СТРОИТЕЛЬНАЯ МЕХАНИКА ИНЖЕНЕРНЫХ КОНСТРУКЦИЙ И СООРУЖЕНИЙ
STRUCTURAL MECHANICS OF ENGINEERING CONSTRUCTIONS AND BUILDINGS

2025. 21(6). 565–584

ISSN 1815-5235 (Print), 2587-8700 (Online)

HTTP://JOURNALS.RUDN.RU/STRUCTURAL-MECHANICS

РАСЧЕТ ТОНКИХ УПРУГИХ ОБОЛОЧЕК
ANALYSIS OF THIN ELASTIC SHELLS

DOI: 10.22363/1815-5235-2025-21-6-565-584
EDN: FQCZQA

Research article / Научная статья

Strength, Stability and Dynamics of Rigid Shells: Analysis of Recent Research

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Received: May 12, 2025
Revised: September 18, 2025
Accepted: October 2, 2025

Abstract. Many numerical methods of analysis of rigid shells, such as the displacement-based finite element method (FEM), finite difference energy method, method of separation of variables, kinematic method of the theory of limit equilibrium, and so on, were proposed and tested until 2000. Most problems of static and dynamic analysis of canonical shells were successfully solved at the same time. All these methods were used actively after the 2000, too. However, new problems began to appear before structural engineers, architects, and builders. These problems are associated with multi-layer shell walls, with the emergence of new composite construction materials, and therefore, with the solution of physically nonlinear problems. Geometricians presented several hundred new forms of middle surfaces of shells, and that is why the need to select optimal forms from several alternatives using criteria of optimality came into existence. The selection of necessary computing software from many of their types began to be a problem. New problems demanded new methods of approach for their solution. In this paper, a critical evaluation of proposed solutions on strength, stability, and vibration analysis of shells was conducted in connection with new problems that appeared after the year 2000. Rigid shells in the form of analytical surfaces, designed using the canon of parametric architecture, were taken as an example. Analytical middle surfaces of shells, which attracted the attention of architects after 2000, are pointed out, and suitable methods of analysis of these shells are noted for the first time. The review was compiled based on 112 fundamental scientific works published after 2000. Other scientific reviews devoted to the investigation of joint problems of geometry, application, and calculation of assembled rigid thin-walled shells with analytical middle surfaces were not found.

Keywords: shell strength, shell stability, free and forced vibrations of shell, finite difference energy method, method of separation of variables, analytical surface, classification of surfaces

Conflicts of interest. The authors declare that there is no conflict of interest.

Authors' contribution: *Krivoshapko S.N.* — conceptualization of the research, collection and scientific analysis of materials, drafting the text, and synthesis of the results; *Chiadighikaobi P.Ch.* — was responsible for the scientific processing of sources.

For citation: Krivoshapko S.N., Chiadighikaobi P.Ch. Strength, stability and dynamics of rigid shells: analysis of recent research. *Structural Mechanics of Engineering Constructions and Buildings.* 2025;21(6):565–584. http://doi.org/10.22363/1815-5235-2025-21-6-565-584 EDN: FQCZQA

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Прочность, устойчивость и динамика жестких оболочек: анализ современных исследований

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Поступила в редакцию: 12 мая 2025 г.
Доработана: 18 сентября 2025 г.
Принята к публикации: 2 октября 2025 г.

Аннотация. Многие численные методы анализа жестких оболочек, такие как метод конечных элементов (МКЭ) в перемещениях, метод конечных разностей энергий, метод разделения переменных, кинематический метод теории предельного равновесия и т.д., предложены и апробированы до 2000 г. Большинство задач статического и динамического анализа канонических оболочек были успешно решены одновременно. Все эти методы активно использовались и после 2000-х гг. Однако перед инженерами-конструкторами, архитекторами и строителями стали возникать новые проблемы, связанные с многослойными стенками оболочек, с появлением новых конструктивных композиционных материалов, а следовательно, и с решением физически нелинейных задач. Геометры представили несколько сотен новых форм средних поверхностей оболочек, и поэтому возникла необходимость выбора оптимальных форм из нескольких аналогов с использованием критериев оптимальности. Выбор необходимых вычислительных комплексов из множества их типов стал проблемой. Новые проблемы требовали новых подходов к их решению. Проведен критический анализ предлагаемых решений по выполнению анализа оболочки на прочность, устойчивость и вибрацию в связи с новыми проблемами, появившимися после 2000-го года. В качестве примера взяты жесткие оболочки в виде аналитических поверхностей, спроектированные с использованием канона параметрической архитектуры. Выделены аналитические средние поверхности оболочек, которые привлекли внимание архитекторов после 2000 года, и впервые отмечены подходящие методы анализа этих оболочек. Обзор составлен на основе 112 фундаментальных научных работ, опубликованных после 2000-х годов. Других научных обзоров, посвященных исследованию совместных задач геометрии, применению и расчету сборных жестких тонкостенных оболочек с аналитическими средними поверхностями, авторами найдено не было.

Ключевые слова: прочность оболочки, устойчивость оболочки, свободные и вынужденные колебания оболочки, конечно-разностный энергетический метод, метод разделения переменных, аналитическая поверхность, классификация поверхностей

Заявление о конфликте интересов. Авторы заявляют об отсутствии конфликта интересов.

Вклад авторов: *Кривошапко С.Н.* — концепция исследования, сбор и научный анализ материалов, написание текста, обобщение результатов; *Чиадигикаоби П.Х.* — научная обработка источников.

Для цитирования: *Krivoshapko S.N., Chiadighikaobi P.Ch.* Strength, stability and dynamics of rigid shells: analysis of recent research // Строительная механика инженерных конструкций и сооружений. 2025. Т. 21. № 6. С. 565–584. http://doi.org/10.22363/1815-5235-2025-21-6-565-584 EDN: FQCZQA

1. Introduction

The interest in designing and building shell structures, rigid shells, and facilities of curvilinear forms has increased considerably over the last 25 years. Some experts associate it with the emergence of convenient computer software for accurate analyses of these structures, others prove the advantages of human perception of curvilinear forms [1], and the others foretell the rebirth of the “golden age” of shells [2]. Broad information about research on the structural mechanics of thin shells performed until the middle of the 1960s is presented in the book by Prof. V.G. Rekach [3], who himself took an active part in working out methods of analysis of thin-walled shells of complex geometry using linear shell theory. More and more structural engineers are drawn into the investigation of the structural mechanics of thin shells in

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geometrically and physically linear and nonlinear formulations of the problem. Scientists and structural engineers from different countries made valuable contributions to the development of the shell theory, mentioned in study [4]. The interest in the study of theories of analysis and strength characteristics of construction materials for spatial structures appeared at that time [5]. Plywood, corrugated steel, multi-layer wooden panels, and reinforced plastics were used as construction material of thin rigid shells besides reinforced concrete. Study [6] contains information about increasing interest in the development of new methods of shell analysis in connection with the emergence of numerical methods of calculation and the growing use of computers.

During the "golden age" of shells between the 1920s and 1960s, thin shells were built and designed in great numbers in practically every country. But the majority of shells, used in real structures and designed based on analysis from the geometrical point of view, concerned a limited number of surfaces that are cylindrical, conical, spherical, and torus surfaces, right translational surfaces, paraboloids of revolution, hypars, ellipsoids of revolution, and some others [7]. Shells having more complex forms, generally, were implemented on the basis of experiments or intuition of architects and engineers with the application of the simplest calculation methods.

The list of geometrical shapes for the design of thin shells considerably increased because of the computerization of calculating processes, accurate definition of taken hypotheses for the shell analyses, and the emergence of new construction materials closer to the 21st century [2].

This paper aims to highlight geometrical shapes of thin rigid shells for which methods of strength analysis were obtained. The objectives of this study are:

i. To investigate methods of calculating critical forces that cause local or general buckling.

ii. To explore methods of resistance of shells to dynamic actions.

iii. To highlight analytical, semi-analytical, and numerical methods of shell analysis from the sets of methods proposed in the 20th century, which structural engineers continue to use.

iv. To expose new approaches to strength analysis of thin rigid shells in the 21st century.

2. Methodology

This study used a qualitative approach to achieve the aim and objectives. The study examined the strength, stability, and dynamic of rigid shells after 2000.

3. Shells with Ruled Middle Surfaces

Ruled shells can have only zero or negative Gaussian curvature. There are no ruled shells of positive Gaussian curvature.

3.1. Shells with Ruled Middle Surfaces of Zero Gaussian Curvature

3.1.1. Cylindrical Shells

Cylindrical shells have been known since ancient times. They have not lost their relevance at present. A voluminous review of investigations on cylindrical shells has been conducted in study [8], where the author considers, in general, the works before 2000.

At the present time, cylindrical shells with horizontal and vertical axes are investigated with the help of typical computer software. For example, the derivation of the governing equations for analysis of three-layered vertical cylindrical shells under axisymmetric loading with creep taken into account and with the fixed lower edge was given in study [9]. The solution is performed with the help of the finite element method (FEM) together with Euler's method. It was proved that the creep of the middle layer has a positive influence on the strength of the examined shell. Another approach was taken in the study [10], where the presented analytical method allows for solving two types of engineering problems, which are the determination of deviation of dimensions of a shell under known parameters of thermal and pressure loading or the determination of ultimate load when limiting deviations are given. Examination of the publications of the last 20 years shows that, at the present time, analysis of cylindrical shells is connected with taking into account multi-layer walls, geometric and physical nonlinearity, new types of external static and dynamic [11] loading, and assumption of accurate hypotheses [12].

Thus, the newest research shows that structural engineers can calculate any cylindrical shell subjected to various loads without difficulty.

Parabolic cylindrical shells. Parabolic cylindrical shells are the most popular after circular cylindrical shells, especially as roofs of warehouses, industrial workshops, and exhibition pavilions. At the present time, additional investigations of these shells are needed. The dynamic behavior of a thin concrete shell resisting jointly with support columns with the application of FEM has been studied [13]. FEM has been used [14] for the evaluation of the influence of cracks on static and dynamic behavior of reinforced concrete roofs. The account of settlement of one of four corner supports under columns is one more direction in the investigation of parabolic cylindrical shells [15].

3.1.2. Conical Shells

Conical shells and cylindrical shells are some of the first spatial structures that humans learned to build. Conical thin shells always attracted the attention of architects (Figure 1).

Figure 1. Conical pavilions of Austria at "EXPO-2020" exhibition
Source: compiled by Available from: https://www.constructionweekonline.com/cloud/2021/07/07/SwgTuR50-wam-expo-2020-dubai-Austria-Pavilion-1200x801.png (accessed: 04.03.2025).

The investigation of conical shells is realized involving more complex mathematical technique, which allows solving more complex problems as close as possible to present-day reality. Study [16] addresses the issue of thermal stresses and strains in a thin-walled conical shell subjected to uniform heat flow along its side surfaces and at both ends of the thermal insulation shell. The governing equations were derived using a semi-coupled static thermoelastic equation and an energy equation. The Galerkin finite element method was employed. Dissertation [17] deals with a right circular conical shell of variable thickness. A system of three partial differential equations in terms of displacements that was reduced to a system of ordinary differential equations, which was solved by a finite difference method, was derived in this work [17]. The mathematical model of the refined theory depends on three-dimensional equations of the theory of elasticity using approximation of the stress state components by polynomials for the normal coordinate of the middle surface by two exponents higher relative to the classic Kirchhoff — Love theory. It shows that the components of the stress state calculated based on the presented model near the zone of distortion of the stress-strain state differ from alternatives of the classic theory.

The vibrational behavior of shells in the form of truncated cones containing ideal compressible fluid is studied in [18]. The dynamic behavior of the elastic structure is investigated based on the classical shell theory, the constitutive relations of which represent a system of ordinary differential equations written for new unknowns. Small fluid vibrations are described in terms of acoustic approximation using the wave equation for hydrodynamic pressure written in spherical coordinates. Its transformation into the system of ordinary differential equations is carried out by applying the generalized differential quadrature method. The formulated boundary value problem is solved by Godunov's orthogonal sweep method. The natural frequencies of shell vibrations are calculated using the stepwise procedure and the Muller method. The accuracy and reliability of the obtained results are estimated by making a comparison with the known numerical and analytical solutions. For conical, straight, and inverted shells, a numerical analysis has been performed to estimate the possibility of finding configurations, at which the lowest natural frequencies exceed the corresponding values of the equivalent cylindrical shell.

Hence, the calculation of thin conical shells of general type subjected to the action of different factors can be fulfilled, but it is carried out with some difficulties.

Besides the right circular conical shell, the right elliptical conical shell finds application in practice. S.N. Krivoshapko obtained analytical formulae for the calculation of tangential forces using the momentless shell theory.

3.1.3. Developable Shells

Over 400 scientific papers of 280 authors are devoted to the study of the geometry of developable surfaces. The least scientific papers are devoted to the investigation of problems of strength, stability, and dynamics of developable shells. A vast majority of them appeared after 2000. The opening period of the study of developable shells was described in study [19], where it was noted that the main difficulty in the determination of the stress-strain state of developable shells involved the necessity of using a system of 20 governing equations of thin shells given in non-orthogonal conjugate curvilinear coordinates. It allowed to use analytical methods of calculation of the moment shell theory.

The situation improved after the emergence of computer software using numerical methods of analysis of shells with middle surfaces given in arbitrary curvilinear coordinates. Now, a vast majority of studies is devoted to strength analysis of developable shells of equal slope and in the form of evolvent helicoids.

Equal Slope Developable Surfaces with Directrix Ellipse at Base. The problem is solved most simply for equal slope developable shells because their middle surfaces can be defined in principle curvature lines. Firstly, an analysis for the determination of uniformly distributed critical load for an equal slope developable shell with a directrix ellipse at the base has been made in the study [20]. The study calculated a steel shell subjected to vertical load in the form of its weight with the help of the LIRA 9.4 computer software which uses displacement-based FEM. The cases when the low edge is fixed or simply supported were solved in the study. The first two buckling modes for the first case of supports and three modes for the simply supported shell were found. It was shown that the buckling of the shell takes place before exhausting its strength [20].

Static analysis of a developable shell of equal slope with a directrix ellipse at the base was presented first in studies [21; 22]. This analysis was made with the help of two numerical methods. Displacement-based FEM and finite difference energy method of calculation of the action of static external load were used.

Evolvent helicoidal shells. Evolvent helicoid (developable helicoid, Archimedes' screw) is the most known developable surface with a helical edge of regression [23] (Figure 2).

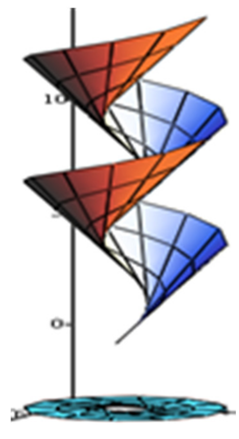


Figure 2. Developable helicoid
Source: compiled by V.N. Ivanov.

All results on the investigation of the strength parameters of the shell in the form of developable helicoid appeared effectively at the end of the 1990s and after the 2000. The approximate method of small parameter was used for the calculation of a long developable helicoid with fixed helical edges due to the given displacement of the inner contour [24] and due to the action of its weight [25].

Firstly, study [26] carried out a FEM investigation of the influence of restraining rectilinear edges of a shell in the form of a developable helicoid on the stress-strain state in its middle zone. It was shown that a very small near-support zone reacts on the restraining method of the straight edges of the helicoid. So, the study confirmed that for a long evolvent helicoidal shell, the reduction of the governing system of equations in partial derivatives to three ordinary differential equations with one independent parameter makes sense if the length of the helical edge is greater than its rectilinear edge.

3.1.4. Parabolic Bending of Elastic Thin Plates into Developable Shells

The main theoretical investigations on parabolic bending of plane metal sheets into developable structures are conducted at the Academy of Engineering of the Peoples Friendship University of Russia (RUDN University), Moscow. The main results of the investigation are outlined in study [27], while several works from other organizations can be examined [28; 29]. Seemingly, the results of investigations on this problem are not interesting for engineers. However, neglect of inner normal stresses appearing in the process of bending the sheet can have great consequences. Bending of real plane sheets from plywood, aluminum, or steel into accurate developable shells with the preservation of rectilinear generatrices is impossible because of the presence of the Poisson's ratio of sheet material according to study [30], which was the first to point this out.

The researchers of [31] realized a series of experiments devoted to the bending of copper plates 5 millimeters thick into cylindrical shells. They constructed experimental curves using the results of the experiments. However, the process of bending is why it is necessary to compare experimental results with theoretical results with a certain portion of skepticism.

Metal sheet bending is realized by different methods, and every one of them has specific advantages. One may obtain conical or pipe hardware of different forms with the help of cold bending. This method would be suitable to receive a bend of a big radius in any case.

3.2. Shells with Ruled Middle Surfaces of Negative Gaussian Curvature

Ruled surfaces of negative Gaussian curvature may also be called oblique ruled surfaces or ruled saddle surfaces. Conoids, hyperbolic paraboloids, and one-sheet hyperboloids are twice-ruled surfaces. Hyperbolic paraboloids and one sheet hyperboloids are considered as a part of the "second order surfaces".

3.2.1. Conoids and Cylindroids

Shells in the form of conoids and cylindroids are seldom used in comparison with conical, spherical, umbrella shells or parabolic shells of revolution (Figure 3) or the form of one-sheet hyperboloids of revolution. The general information about conoids and cylindroids is given [32]. The visualization of new conoidal surfaces has been presented in [33]. Here, only papers published recently on the analysis of thin shells in the form of conoids and cylindroids are mentioned, but we shall not repeat known information published before 2000, which is contained in study [32]. It is noted in study [34] that composite conoidal shells are very important for the aerospace field, and that is why it is necessary to investigate the natural vibration of such structures. The investigation of the conoidal shell resting only upon columns is presented [35]. Computer software are used for the design of cylindroids [36].

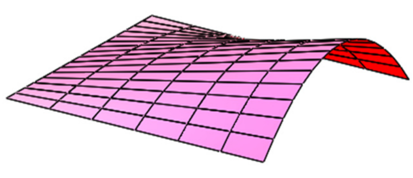


Figure 3. Right parabolic conoid
Source: compiled by V.N. Ivanov.

Almost all scientific papers devoted to the analysis of conoidal shells were published by scientists from India.

3.2.2. Right Helicoidal Shells

The right helicoid is the only ruled surface from a family of minimal surfaces. This helical shell of negative Gaussian curvature found a wide application after the beginning of the 20^{th} century, especially in mechanical engineering. In buildings, they are used, primarily, as ramps for car parking lots. The growing application of right helicoids calls for the elaboration of reliable methods of analysis. In the first period, these were approximate analytical methods but later, reliable numerical methods appeared on the basis of FEM [23].

Now, analytical methods with the application of equations of the linear theory of shells are used only for static analysis of long right helicoidal shells, when the system of governing equations is reduced to ordinary differential equations [37]. In 1957, V.G. Rekach, using two partial differential equations of E. Reissner, obtained one partial differential equation of the eighth order, which he proposed to solve with the help of Fourier series. But only in 2008, M.I. Rynkovskaya [38] composed a computer program with the help of the MathCad computer language and derived numerical results after the correction of equations obtained by Rekach.

The kinematic method of the theory of limit equilibrium was used for the evaluation of the bearing capacity of a right helicoidal shell in study [39]. Examples of calculation of the upper limit of ultimate load are presented in it. The influence of methods of restraint on the bearing capacity of the shell is discussed. One can continue quoting modern works, for example [40–42], but the review of the investigations shows that these ruled shells continue to be used. Study [41] is devoted to the shape optimization of right and developable helicoids in Comsol Multiphysics. Engineers expect more accurate methods of analysis using the principles of the theory of elasticity and shell theory, but one can consider that now problems on strength, stability, and dynamics of right helicoids are solved quite enough.

Having taken the importance of accurate results for mechanical engineering, study [43] proposes to use the results obtained experimentally.

3.2.3. Oblique Helicoidal Shells

The oblique helicoid found application in mechanical engineering, but it is known much less than the right helicoid [23]. Several works on static analysis, published after 2000, are available; for example, one can read paper [44].

4. Shells of Revolution

Shells of revolution enjoy the greatest popularity among architects and mechanical engineers due to their attractiveness, varieties of geometrical forms [45], and due to satisfaction of the majority of techno-logical, economic, ergonomic, and other desires in the design of large-span buildings and to structures of small size. For example, a selection of shells of revolution using 23 criteria of optimality is proposed in the paper [46]. Research paper [47] proposes to extend geometrical investigations of surfaces of revolution. It will be conducive to the introduction of shells of revolution into different fields of engineering and construction.

All known surfaces of revolution, described in the scientific literature on the geometry of surfaces, are presented in study [45] in graphical form. Part of them found application or were recommended for application in engineering and construction, and that is why the necessity appeared for their subsequent investigation on strength, stability, nature, and forced vibrations.

It is easier to obtain results using the momentless shell theory. Simple formulae for the calculation of longitudinal (meridional) and circular stresses in cylindrical, conical, spherical, ellipsoidal, and toroidal shells loaded by inner pressure are given in [48].

The emergence of numerical methods of analysis allows the calculation of layered shells of revolution [49], shells of revolution with pre-defined geometric characteristics [50], shells with the creep of shell material [9], and shells strengthened by frames.

4.1. Spherical Shells

Sphere is one of the canonical forms used in all branches of manufacturing industry, engineering, and construction. Several hundreds of scientific works are devoted to the investigation of geometry, strength, stability, vibration, and application of thin-walled and thick-walled structures in the form of spherical surfaces. Those who are interested in the achievements of scientists in the field of strength, stability, and dynamics of structures in the form of spherical surfaces can begin investigating these problems by studying the following literature [3, 6, 51, 52]. Practical interest in rigid spherical shells does not go out [17; 53–55]. Almost all new construction materials were used in spherical shells. Spherical structures are present in seven of nine types of spatial curvilinear structures and shells, but rigid spherical shells are among the group “rigid shells.” The information presented above shows that designers will not have any problems with designing spherical shells.

4.2. Paraboloids of Revolution

Domes in the form of paraboloids of revolution are not new. These shells are a particular case of elliptic paraboloids. The paraboloid of revolution is also considered to be among surfaces of the class “surfaces of right translation”. The results of the investigation of shells in the form of a paraboloid of revolution can be divided into the following groups: momentless theory of shell analysis, moment theory of shell analysis, natural and forced vibrations, stability of shells, experimental research, temperature actions, problems of elasticity theory, and application of paraboloids of revolution in analytical geometry, architecture, building, using optical properties of paraboloids of revolution. The solution to the indicated problems has been the aim of many investigations in the last 30 years. For instance, FEM and Vlasov’s shallow shell theory have shown good agreement in terms of stresses [56]. A paraboloid-shaped resonator with characteristics of high-overload resistance is proposed in the study [57]. A model of a paraboloid-shaped shell was established by using the energy method. The characteristics of natural frequency and vibration modes were analyzed, and the effects of structure parameters on the natural frequency were obtained by using the finite element method. Finally, experiments on actual resonators proved that this method is reasonably practicable.

Study [58] established that a significant contribution to the overall stress state of the paraboloid of revolution shell with radial waves is made by normal and tangential stresses. The influence of bending stresses acting in the circumferential direction is also great. The minimum load parameter when the shell buckles is more than three times the margin.

4.3. One-Sheet Hyperboloids of Revolution

One sheet hyperboloid of revolution is the only ruled surface of revolution of negative Gaussian curvature. This form can be seen in hundreds of built cooling towers [59; 60]. The majority of investigations of structures in the form of a one sheet hyperboloid of revolution are devoted to lattice shells. However, interesting results can be obtained if one analyzes rigid shells using the physically non-linear theory. The calculation of a shell subjected to wind load and its weight gives excessive values of displacements in comparison with displacements obtained with the help of the physically linear theory [61].

4.4. Pseudospherical Shells

Pseudospherical surfaces (Figure 4) are regarded by geometricians, physicists, and landscape designers. However, there are proposals for their applications in building shells of negative Gaussian curvature. These gave rise to the elaboration of methods of their analysis. In the 21st century, study [62] shows the results of theoretical investigations on the strength and stability of pseudospherical shells subjected to the action of an external distributed load. Both geometrical problems of the middle surfaces of the considered shells and the momentless theory of shell analysis were investigated. Later, the momentless shell theory proved useful for the determination of critical load. The results of analytical and numerical calculations by using FEM are presented in tabular form. Numerical analyses have been performed with the help of the ANSYS computer software. The authors showed that the buckling behavior of thin shells is untypical for the majority of shell structures [62].

Detailed study of numerical methods of calculations and the emergence of standard computer programs of strength analysis allowed using geometrically nonlinear shell theory [63; 64].

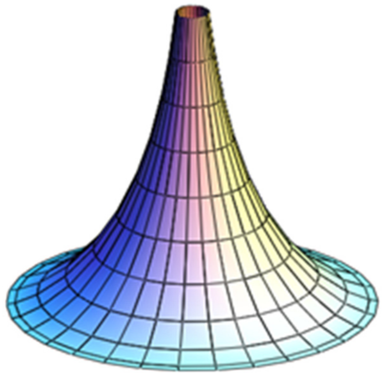


Figure 4. Pseudospherical surface
Source: compiled by V.N. Ivanov.

4.5. Circular Toroidal Shells

Circular toroidal shells can be seen in aerospace engineering, in objects of nuclear power engineering [65], and in other branches of mechanical engineering [66]. In study [65], an analysis of a smooth toroidal shell is presented. Godunov's method of orthogonal sweep was used for the numerical solution of the system of differential equations (a boundary element method) at the finished stage. A method of designing a solution of composite pressure vessels of high pressure of toroidal form subjected to inner pressure is proposed in study [66]. The asymptotic method was evaluated in the process of studying the vibration of the toroidal shell. This form is rarely used in construction fields. Seemingly, only roofs from toroidal fragments are used in construction and architecture.

4.6. Catenoidal Shells

A catenoid is formed by the rotation of a catenary curve around the axis of the surface of the revolution (Figure 5). Soap film pulled over two wire circles, the planes of which are perpendicular to the axis of rotation, which passes through the centers of the circles, takes the shape of a catenoid. Several buildings in the form of catenoids are available, but their extended application is put off for the future. At the present time, theoretical investigations of the strength of these smooth [67] and ribbed [68] shells are in progress. The catenoid is the only minimal surface of revolution.

Orthographic representations of displacements, internal normal forces, and bending moments have been compared [67]. A conclusion was made that the catenoid shell, based on these parameters, possesses some advantages over four other shells of revolution loaded by the same linear vertical load applied to the upper shell edge.

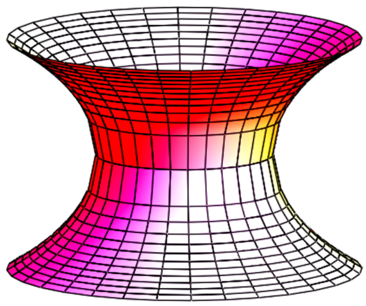


Figure 5. Catenoid
Source: compiledby by
S.N. Krivoshapko, V.N. Ivanov [67].

4.7. Untypical Shells of Revolution

It was pointed out earlier that all known shells of revolution are listed in study [45]. One can design an infinitely large number of new surfaces of revolution in addition to the list presented in study [45]. Some authors created shells of revolution unknown before. They affirm that these shells are necessary for some branches of science, engineering, and construction. For example, the supercritical behavior of any untypical shells of revolution was investigated [69]. A momentless analysis of the hyperbolic-parabolic shell of revolution is proposed and a parabolic-logarithmic shell of revolution is introduced into practice [70]. Generally, the study of these untypical shells of revolution finished after the first published paper.

4.7.1. Egg-shaped Shells

There is no such name in the classic theory of surfaces. However, some authors use this name for a surface. For example, buckling of the egg-shaped shell of revolution was studied in [71].

4.7.2. Shells of Barreled Shapes

The situation is similar for barreled surfaces that are surfaces of rotation of a plane curve around the axis of rotation, but the axis of symmetry of the generatrix curve, that is a meridian, is perpendicular to the axis of rotation. A so-called barreled reservoir has been considered in study [72].

5. Shells With Cyclic Middle Surfaces

A cyclic surface is a surface formed by the motion of a circle of variable (Figure 6) or constant radius in space in some pattern. Prominent geometricians A.P. Norden, V.F. Kagan, V.I. Shulikovskiy, A.M. Yakubovskiy, and others have studied cyclic surfaces.

A valuable contribution to the study of the strength of shells with cyclic middle surfaces was made by Prof. V.N. Ivanov [73] and other researchers [74–76]. Prof. V.N. Ivanov paid attention to these shells in 1971 [77] and compiled a complete classification of cyclic surfaces, specified the names of some surfaces, formed many groups and subgroups of cyclic surfaces, and developed computer software for their analysis using the finite difference energy method. The researchers considered the finite difference energy method more convenient for the calculation of cyclic shells.

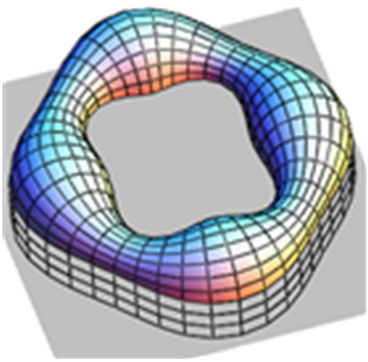


Figure 6. Catenoid
S o u r c e: compiled by V.N. Ivanov [73].

Method of curvilinear nets, momentless shell theory [77], finite difference method, the equations of linear shell theory in complex form, and the method of complex limitations for the determination of natural frequencies have been used for the analysis of cyclic shells. More than 150 monographs, dissertations, scientific papers, and conference proceedings contain information on geometry (80%), analysis of strength (15%), and vibrations of thin shells with cyclic middle surfaces.

Pipe shells with a bent axis. Study [78] is the first work on the analysis of pipe shells. Several early works devoted to the calculation of cyclic shells are listed in bibliography [3]. Several alternatives of the arrangement of pipe shells with curvilinear center lines are presented in study [79]. Many studies are devoted to static analysis and dynamic behavior of pipes with curvilinear axis. They are not considered in this study because general attention is given to the calculation of rigid, large-span shells.

6. Helical and Helical-Type Shells

The most complete information on helical ruled and cyclic shells is given in study [23] with 181 sources. Results of scientific works published in the 20th century are analyzed in the study.

Investigation of helical and helical-type shells did not stop after the year 2000. Performed investigations showed that engineers and designers, in general, engage with ruled helical surfaces and with helical pipe surfaces, but geometricians have some more classes and groups of surfaces with helical or spiral directrix curves in their toolbox.

Ruled helical shells were considered earlier in the "Evolvent Helicoidal Shells", "Right Helicoidal Shells", and "Oblique Helicoidal Shells" sections.

6.1. Helical Pipe Shells

Helical pipe shells find extensive use besides helicoidal ruled shells. Before the year 2000, structural engineers used the method of model equations [52], the method of separation of variables, and the method of the curvilinear net, which is a generalization of the method of finite differences. After the year 2000, in study [80], an analysis of a pipe shell with a bent axis was presented. V.S. Lyukshin distinguished five types of circular helical surfaces, one of which is shown in Figure 7.

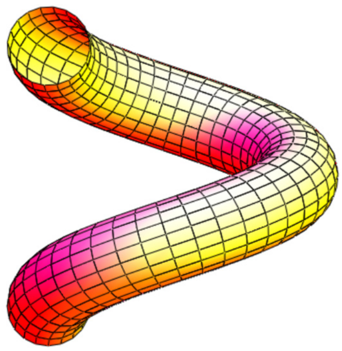


Figure 7. Helical pipe surface with the generatrix circle perpendicular to the axis of the directrix helix
Source: compiled by S.N. Krivoshapko.

7. Shells with Kinematic Middle Surfaces

The total classification of kinematic surfaces with congruent generatrix curves is given in study [81], containing 38 sources.

7.1. Shells of Right Translation

Fundamental surfaces of right translation that found application in architecture and construction are listed in study [45]. In study [82], examples of using surfaces of right translation in the construction of compound shells were shown. In general, all problems of the design of translation shells were solved after the emergence of design computer programs. Some problems associated with the emergence of new construction materials, taking into account the orthotropy of wooden roofs [83], and with strengthening of the bearing capacity, are solved as new ideas arise [84].

7.2. Velaroidal Shells

Velaroidal surfaces are formed by plane curves of variable curvature, but surfaces of right translation are formed by a plane rigid curve. Formed velaroidal surfaces are supported by straight lines of plane rectangular contour (Figure 8). About ten velaroidal surfaces are presented and studied in scientific and technical literature. In study [85], velaroidal shells are considered from a standpoint of their architectural potential. Scientific paperdevoted to the strength analysis of velaroidal shells began to appear, too [86].

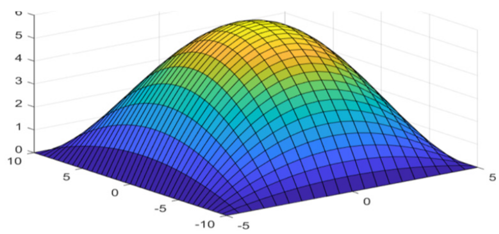


Figure 8. Velaroidal shell
Source: compiled by E. Tupikova, M. Berdiev [86].

7.3. Shells of Diagonal Translation with Generatrix Curve of Variable Curvature

Interesting surfaces of diagonal translation with generatrix curves of constant and variable curvature of velaroidal type on a rhombic plane were proposed for introduction in building shells in study [87].

8. Second-Order Algebraic Surface Shells

The second-order surfaces (quadrics) form a class of surfaces consisting of 17 names. Five non-degenerate non-disintegrate surfaces will be considered in this section. Four degenerate non-disintegrate surfaces that are elliptical, hyperbolical, parabolic cylinders, and elliptical cones were considered in section "Shells with Ruled Middle Surfaces of Zero Gaussian Curvature". The remaining three imaginary surfaces and five degenerate disintegrated surfaces do not find application in the form of rigid shells.

8.1. Ellipsoids

A more detailed description of the results of investigations on the analysis of thin shells in the form of triaxial ellipsoid, published before 2000, is presented in study [88].

In the second ten years of the 21st century, a group of scientists from Volgograd State Agrarian University [89] worked actively for the improvement of a method of analysis of ellipsoidal shells with the help of FEM. They proposed formulae of representation of the radius-vector of triaxial ellipsoids allowing to perform continuous parametrization of surfaces, but two parameters, used in the formulae, have clear geometrical interpretation. It is very important for the analysis of shell structures in the form of ellipsoids, elliptical cylinders, elliptical cones, and similar to them. In study [90], they derived physical equations without a hypothesis about the separation of deformations into elastic and plastic parts. The condition of proportionality of the components of deviators of stress increments was used for their determination. The application of the equations was performed by using a hybrid prismatic finite element with a triangular base. The method was tried out on a truncated ellipsoid. The study also used a quadrangular curvilinear finite-element with nine degrees of freedom in the node as a discretization element of thin-walled structures of concatenated shells with different properties of materials.

8.2. Elliptical Paraboloids

The application of thin shells in the form of elliptic paraboloids took place both in the 20th century and at the beginning of the 21st century [91], and that is why great attention is paid to investigations on static analysis, buckling, and vibration of elliptical-paraboloid shells. After 2012, studies were devoted to the determination of critical forces [92], the strength analysis [93], the determination of free and forced vibrations [94], and the study of reinforced concrete shells in the form of an elliptic paraboloid with openings [95]. It was determined in study [95] that an opening at the top of a paraboloid reduces shell strength by 14%.

8.3. Hyperbolic Paraboloids

This shell form is very well known. Hyperbolic paraboloids have enjoyed popularity for many years (Figure 9), and their popularity is not going out at present. Hundreds of monographs and scientific manuscripts are devoted to them, examples are [7, 96, 97].

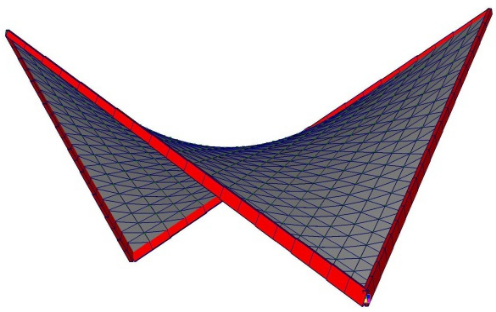


Figure 9. Hyperbolic paraboloid
S o u r c e: compiled by O.O. Aleshina [98].

9. Higher-Order Algebraic Surface Shells

A great number of algebraic surfaces of the higher orders were presented and studied by geometricians. However, a small part of them found application in rigid building shells.

Shells in the Form of Algebraic Surfaces with a Main Frame from Three Superellipses

Taking the main frame of surface in the form of three superellipses placed in three coordinate planes, one can construct three analytical surfaces by the motion of every superellipse with a simultaneous change of its curvature along another superellipse. A surface formed by the motion of the superellipse of variable curvature in the planes parallel to the horizontal coordinate plane is shown in Figure 10, *a*. If one of the superellipses is taken in the form of a rhombus, then one of the obtained surfaces will be a cylindroid [98] (Figure 10, *b*).

In the 2020s, studies were published where the results of static analyses of shells loaded by their weight and which have main frames from three ellipses (Figure 10, *a*) [98]. A thin ruled shell with elliptical base where two of three superellipses of the main frame degenerate into rhombuses (Figure 10, *b*) is considered in study [99]. In study [100], a shell with the middle algebraic surface on a rhombic base is studied.

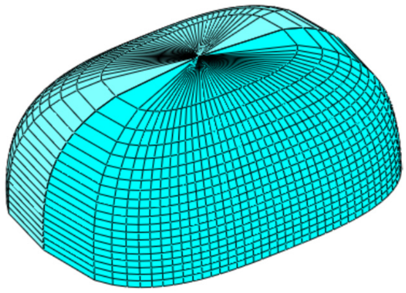
Superellipsoidal surface is defined by implicit equation

$$(x/a)^{2n} + (y/b)^{2n} + (z/c)^{2n} = 1,$$

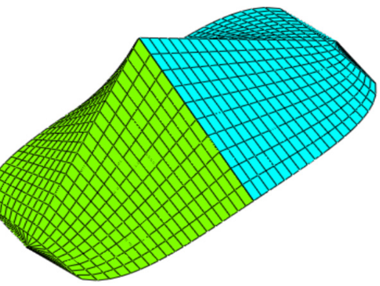
which describes a sphere ($n = 1$, $a = b = c$), ellipsoid ($n = 1$), or cuboid ($n = \infty$). Buckling of the shell with a pointed middle surface subjected to external pressure is studied [101].

Unfortunately, all researchers calculating shells in the form of algebraic surfaces with main frames from three superellipses did not mention the order of decomposition of the vectors of inner forces and moments along the local non-orthogonal or orthogonal axes of the shells. This problem appears because the authors use non-orthogonal curvilinear coordinates. That is why it cannot be determined what internal forces or “pseudo-forces” and moments or “pseudo-moments” were derived.

The same questions can be put before the authors of papers on velaroidal shells (Section 6.2), developable shells (Section 2.1.3), and oblique helicoidal shells (Section 2.2.3).



a



b

Figures 10. Algebraic surfaces with the frames from three superellipses
S o u r c e: compiled by V.N. Ivanov.

10. Umbrella Shells and Umbrella-Type Shells

V.A. Lebedev was the first to pay serious attention to the advantages of umbrella shells in 1954. In 1958, he published a monography [102] in which the information on methods of formation of these shells was generalized, a method of static analysis was proposed, and recommendations on the application of umbrella domes in buildings were presented. Umbrella shells (Figure 11) enjoy popularity as in former times [103]. Structural engineers defined more exact methods of their analysis and optimization of their form [104].

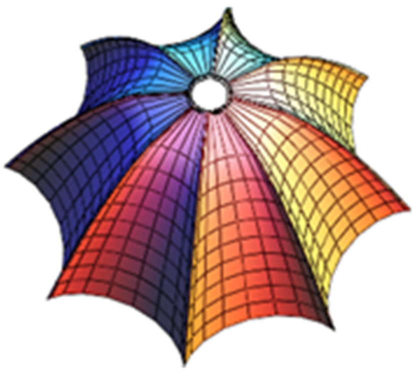


Figure 11. Umbrella surface
S o u r c e: compiled by V.N. Ivanov.

Umbrella-type surfaces are cyclically symmetrical surfaces containing several identical fragments. Complete umbrella-type surfaces and all identical surfaces compiling a complete surface are defined by the same analytical equation. Practically, there are no examples of strength analysis of umbrella-type shells. Only two works on the calculation of these shells were found. In [105], a finite difference energy method was used, and in study [106], FEM was applied. In study [106], a comparative analysis of three shells was made. A shell in the form of a paraboloid of revolution and two umbrella-type shells with radial waves formed by cubic parabolas and sinusoidal curves were studied. The static calculation has been performed for the action of self-weight.

11. New Analytical Forms of Thin Structural Shells, Proposed for Architects

There are many scientific studies devoted to applying new forms of thin shells for public and industrial buildings. The authors of some works adduced the methods of calculation of thin shells with the help of displacement-based FEM, finite difference energy method, method of separation of variables, membrane theory, and so on.

Study [74] elaborated the algorithm of analysis of thin-walled shells in the form of Joachimsthal's canal surfaces with the help of the finite difference energy method. Static calculation of the shell in the form of the considered analytical surface for the action of self-weight and with one fixed lower edge (Figure 12) has been carried out [74]. Joachimsthal's canal surface is a cyclic surface with a plane directrix line of the centers of generatrix circles of changing radius lying in the planes of the pencil [107]. Three scale models of gypsum shells in the form of canal surfaces of Joachimsthal were made [74].

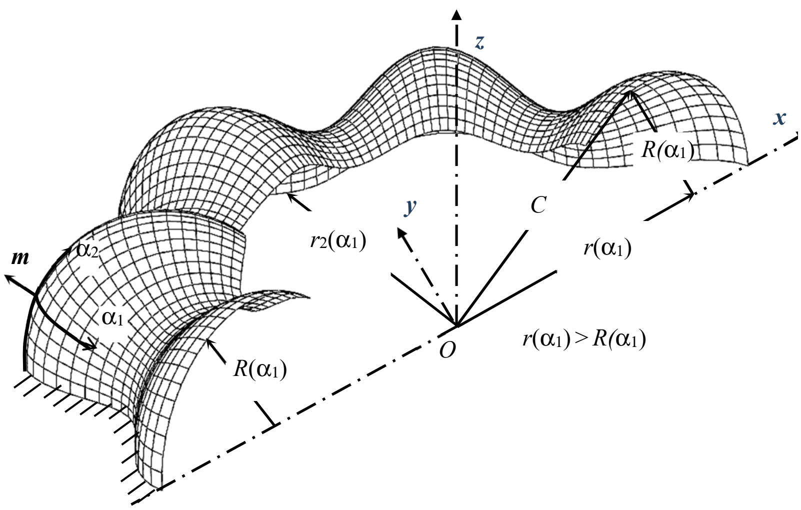


Figure 12. The first type of Joachimsthal's canal surfaces
Source: compiled by V.N. Ivanov.

Figure 13. Reinforced concrete roof of a house, Nepal
Source: compiled by G.P. Lamichhane [76].

The author of [76] verified the finite difference energy method and method of global elements on the example of consecutively joined compartments of thin shells in the form of carved parabolic and sinusoidal surfaces of positive and negative Gaussian curvature. The theoretical elaborations of study [76] were implemented in the real reinforced concrete roof of a house (Figure 13). Carved parabolic and sinusoidal surfaces are formed by the movement of a sinusoidal curve at the normal plane of a quadratic parabola [107].

Carved surfaces of general type are formed by a plane fixed generatrix curve, one point of which moves along any directrix curve, and all the time, the generatrix curve must be in the normal plane of the directrix curve. Static analysis of thin shells in the form of carved surfaces of general type was presented by [75], but the author did not show the real application of these shells in practice.

12. Results

Architects determined that 47 of 600 analytical surfaces, most of which are described in encyclopedia [107], were implemented in architectural projects. Architects and engineers proposed additional analytical surfaces for introduction, which did not find architectural implementation, but study [108] considered that they must be reserved for future application.

Practically all analytical surfaces used before 2000 were implemented in the forms of real building structures at the beginning of the 21st century. Besides the analytical surfaces pointed out above, inclined cylindrical, developable, truncated conical (the new Astronomy Centre of the Royal Observatory, Greenwich, 2007), ordinary helical box-shaped, umbrella, drop-shaped, rotational (roof of the Water Sports Complex, Rostov-on-Don, Russia), velaroidal, wave-shaped (“The Wave” residential complex, Vejle, Denmark, 2006–2015), and wavy (Irina Viner-Usmanova Gymnastics Palace in Luzhniki, Moscow) surfaces, spiral (wedding chapel in the form of two spiral staircases, Onomichi, Japan, 2013) and spiral-shaped surfaces, cylindrical helical strip (fence of helical car parking lot), conoids and cylindroids, nodoids, “The Egg” surface of revolution, circular torus, helix-shaped surfaces, triaxial ellipsoids (the observation site of “Zhivopisny Bridge”, Moscow, 2007), elliptical paraboloids (lattice roof of the “Shachtar Stadium”, Donetsk, DNR, 2009), and one-sided ruled surface (340 m long “Mobius Strip” landscape attraction, VDNKh, Moscow, 2023) were used after the year 2000. But all thin shells with the mentioned analytical middle surfaces do not belong to the “rigid shells” group. They form the remaining eight structural types of shells. The additional information on the classification of curvilinear structures can be acquired in paper [2].

Having summarized the written above, one can say, that many earlier review papers with narrow focus of investigations of rigid thin-walled shells were published in the beginning of the 21st century, for example, on strength analysis of cylindrical shells [8], on geometrical modelling of shell shapes [45, 81], on architectural styles as applied to shells, on the application of shells in architecture [7], on distinguished structural scientists [4], on the reserve of shells for the future [106], and so on. In the presented manuscript, the researchers have tried, for the first time, to join all solved problems devoted to shell design in the last 25 years in one review. Some problems unsettled today were noted too.

In this review, for the first time, thin shells, formed by analytical surfaces, and for which methods of strength analysis under static external load were developed and verified, are presented. The critical buckling parameters and buckling modes are given. The papers where one may explore the methods of determination of natural frequencies, are presented. The subject-matter of review [109] is the closest one to the topic of the presented study. The authors of that paper reviewed research on the compound thin-walled shell structures composed of regular shell elements joined together along their common boundaries. They considered shell junctions of simple forms.

Methods and instructions, pointed out in this review, are contained in the 112 given sources. These works show the main directions in investigations of the last 25 years in structural mechanics of thin rigid shells with analytical middle surfaces. The sources, used in the review, show a high potential of rigid shell structures that are claimed at the beginning of the 21st century [2]. For example, additional information on the structural mechanics of thin shells is given in works [4] with 96 references and in [110] with 1163 references. Review paper [4] mentions the researchers who improved the theory, calculation methods in aspects of strength, stability, and vibrations of thin elastic shell structures. In this review, the results of contemporary research of thin shell theory are presented. The combined stress state exists widely in nature and engineering. Hence, strength of materials and shell structures under the combined stress state is a general problem. Strength theory is of great significance in theoretical research and engineering application. Review paper [110] presents a survey of the advances in strength theory (yield criteria, failure criterion, etc.) of materials, including concrete, iron, polymers, and wood, under combined stress and gives a method of selecting reasonable failure criterion for application in research and engineering.

Design of shell structures begins with the process of “form finding,” when architects and designers explore geometric shapes that naturally distribute loads and stresses. This process involves finding a form that minimizes material usage while maintaining structural integrity. The second stage is the selection of construction materials for the shell. The third stage contains the structural analysis of shell structures that involves complex mathematical modelling to understand how stresses and strains are distributed throughout the shell. So, architectural principles such as form finding, curvature, and attention to visual appearance converge with engineering principles like structural analysis, material selection, and construction techniques to bring these shell structures to life [111]. Architects, designers, and structural engineers can use the materials of this review at every stage of the design process of buildings. It is important to recognize that concrete thin shells have only had a lifespan of less than a century, quite a short time in comparison to other systems [112], but due to construction economy and the inherent beauty of these shapes, they can be widely used in recent years.

13. Conclusion

Nine structural types of shells are known. Researchers limit themselves to the scientific analysis of existing methods of analysis on strength, stability, and dynamics of only rigid building shells. Some types of designs can be implemented only in large-span curvilinear structures. These are exhibition pavilions, planetariums, trade centers, warehouses, cultural facilities, and many others. Having united the experience and knowledge of geometricians, architects, structural engineers, and IT specialists, one can realize virtually any idea in construction and mechanical engineering fields. Curvilinear structures built in the last 25 years, the reliability of which is secured by correct static and dynamic analyses, confirm this assumption. The safety of buildings depends on structural engineers, on rightly defined problems for engineers, on the information related to the design objective. Presented materials show that, today, there are many problems as applied to rigid shells that must be solved in the future or already solved.

It was established that the greatest number of rigid building shells was constructed in the form of cylindrical and conical surfaces, surfaces of revolution and hypars. Accordingly, the greatest number of analytical and numerical methods of analysis on the action of static, thermal and dynamic external loading is devoted to these single or multilayered shells from new construction materials in linear or non-linear definition of the problem.

Using the materials of this study, researchers can independently make a list of unused surfaces for the design of shell forms in architecture and engineering and can propose methods of analysis of shells of unusual form.

This scientific review can help researchers to select future exploration directions and can help clients to define design objectives more accurately.

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