsymbol{\rho}_v = \boldsymbol{e}(u);\ \boldsymbol{\rho}_{vv} = 0;$$

$$\boldsymbol{\rho}_{uu} = \left[(s' - vk_s\cos\theta)' + vk_s(\chi_s + \theta')\sin\theta\right]\boldsymbol{\tau} + k_s(s' - vk_s\cos\theta)\boldsymbol{\nu} - v(\chi_s + \theta')^2\boldsymbol{e} + v(\chi_s' + \theta'')\boldsymbol{g};$$

$$\boldsymbol{\rho}_{uv} = -k_s\cos\theta\boldsymbol{\tau} + (\chi_s + \theta')\boldsymbol{g};$$

$$\boldsymbol{m} = \frac{(\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v)}{|\boldsymbol{\rho}_u \times \boldsymbol{\rho}_v|} = \frac{1}{A}\left[-v(\chi_s + \theta')\boldsymbol{\tau} + (s' - vk_s\cos\theta)\boldsymbol{g}\right]. \quad (4)$$

The geometrical characteristics of the normal ruled surfaces

$$E = A^2 = (s' - vk_s\cos\theta)^2 + v^2(\chi_s + \theta')^2;\ G = 1;\ F = 0.$$

$$L = (\boldsymbol{\rho}_{uu}\boldsymbol{m}) = \frac{v(\chi_s + \theta')}{A}\left\{-\left[(s' - vk_s\cos\theta)' + vk_s(\chi_s + \theta')\sin\theta\right] + (s' - vk_s\cos\theta)\right\} + \frac{k_s\sin\theta}{A}(s' - vk_s\cos\theta)^2;$$

$$M = (\boldsymbol{\rho}_{uv}\boldsymbol{m}) = \left[vk_s\cos\theta + (s' - vk_s\cos\theta)\right]\frac{\chi_s + \theta'}{A} = s\frac{\chi_s + \theta'}{A};\ N = (\boldsymbol{\rho}_{vv}\boldsymbol{m}) = 0. \tag{5}$$

It's seen from the formulas of the geometrical characteristics of the normal cycle surfaces that the coordinate system of the normal ruled surfaces is orthogonal but it isn't conjugated in common, the generating straight lines are not the lines of principle curvatures and the normal ruled surfaces are not developable surfaces in common.

The normal developable surfaces

The normal ruled surfaces will be developable if geometrical coefficient M will be equal zero

$$M = 0 \quad \to \quad \theta(u) = -\int \chi_s du + \theta_0. \tag{6}$$

So, for forming torus normal surface the function of rotation of the generating right line at the normal plane of the base directrix must be linked up with the curvature of the base curve.

Consider some examples.

A helix is a base directrix [16]:

$$\boldsymbol{r}(u) = a\boldsymbol{h}(u) + bu\boldsymbol{k}, \tag{7}$$

$\boldsymbol{h}(u) = \boldsymbol{i}\cos u + \boldsymbol{j}\sin v$ is a vector function of the circle of the unit radius,

$$\chi = \frac{b}{s'^2},\quad s' = \sqrt{a^2 + b^2},\quad \chi_s = s'\chi = \frac{b}{s'},\quad \theta(u) = -\chi_s u + \theta_0. \tag{8}$$

We have for conical spiral [16]

$$\boldsymbol{r}(u) = e^{pu}\left[a\boldsymbol{h}(u) + b\boldsymbol{k}\right], \tag{9}$$

$$\chi = \frac{bp}{s'^2}e^{-pu},\quad s' = s'_0 e^{pu},\quad s'_0 = \sqrt{a^2(1 + p^2) + b^2p^2},\quad \chi_s = s'\chi = -\frac{bp}{s'_0},\quad \theta(u) = -\chi_s u + \theta_0. \tag{10}$$

As it's seen from the received formulas of those base curves, the angle of rotation of the generating straight line at the normal plane of the helix and conical spiral are proportional to coordinate parameter of the rotation u. The initial angle $\theta = \theta_0$ in formulas (8), (10) links an initial coordinate of the base line $u = 0$. If there is considered the section with the initial parameter $u_0 \neq 0$ then it's better to use the formula

$$\theta(u) = -\chi_s(u - u_0) + \theta_0. \tag{11}$$

So the turn of the generating straight line at angle θ_0 will be at the beginning of the section.

The figures of the normal developable surfaces with a base helix and a conical spiral are shown at Figure 2.

The parameters of the base curves:

a) helix: a = 2,5: b = 1: u = 0÷6π: v = 0÷2;

b) conical spiral: a = 2: b = 3: p = 0,15; u= π÷2,5π: v = 0÷6. $\theta_0 = -\pi/2$.

A developable normal surface with a flat base curve ($\chi = 0$, $\theta = \theta_0$) and the generating straight line that moves at the normal plane of the base directrix with constant angle to the plane of the base curve will be the surface of constant slope. If $\theta_0 = 90°$, then a cylindrical surface will be.

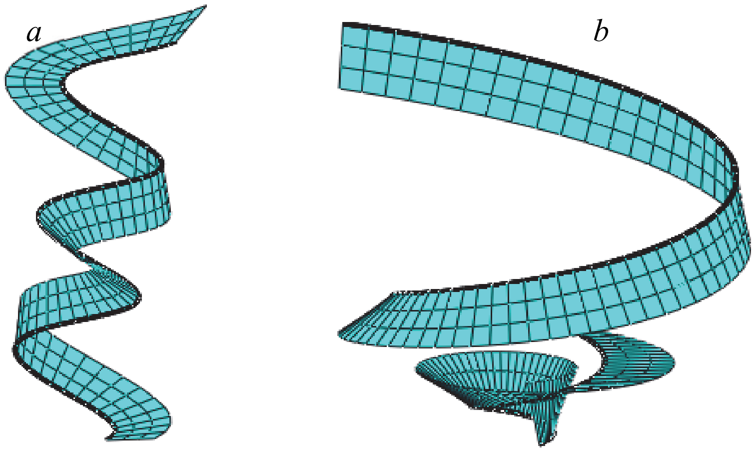

Figure 2. Developable normal ruled surface base curves:
a – helix; *b* – cylindrical spiral

If $\theta_0 = 0$ and the generating straight line moves at the plane of the flat base curve then the surface degenerates into plane. At the plane there is forming the trapezium-curved orthogonal coordinate system, the system of the curves parallel to the base directrix lines and the system of orthogonal to the base curve straight coordinate lines. The method of forming the surfaces on the trapezium-curved plans was regarded in a paper [22; 23].

At Figure 3, the surfaces of constant slope with different base curve and different angle of slope of the generating straight lines are shown. At the top row, the trapezium-curved plans ($\theta_0 = 0$) with base curves analog to the base curve of surfaces at lower row are shown.

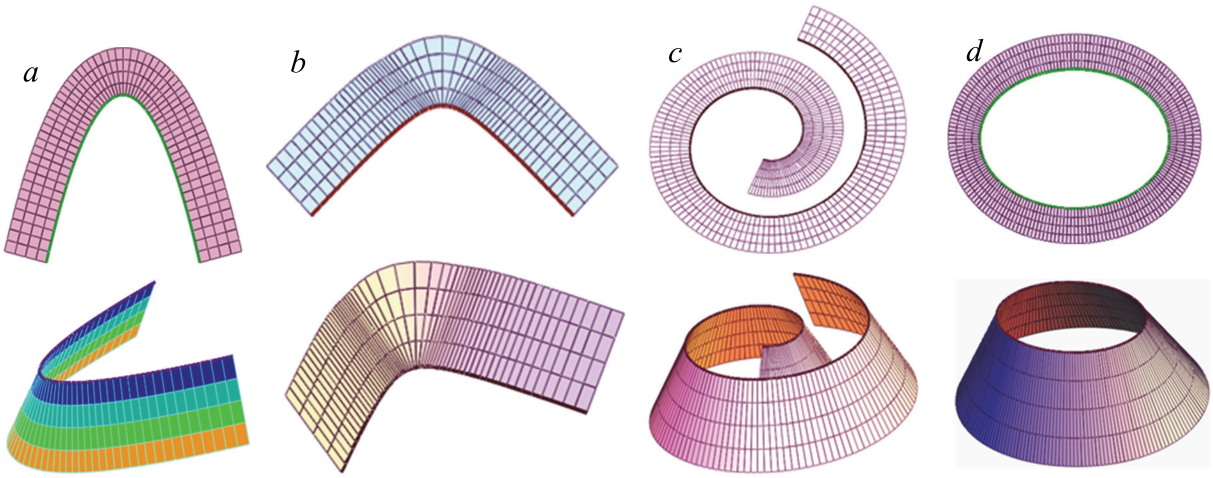

Figure 3. Trapezium-curved plans and surfaces of constant slope base curves:
a – parabola; *b* –hyperbole; *c* – evolvent of the circle; *d* – ellipse

If the directrix is a straight line then the surface of straight conoid formed [24; 25].

The screw normal ruled surfaces and curves

Consider the normal ruled surfaces which are formed by the generating straight line rotating at the normal plane of different base directrix proportional to the coordinate parameter *u*

$$\theta(u) = 2k\pi \frac{u - u_0}{u_1 - u_0} + \theta_0, \; u = u_0 \div u_1, \tag{12}$$

k is a number of the fool rotations of the generating straight line when it's moving along the base curve (in may be not hole number).

The coordinate curve of a surface is formed with constant coordinate parameter $v = v_0$.

If the base curve is a straight line, then the usual helical surface is formed.

At Figure 4, the normal screw surfaces with a base parabola $x = u$, $y = au^2$, $a = 0.25$, $u = –8÷8$ and with different number of rotation of the generating straight line, $v = 0÷3$, $\theta_0 = 0$ are shown.

At Figure 5, the normal screw surfaces with a base ellipse $x = a \cos u$, $y = b \sin u$, $a = 5$, $b = 3$, $u = 0÷2\pi$ and with different number of rotation of the generating straight line, $v = 0÷1.5$, $\theta_0 = \pi$ are shown.

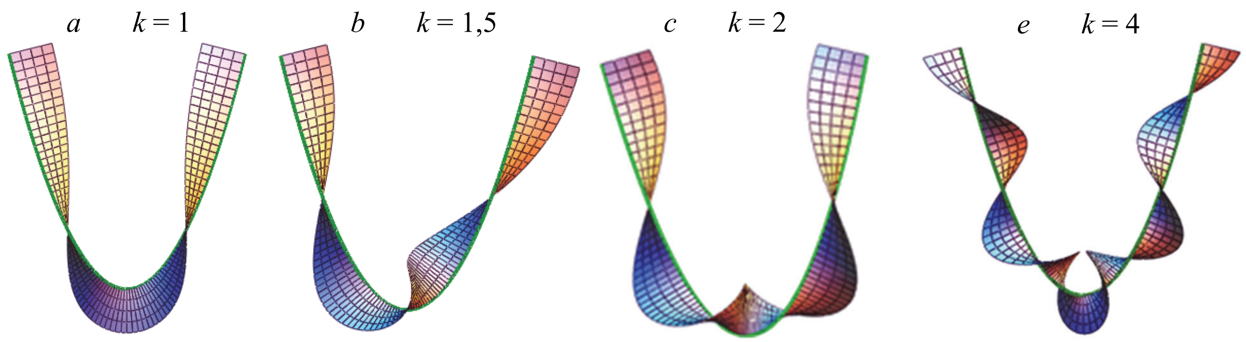

Figure 4. Helical normal ruled surfaces with base parabola

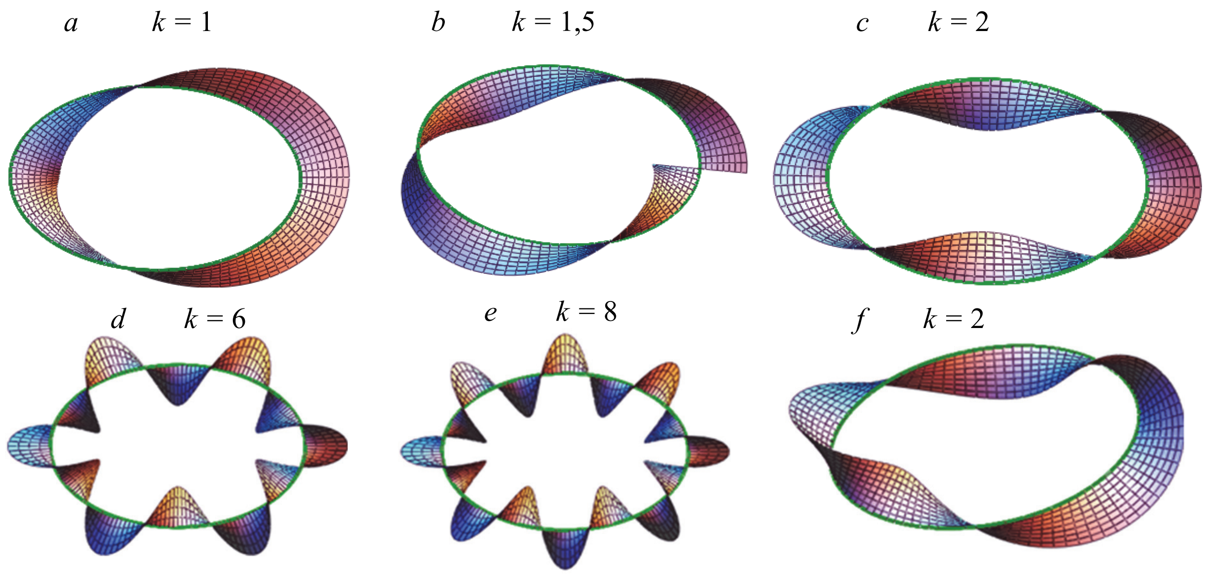

Figure 5. Helical normal ruled surfaces with base ellipse:
a–*e* – $\theta_0 = \pi$; *f* – $\theta_0 = 2/3\pi$;

At Figure 6, the normal screw surfaces with a base sine $y = a \sin\pi\, u/b$, $a = 2$, $b = 4$, $u = 0÷mb$ and with different number of rotation of the generating straight line, $v = 0÷1$, $\theta_0 = 0$ are shown; m is a number of half-waves of the sine. Figure 6, *g:* $v = 0.5÷1.5$; Figure 6, *h*: $v = 1$ is a screw curve.

At Figure 7, the normal screw surfaces with a base evolvent of the circle $x = a(\cos u+u \sin u)$; $y = a(\sin u – u \cos u)$; $a = 0.5$; $u = 0.5\pi÷3.5\pi$; $v = 0÷1$; $\theta_0 = \pi$ are shown.

At Figure 8, the normal screw surfaces with a base space helical curve $x = a \cos u$; $y = a \sin u$; $z = bu$; $a = 2.5$; $b = 1$ are shown; $u = 0÷2\pi$; $\theta_0 = \pi$; *a–f:* $v = 0÷2$; *g*: $v = 0,5÷2$; *a*) $k = 0$ is a right helix; *g*) $v = 2$ is a screw curve.

At Figure 9, the normal screw surfaces with a base conical spiral $x = ae^{pu}\cos u$; $y = ae^{pu}\sin u$; $z=be^{pu}$ are shown; $a = 2$; $b = 3$; $p = 0.15$; $u = 2\pi÷4\pi$; $\theta_0 = \pi/2$; *a–f:* $v = 0÷6$.

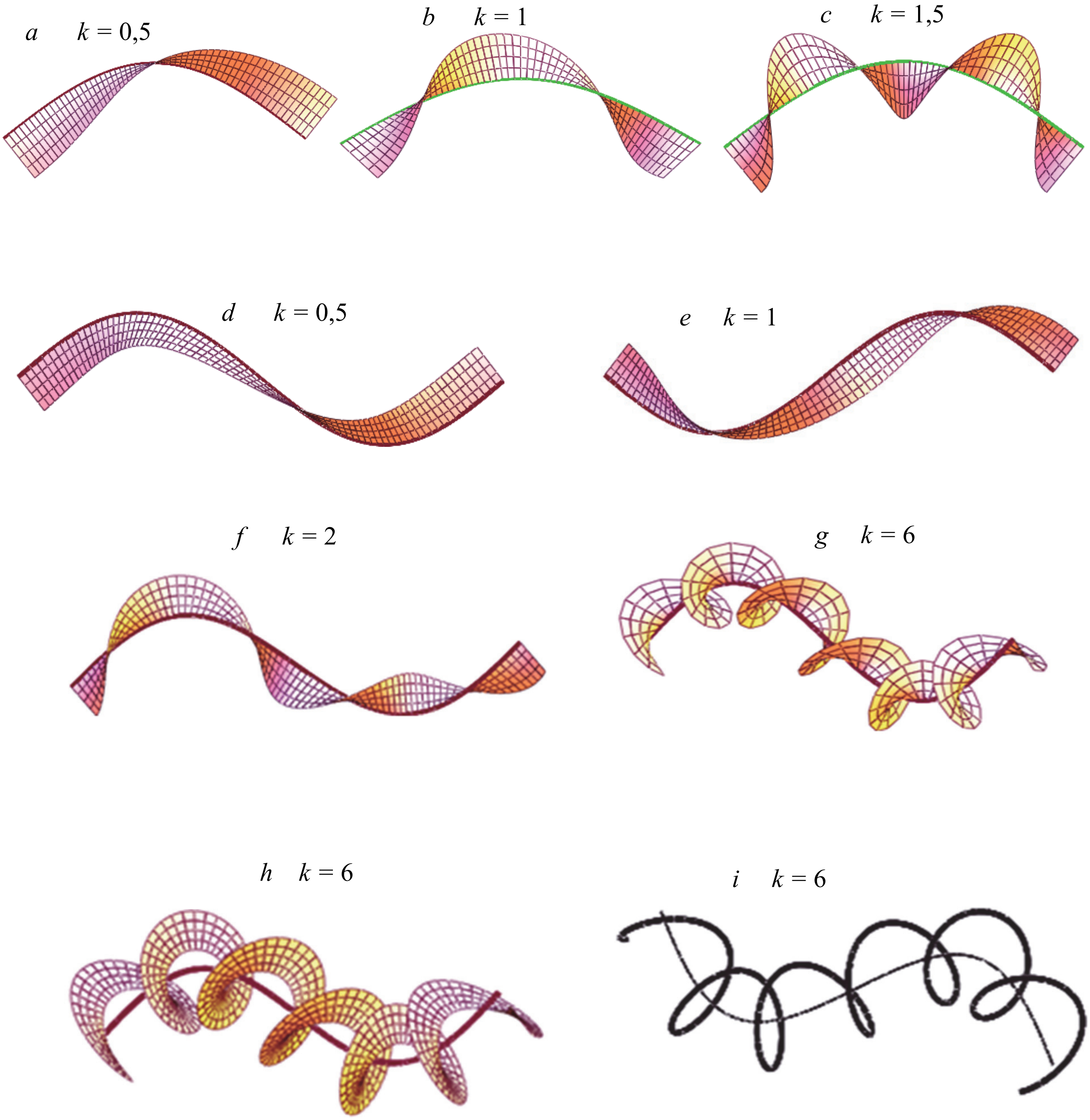

Figure 6. Screw normal ruled surfaces with a base sine
a–*c* – on one halfwave; *d*–*i* – on two halfwaves

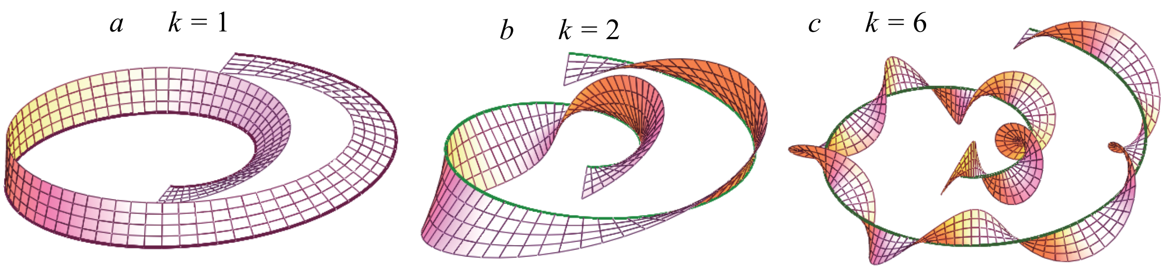

Figure 7. Screw normal ruled surfaces with a base evolvent of the circle
a–*b* – on one halfwave; *c* – on two halfwaves

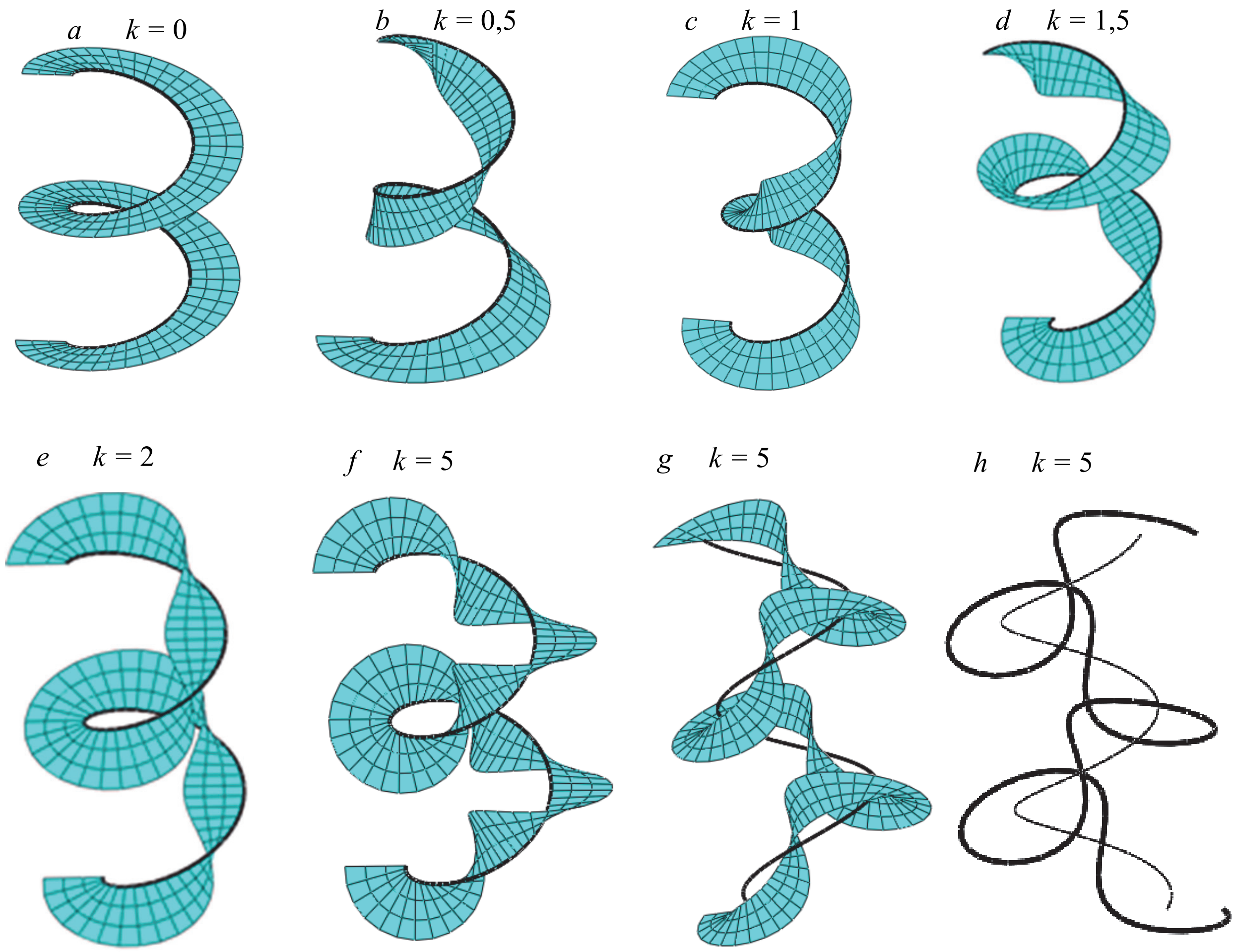

Figure 8. Helical normal ruled surfaces with base helix

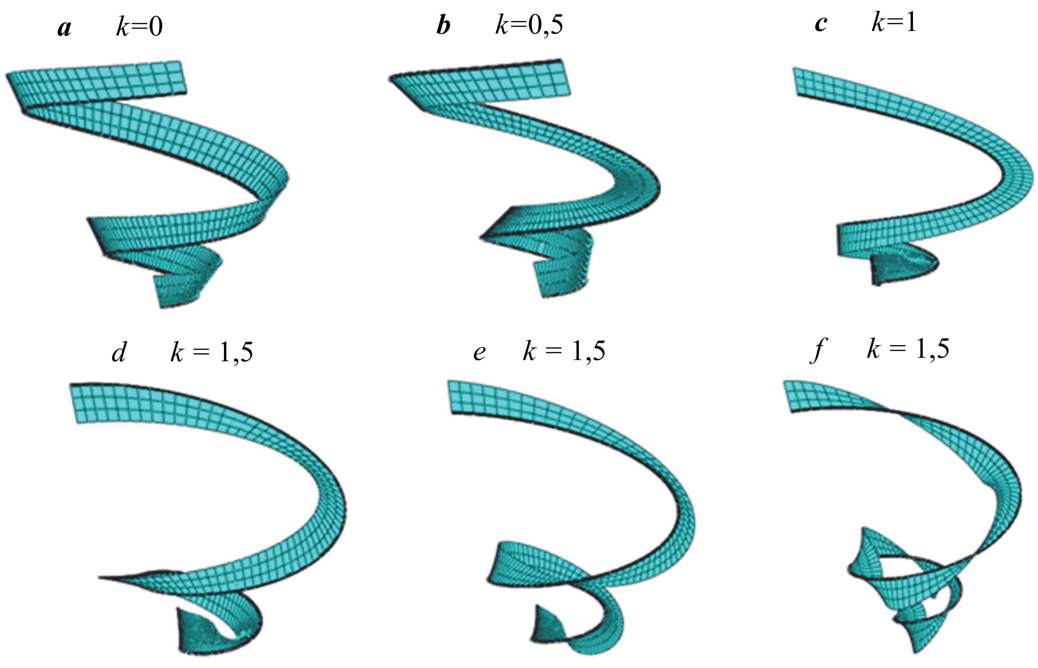

Figure 9. Screw normal ruled surfaces with a base conical spiral

The wave normal ruled surfaces

Consider the subclass of normal ruled wave surfaces with cosine function of the angle of rotation of the generating straight line at the normal plane of the base directrix curve

$$\theta(u)=\theta_0+c\left[\cos\left(k\pi\frac{u-u_0}{u_1-u_0}\right)+p\right], \quad (12)$$

c is an amplitude of oscillation of the angle parameter of the generating straight line at the normal plane of the base curve when it's moving along the base directrix $u=u_0\div u_1$; k is a number of half-wave of the cosine; p = 0, 1, –1 is a parameter for tree types of the wave normal ruled surfaces.

If parameter c = 0 and the base directrix is a plane curve then with the help of the equation (12), the surface of constant slope will form. If c > 0 and p = 1, then the wave surface will form which will touch the surface of constant slope outside. If p = –1, the wave normal ruled surface will touch the surface of constant slope inside. If p = 0, the wave normal ruled surface will be symmetrical about the surface of constant slope. The surfaces of constant slope may be called the supporting surfaces of the wave normal ruled surfaces. As it was shown above, if θ_0 = 0 or θ_0 = π, then a surface of constant slope degenerates into the trapezium curved plane region. If θ_0 = π, a cylindrical surface will be received.

At Figure 10, the wave normal ruled surfaces with a base parabola $x = u$, $y = au^2$, a = 0.25, u = –8÷8; v = 0÷8; θ_0 = π, and c = π/4 are shown with different parameters k, p.

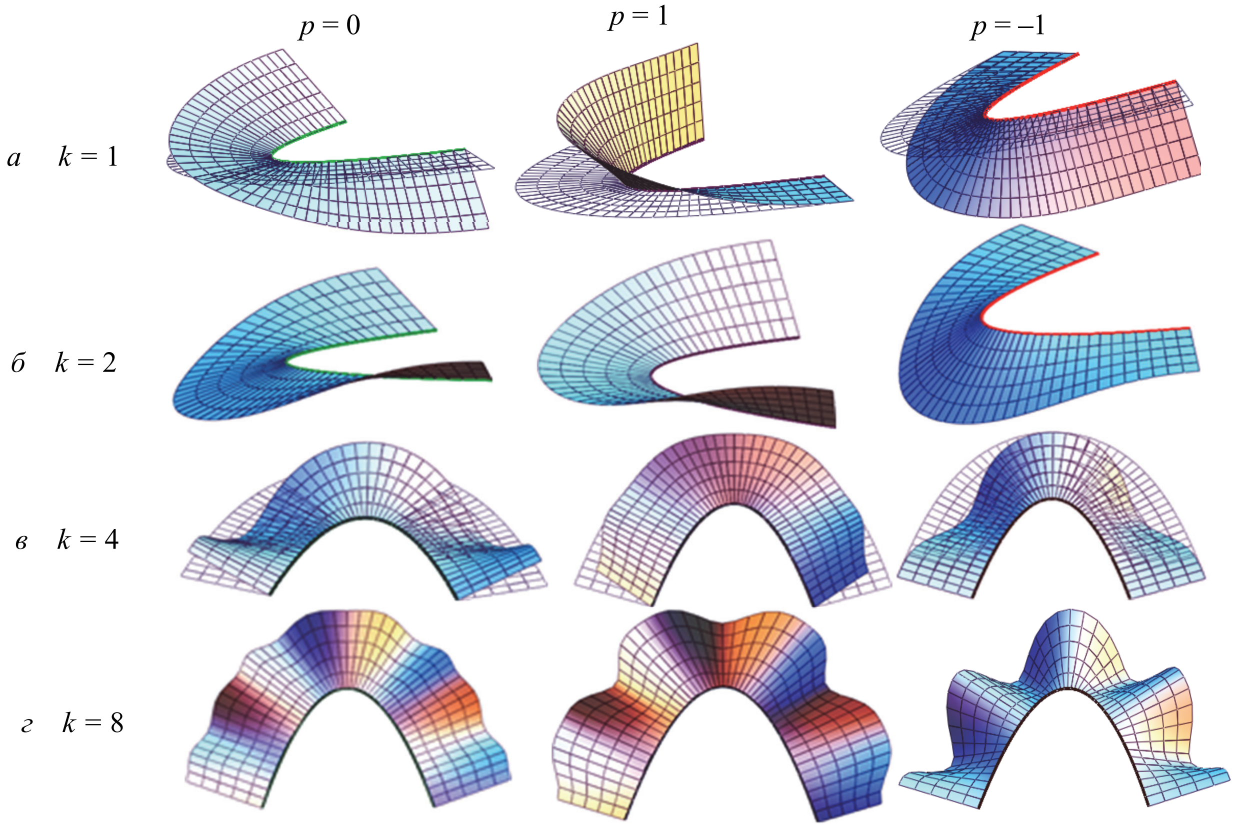

Figure 10. Wave normal ruled surfaces with base parabola

On some figures, the wave normal ruled surfaces are shown together with supporting surfaces of constant slope.

At Figure 11, the wave normal ruled surfaces on the parabolic cylinder with θ_0 = π/2 are shown.

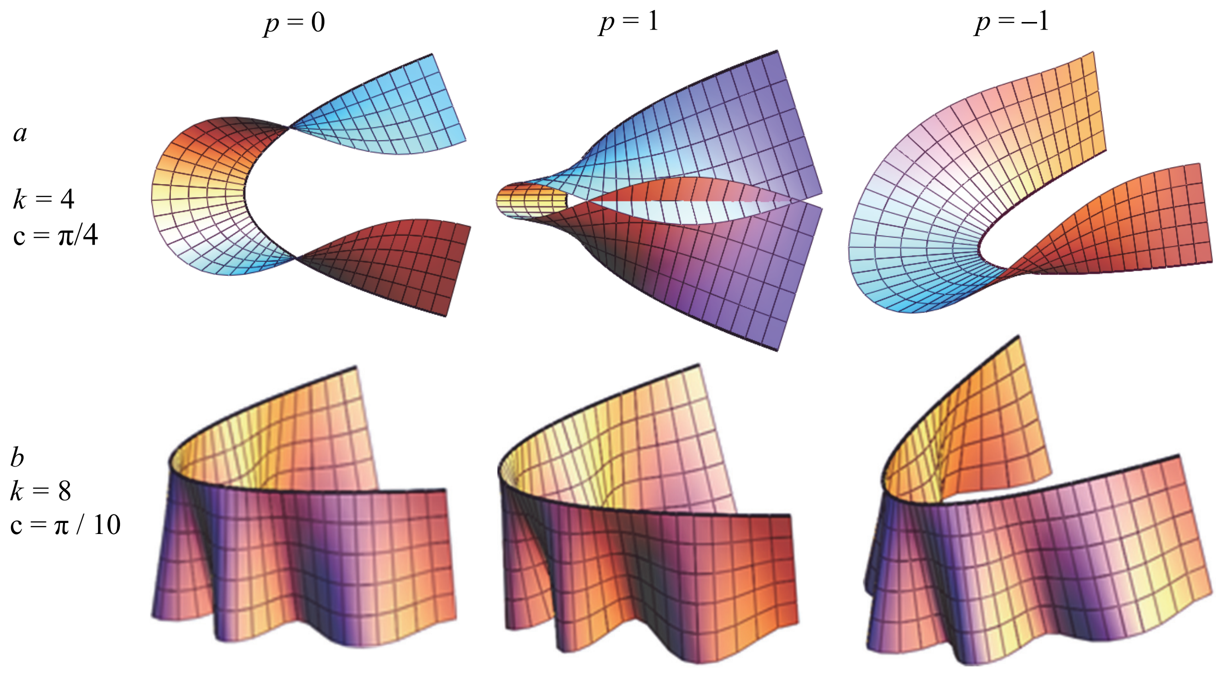

Figure 11. Wave normal ruled surfaces with a base parabolic cylinder, $\theta_0 = \pi\theta_0 = \pi$

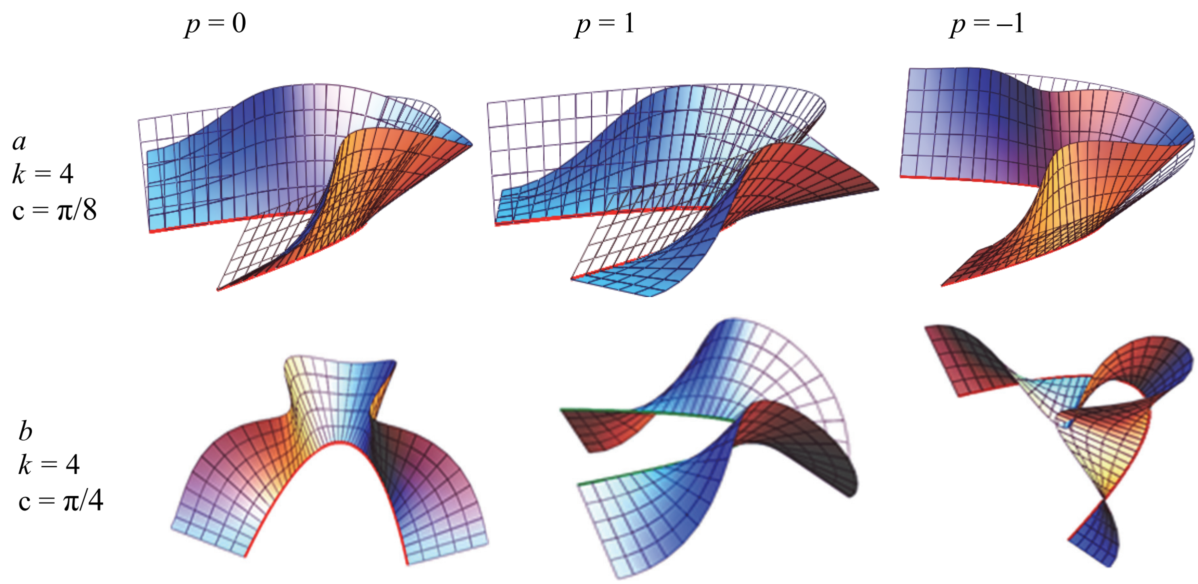

Figure 12. Wave normal ruled surfaces with a base parabola, $\theta_0 = 0.75\pi$

At Figure 12, the wave normal ruled surfaces on a base parabola with $\theta_0 = \pi/2$ are shown.

Consider the wave normal ruled surfaces with a base ellipse.

At Figure 13, the wave normal ruled surfaces with a base ellipse $x = a \cos u$; $y = b \sin u$; $a = 3$; $b = 2$ and parameters $\theta_0 = \pi$, $c = \pi/4$ are shown.

At Figure 14, the wave normal ruled surfaces with base ellipse $x = a \cos u$; $y = b \sin u$; $a = 3$; $b = 2$ and parameters $\theta_0 = \pi$, $c = \pi/8$ are shown. The figures of the wave surfaces are shown together with a supporting ellipsoidal cylinder.

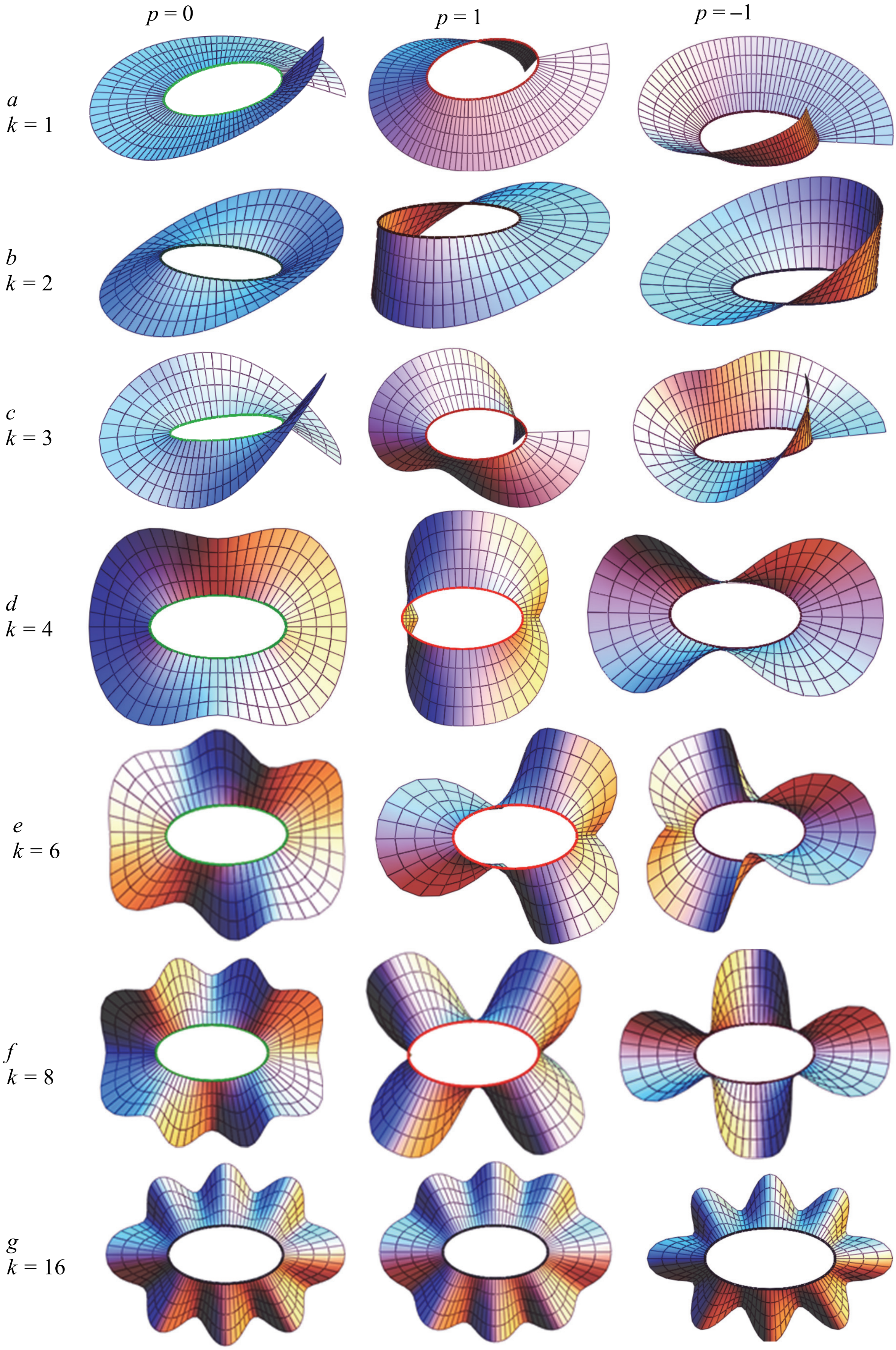

Figure 13. Wave normal ruled surfaces with a base ellipse, $\theta_0=\pi$, $c=\pi/4$

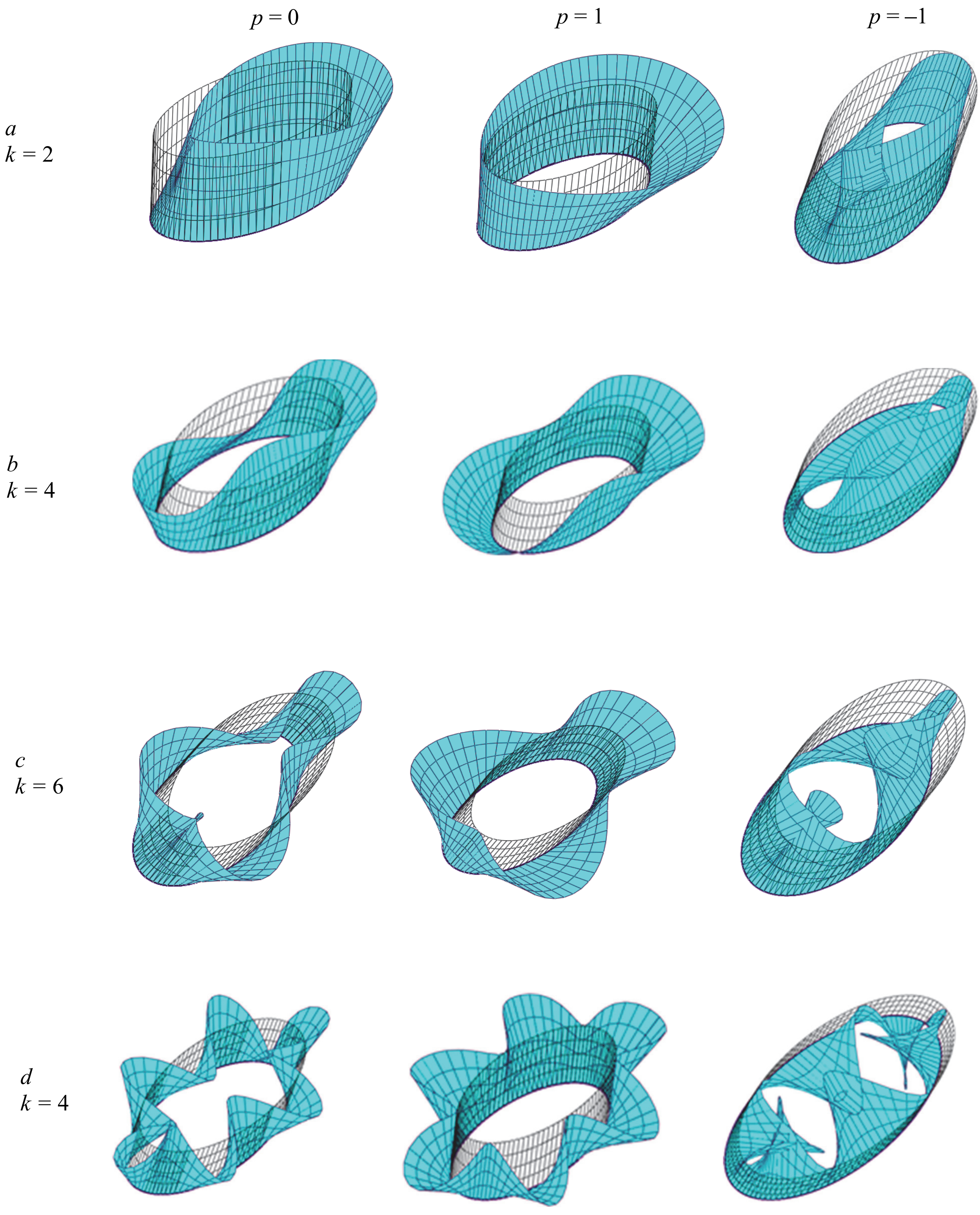

Figure 14. Wave normal ruled surfaces with base ellipse, $\theta_0 = \pi/2$, $c = \pi/8$

At Figure 15, the wave normal ruled surfaces with a base ellipse $x = a \cos u$; $y = b \sin u$; $a = 3$; $b = 2$ and parameters $\theta_0 = 0{,}75\pi$. $c = \pi/8$ are shown. The figures of the wave surfaces are shown together with a supporting surface of constant slope.

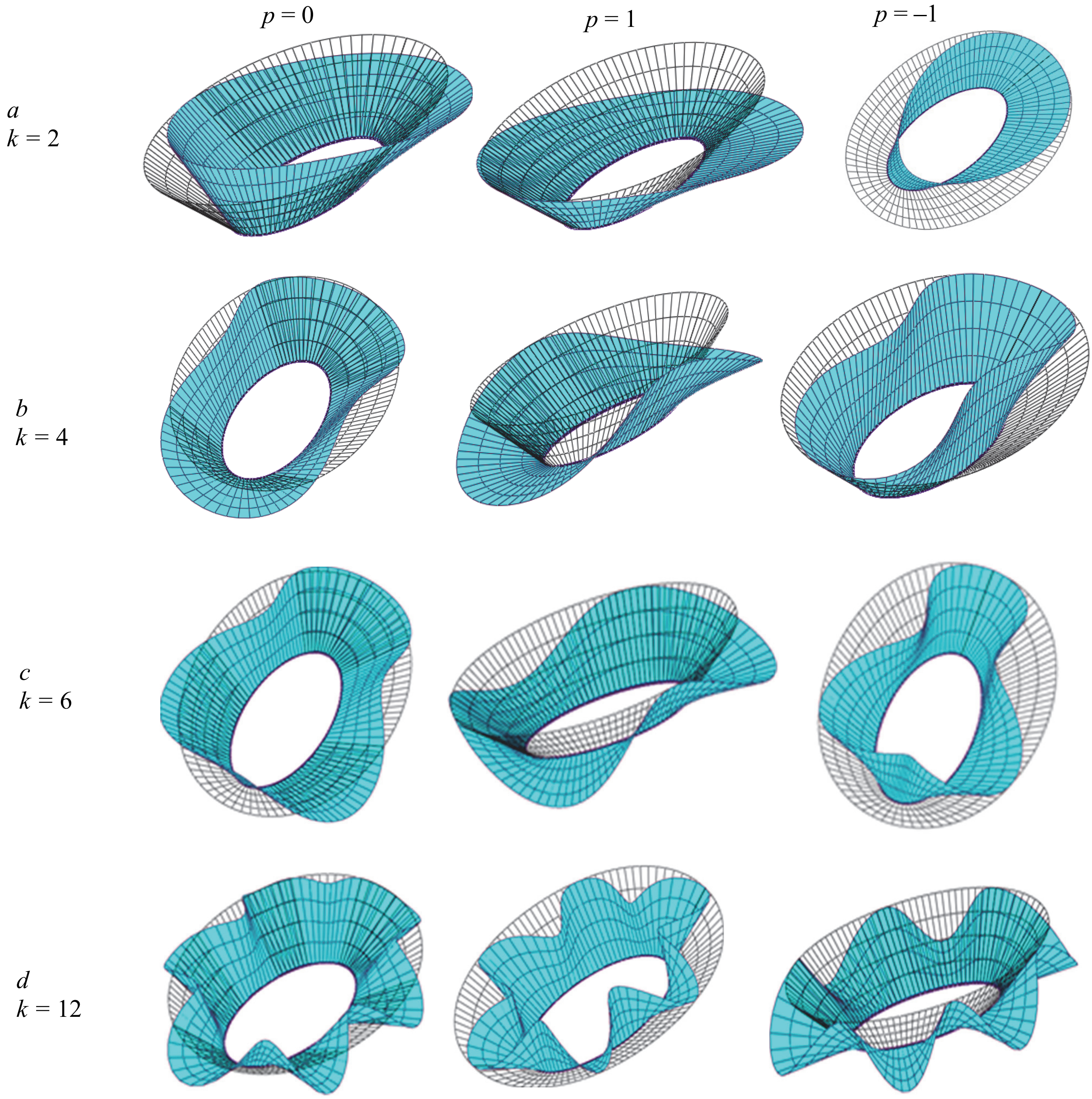

Figure 15. Wave normal ruled surfaces with a base ellipse, $\theta_0 = 0{,}75\pi$, $c = \pi/8$

It's seen from the shown figures of the wave normal ruled surfaces that the modules of waved normal ruled surfaces with parameters $p = 0$ и $p = -1$ may intersect themselves and it depends of amplitude or cosine and the length of the generating straight lines.

The wave normal ruled surfaces with uneven number of oscillation $\kappa = 1, 3, 5, \ldots$ will be unlocked (Figure 13, *a, b*). If $\kappa = 2.6, 10, \ldots$ ($\kappa/2$ is uneven number) then the locked surface with one xOz plane of symmetry will be formed; if $\kappa = 4.8, 12, \ldots$ then the locked surface with two planes of symmetry xOz, yOz will be formed.

Conclusion

The manuscript considers the geometry of normal ruled surfaces that are formed by moving straight generating line at the normal plane of any base directrix curve. The generating straight line rotates at any law given in advance at the normal plane of the base directrix curve. The vector equation of the surfaces and the coeffi-

cients of the fundamental forms of these normal ruled surfaces are presented. It is shown that curvilinear coordinate system of the normal ruled surfaces is orthogonal but non-conjugated in common. So, the generating straight lines of the normal ruled surfaces are not the lines of principle curvatures of the surfaces in general. The condition when the coordinate system will be conjugated and the normal ruled surface will be developable is received. The figures of the normal ruled torus surfaces with plane base directrix curves which are the surfaces of constant slope and figures of torus normal ruled surfaces with space base curves such as helix and conical spiral are shown.

There are considered the subclasses of the normal ruled surfaces:

a) the screw normal ruled surfaces, that are formed by the movement of the generating straight line at the normal plane of the base directrix curve with linear law of rotation of the generating line due to the coordinate parameter of the base curve. If the base curve is a straight line one can receive the classic helix;

b) the wave normal ruled surfaces, that are formed by the oscillation of the generating straight line at the normal plane of the base directrix.

In a paper, the figures of screw and wave normal ruled surfaces with different parameters are presented.

The figures of the surfaces are realized with using of the program complex MathCAD.

It is shown the possibility to construct the different forms of the normal ruled surfaces, which may be used in building, machine building, air production and other technic area.

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