Строительная механика инженерных конструкций и сооружений
Structural Mechanics of Engineering Constructions and Buildings

2025. 21(5). 399–413
ISSN 1815-5235 (Print), 2587-8700 (Online)
HTTP://JOURNALS.RUDN.RU/STRUCTURAL-MECHANICS

DOI: 10.22363/1815-5235-2025-21-5-399-413
EDN: DRENKL

Research article / Научная статья

Analysis of Geometry and Strength of Shells with Middle Surfaces Defined by Two Superellipses and a Circle

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Received: August 11, 2025
Revised: October 2, 2025
Accepted: October 12, 2025

Abstract. In this study, thin shells in the form of algebraic surfaces defined by a geometric frame of three plane superellipses lying respectively in three coordinate planes are considered. As the main focus of the study, the case when the horizontal superellipse is a circle is examined. It is shown that depending on the type of the other two superellipses, it is possible to obtain a conical surface, or a surface of negative Gaussian curvature, including conoids, or surfaces of positive Gaussian curvature. The construction of 12 particular cases of such surfaces with a circular base is illustrated. Six of them are investigated in detail using the methods of differential geometry, i.e. expressions of the fundamental quadratic forms are obtained, for the first time. Out of the 12 presented shell shapes, two ruled shells of zero and negative Gaussian curvature (conical and cylindroidal respectively) with the same geometric frame were selected for comparative static analysis. The two shells were analyzed for uniform distributed load using displacement-based FEM implemented in the SCAD software. It is shown that despite the two shells having identical geometric frames, the conical shell demonstrated better performance over the most strength parameters.

Keywords: circular base, algebraic surface, cylindroid, cone, static analysis, FEM

Conflicts of interest. The authors declare that there is no conflict of interest.

Authors' contribution: *Karnevich V.* — investigation, formal analysis, software, visualization, writing — original draft; *Mamieva I.A.* — methodology, validation, writing — review & editing. Both of the authors read and approved the final version of the article.

Acknowledgements: The authors sincerely thank Prof. Sergey N. Krivoshapko for his swift, undemanding and engaged assistance and comprehensive commentary at any stage of preparing this paper, and for his suggestions of future research. His help and advice always extended beyond the original inquiry and was not bounded by subject or time. It was his broad and deep knowledge of the topic, easily understandable explanation and enthusiastic delivery that originally sparked and further fueled the interest in the study of thin-walled structures and fundamentally encouraged pursuing science.

For citation: Karnevich V.V., Mamieva I.A. Analysis of geometry and strength of shells with middle surfaces defined by two superellipses and a circle. *Structural Mechanics of Engineering Constructions and Buildings.* 2025;21(5):399–413. http://doi.org/10.22363/1815-5235-2025-21-5-399-413 EDN: DRENKL

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Исследование геометрии и напряженно-деформированного состояния оболочек со срединными поверхностями, заданными двумя суперэллипсами и окружностью

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Получено: 11 августа 2025 г.
Доработана: 2 октября 2025 г.
Принята к публикации: 12 октября 2025 г.

Аннотация. Рассмотрены тонкие оболочки в форме алгебраических поверхностей с геометрическим каркасом из трех суперэллипсов, лежащих в трех координатных плоскостях, в случае, когда горизонтальный суперэллипс представляет собой круглое основание. Показано, что в зависимости от формы остальных двух суперэллипсов можно получить коническую поверхность, поверхность отрицательной гауссовой кривизны, включая коноиды, или поверхность положительной гауссовой кривизны. Проиллюстрировано построение 12 примеров таких поверхностей на круглом основании. Из них 6 поверхностей впервые исследованы подробно методами дифференциальной геометрии, получены их коэффициенты квадратичных форм. Из 12 представленных форм оболочек для сравнительного статического расчета выбраны две линейчатые оболочки нулевой и отрицательной гауссовой кривизны (коническая поверхность и цилидроид) с одинаковым геометрическим каркасом. Расчет оболочек с равномерно распределенной нагрузкой производился с использованием метода конечных элементов (МКЭ) в перемещениях, реализованном в программном комплексе SCAD. Показано, что, несмотря на одинаковый геометрический каркас этих двух оболочек, по большинству параметров НДС лучшие показатели у конической оболочки.

Ключевые слова: круглое основание, алгебраическая поверхность, цилиндроид, коническая поверхность, статический расчет, МКЭ

Заявление о конфликте интересов. Авторы заявляют об отсутствии конфликта интересов.

Вклад авторов: *Карневич В.В.* — исследование, анализ, программное обеспечение, визуализация, написание оригинального проекта; *Мамиева И.А.* — методология, валидация, написание, рецензирование и редактирование текста. Оба автора ознакомлены с окончательной версией статьи и одобрили ее.

Благодарности: Авторы всегда будут помнить профессора Сергея Николаевича Кривошапко за его отзывчивость, готовность прийти на помощь и глубокий интерес к работе над статьей. Особую признательность хочется выразить за его детальные комментарии и ценные рекомендации на всех этапах подготовки текста, а также за конструктивные идеи касательно дальнейших научных изысканий. Сергей Николаевич, оказавший значительное влияние на развитие науки в области тонкостенных конструкций, благодаря своему богатому опыту, широте взглядов и способности доступно преподносить сложный материал, сыграл ключевую роль в пробуждении интереса к исследованию поверхностей и оболочек.

Для цитирования: *Karnevich V.V., Mamieva I.A.* Analysis of geometry and strength of shells with middle surfaces defined by two superellipses and a circle // Строительная механика инженерных конструкций и сооружений. 2025. Т. 21. № 5. С. 399–413. http://doi.org/10.22363/1815-5235-2025-21-5-399-413 EDN: DRENKL

1. Introduction

In descriptive geometry, the frame of a surface is a set of lines, which define the surface. Surfaces constructed from a geometric frame of three curves lying respectively in three coordinate planes are widely used in shipbuilding for the design of hulls of above- and under-water vessels. In [1], the author discusses issues of modelling hull surfaces with discrete points and the computational advantages and geometric intuitivity of using parametric representation in the surface modeling. In [2], thirteen analytical surfaces for preliminary stages of hull shape selection and different methods of their construction are presented. There were suggestions of using superellipses as the plane curves of the geometric frame [3–6], which allow to significantly expand the number of shapes for ship hulls by varying the parameters of the superellipses.

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Paper [7] presents parametric equations and a technique for generating complex submarine hull shapes, which are composed of fragments of surfaces defined by a frame of superellipses. In [8], thin shells with middle surfaces containing three plane superellipses as the geometric frame were originally suggested to be used in construction and architecture.

In [5; 9], the curves defining the considered surfaces are expressed in the following form:

- the first curve of the geometric frame in the xOy plane ($z = 0$):

$$|y|^r = W^r \left(1 - \frac{|x|^t}{L^t}\right), \tag{1}$$

- the second curve of the geometric frame in the yOz plane ($x = 0$):

$$|z|^n = T^n \left(1 - \frac{|y|^m}{W^m}\right), \tag{2}$$

- the third curve of the geometric frame in the xOz plane ($y = 0$):

$$|z|^s = T^s \left(1 - \frac{|x|^k}{L^k}\right), \tag{3}$$

where for convex curves $r, t, n, m, s, k > 1$; for concave curves $r, t, n, m, s, k < 1$. Curves (1)–(3) represent superellipses if the exponents within each equation are equal, or arbitrary plane curves otherwise. The exponents in equations (1)–(3) can take on any positive value. In this study, only superellipses are considered to constitute the geometric frame, so $r = t$, $n = m$, $s = k$. If $r = t = 1$, $n = m = 1$, $s = k = 1$, then curves (1)–(3) degenerate into straight lines, and superellipses degenerate into rhombs.

Using the method described in [6; 9], it is possible to derive the explicit equations of three algebraic surfaces with the same geometric frame of curves (1)–(3):

- generated by a family of sections in x = const planes:

$$|z| = T\left(1 - |x|^k / L^k\right)^{1/s} \left[1 - |y / W|^m / \left(1 - |x / L|^t\right)^{m/r}\right]^{1/n}, \tag{4}$$

- generated by a family of sections in y = const planes:

$$|z| = T\left(1 - |y|^m / W^m\right)^{1/n} \left[1 - |x / L|^k / \left(1 - |y / W|^r\right)^{k/t}\right]^{1/s}, \tag{5}$$

- generated by a family of sections in z = const planes:

$$|y| = WW\left(1 - |z|^n / T^n\right)^{1/m} \left[1 - |x / L|^t / \left(1 - |z / T|^s\right)^{t/k}\right]^{1/r}, \tag{6}$$

where $-L \le x \le L,\ -W \le y \le W,\quad 0 \le z \le T.$

The explicit equations of surfaces (4)–(6) can be transformed into parametric equations:

$$x = x(u) = \pm uL,\ y = y(u,v) = vW\left[1 - u^t\right]^{1/r},\quad z = z(u,v) = T\left[1 - u^k\right]^{1/r}\left[1 - |v|^m\right]^{1/n}; \tag{7}$$

$$x = x(u,v) = vL\left[1 - u^r\right]^{1/t},\ y = y(u) = \pm uW,\quad z = z(u) = T\left[1 - u^m\right]^{1/n}\left[1 - |v|^k\right]^{1/s}; \tag{8}$$

$$x = x(u,v) = vL\left[1-u^s\right]^{1/k},\ y = y(u,v) = \pm W\left[1-u^n\right]^{1/m}\left[1-|v|^t\right]^{1/r},\quad z = z(u) = uT, \tag{9}$$

where $0 \le u \le 1$, $-1 \le v \le 1$; u, v are non-dimensional parameters.

The considered surfaces can be referred to as "kinematic surfaces", since they are formed by the motion of a generatrix of variable or constant curvature along a directrix. By taking each of the three superellipses of the geometric frame as the generatrix one-at-a-time, three analytical surfaces are obtained, which are defined by explicit equations (4)–(6) or parametric equations (7)–(9).

Equations (4)–(9) were used in paper [10] for constructing five groups of new ruled surfaces. Some of these ruled surfaces were taken as middle surfaces of thin shells, which were analyzed for dead load in [11].

In scientific literature and in practice, thin shells with a circular base are the most popular. Virtually all shells with a circular base known to date are shells of rotation, for which about three dozens of optimality criteria have been proposed [12]. Less known is the method of defining the geometry of shells where middle surfaces contain three plane curves as the frame, and one of these curves is a circle.

The objective of this paper is to investigate shells with middle surfaces defined by a geometric frame of superellipses in the particular case when the horizontal curve (base outline) is a circle. Some specific groups of such surfaces are analysed in detail using the methods of differential geometry for the first time to demonstrate the geometrical equivalence or distinction of surfaces with the same frame, but different method of generation. In addition, static analysis is applied to shells with middle surfaces from a particular group to identify the differences in the structural behavior.

2. Methods

2.1. Construction of Surfaces Defined by Two Superellipses and a Circle

Assuming that a surface with the frame of superellipses has a circular base in the *xOy* coordinate plane, then the following values of parameters in equations (1)–(9) can be adopted:

$$r = t = 2,\ L = W = R,\quad 0 \le z \le 0,\ -R \le x \le R,\ -R \le y \le R,$$

and *z*-axis is directed upwards. In this case, expressions (7)–(9) can be rewritten as

$$x = x(u) == \pm vR,\quad y = y(u,v) = \pm vR\left[1-u^2\right]^{1/2},\quad z = z(u,v) = T\left[1-u^k\right]^{1/s}\left[1-v^m\right]^{1/n}; \tag{10}$$

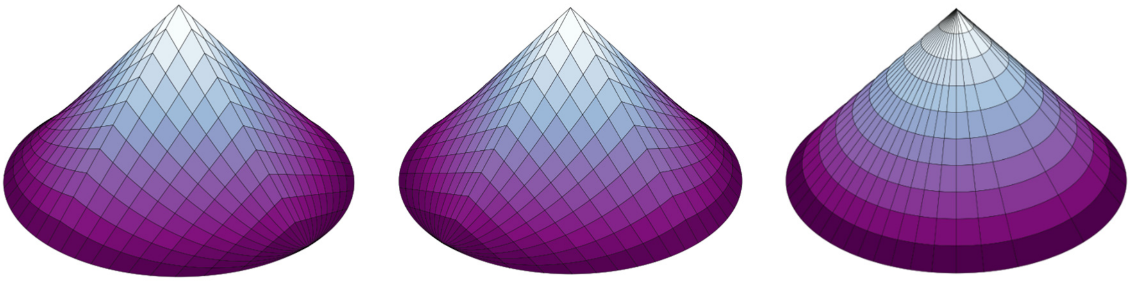
$$x = x(u,v) = \pm vR\left[1-u^2\right]^{1/2},\quad y = y(u) = \pm uR,\quad z = z(u) = T\left[1-u^m\right]^{1/n}\left[1-v^k\right]^{1/s}; \tag{11}$$

$$x = x(u,v) = \pm vR\left[1-u^s\right]^{1/k},\quad y = y(u,v) = \pm R\left[1-u^n\right]^{1/m}\left[1-v^2\right]^{1/2},\quad z = z(u) = uT, \tag{12}$$

where $0 \le u \le 1$, $0 \le v \le 1$; u,v are non-dimensional parameters.

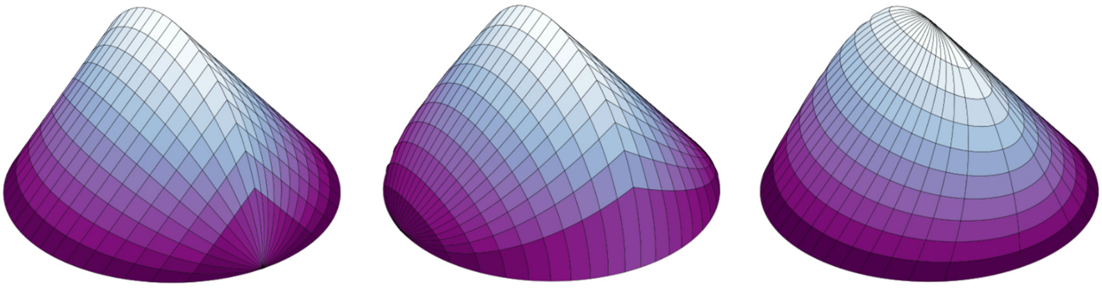
Parametric equations (10)–(12) allow to construct an unlimited number of groups of three surfaces. And in each group, the three surfaces will have the same geometric frame of two half-superellipses in vertical coordinate planes and the same circular base in the horizontal plane.

Several specific groups of three surfaces with the same frame are constructed and illustrated below. Using parametric equations (10)–(12), the first group of three surfaces is constructed for the case of $n = m = s = k = 1$ (Figure 1), the second group of three is constructed with $n = m = 1$, $s = k = 2$ (Figure 2), the third group of three has $s = k = 1$, $n = m = ¾$ (Figure 3), and the fourth group of three is constructed with $n = m = 2$, $s = k = ¾$ (Figure 4). The surfaces are visualized using Matplotlib v3.4.2 plotting library for Python programming language [13].



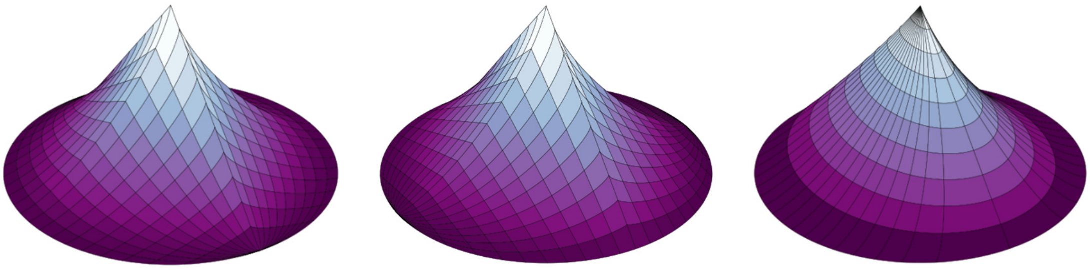
a *b* *c*

Figure 1. Analytical surfaces with a circular base (the 1st group of three where $n = m = s = k = 1$):
a — generated using equations (10); *b* — generated using equations (11); *c* — generated using equations (12)
S o u r c e: compiled by Valery Karnevich.



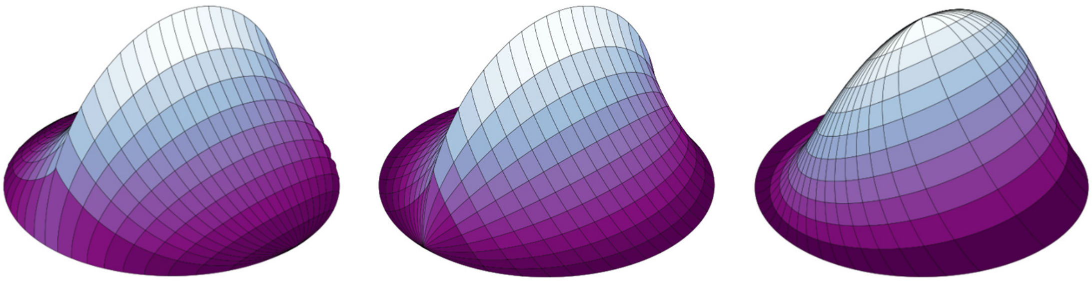
a *b* *c*

Figure 2. Analytical surfaces with a circular base (the 2nd group of three where $n = m = 1$, $s = k = 2$):
a — generated using equations (10); *b* — generated using equations (11); *c* — generated using equations (12)
S o u r c e: compiled by Valery Karnevich.



a *b* *c*

Figure 3. Analytical surfaces with a circular base (the 3rd group of three where $s = k = 1$, $n = m = 3/4$):
a — generated using equations (10); *b* — generated using equations (11); *c* — generated using equations (12)
S o u r c e: compiled by Valery Karnevich.



a *b* *c*

Figure 4. Analytical surfaces with a circular base (the 4th group of three where $n = m = 2$, $s = k = 3/4$):
a — generated using equations (10); *b* — generated using equations (11); *c* — generated using equations (12)
S o u r c e: compiled by Valery Karnevich.

By changing the values of exponents n, m, s, k in equations (10)–(12), it is possible to continue the construction of various surfaces with a circular base. The surfaces demonstrated in Figures 1–4 can be implemented as architectural structures in the form of rigid shells or in the forms of tent coverings. The potential for application of thin shells with middle surfaces shown in Figures 1–4 was originally considered in [14].

2.2. Geometric Analysis

Geometric properties of the first two groups of the presented surfaces (Figures 1 and 2) are examined using the methods of differential geometry.

A two-dimensional manifold (surface) naturally involves the use of two independent parameters. Any analytical surface defined by parametric equations can be expressed in vector form:

$$\boldsymbol{r}=\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k},$$

where u and v are independent parameters. The terminal points of all vectors $\boldsymbol{r}=\boldsymbol{r}(u,v)$ form a surface in space.

Internal and external geometry of a surface is described numerically by the coefficients of the fundamental forms. Coefficients E, G, F of the first quadratic form characterize the internal geometry of a surface, coefficients L, M, N of the second quadratic form characterize the curvature of the surface in space and coefficient K defines the Gaussian curvature [15]:

$$E=A^2=\boldsymbol{r}_u^2,\quad G=B^2=\boldsymbol{r}_v^2,\quad F=\boldsymbol{r}_u\boldsymbol{r}_v;$$

$$L=\frac{(\boldsymbol{r}_{uu}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{A^2B^2-F^2}},\quad M=\frac{(\boldsymbol{r}_{uv}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{A^2B^2-F^2}},\quad N=\frac{(\boldsymbol{r}_{vv}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{A^2B^2-F^2}};$$

$$K=\frac{LN-M^2}{A^2B^2-F^2}.$$

2.3. Static Analysis

Thin shells with the middle surfaces shown in Figure 1 is are selected for a comparative static analysis under uniformly distributed vertical load. The choice of the analysis method is discussed below.

Four stages of creation and development of the theory of plates and shells, which gave rise to mechanism of analysis of spatial roof systems of large-span buildings and structures on a contemporary level, are presented in [16]. The author supposes that the fourth stage of development of the shell theory, design and construction of large-span structures has begun in the 21st century.

Now, a large variety of analytical, semi-analytical, and numerical methods of analysis of shells and shell structures exist. In the previous section, it was shown that the considered middle surfaces of shells can be defined in Cartesian coordinates using algebraic equations (4)–(6) or using parametric equations (10)–(12). Curved coordinate lines u, v of these surfaces can be non-orthogonal ($F\neq 0$) or orthogonal ($F=0$), non-conjugate ($M\neq 0$) or conjugate ($M=0$).

Taking this into account, one may use Goldenveiser's system of 20 governing equations [17] of the thin shell theory for arbitrary curvilinear coordinates containing internal "pseudo-forces" and "pseudo-moments", or the system of governing equations suggested by S.N. Krivoshapko [18] containing internal forces and moments generally used in engineering calculations, or the governing equations of Ya.M. Grigorenko, A.M. Timonin [19] expressed in tensor form. The linear theory of thin elastic shells is an approximate two-dimensional case of three-dimensional linear theory of elasticity [20]. The linear theory of thin elastic shells belongs to classical special two-dimensional models within linear elasticity [21]. The governing equations suggested by these researchers contain coefficients of the fundamental quadratic forms, which have not been previously presented for the specific surfaces examined in this paper.

Relevant literature analysis has shown that these three groups of governing equations of the linear theory of thin shells have been used only in the case of the simplified momentless theory of shells or for the analysis of ruled shells with a number of simplifications in geometry or governing equations. Hence, accurate application of analytical methods for the shells in question cannot be realized at present time.

Several numerical methods were considered for the analysis of the shells in this study. Such included: method of numerical integration of the system of governing differential equations, asymptotic semi-analytical method with a small parameter, finite difference energy method, finite element method in terms of displacements, and others [22]. It was decided to use displacement-based FEM [23]. In the 21st century, such FEM software as LIRA, SCAD, STARK, MicroFE, STADIO, ABAQUS, ADINA, ANSYS, LS-DYNA, COSMOS, MSC/NASTRAN, SOFISTIC, and other were successfully used for similar tasks. It was decided to select SCAD [24], which allows to conveniently define shell geometry using parametric equations and set the mesh discretization step along the curved coordinate lines. By changing the overall dimensions of shells, selecting appropriate exponents of algebraic curves (1)–(3) of the main frame of the shells, and by assuming a particular parameter of optimization, one can select an optimal structure among a large number of shells in automatic mode.

3. Results

3.1. Geometric Analysis

3.1.1. First Group of Three Surfaces

The coefficients of the fundamental forms of the surface in Figure 1, *a* can be expressed in the following form:

$$E = A^2 = \mathbf{r}_u^2 = R^2 + R^2u^2v^2 / \left(1-u^2\right) + T^2\left(1-v\right)^2; \tag{13}$$

$$G = B^2 = \mathbf{r}_v^2 = T^2\left(1-u\right)^2 + R^2\left(1-u^2\right) = B^2\left(u\right); \tag{14}$$

$$F = \mathbf{r}_u\mathbf{r}_v = -R^2uv + T^2\left(1-u\right)\left(1-v\right); \tag{15}$$

$$L = -\frac{R^2Tv\left(1-u\right)}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{3}{2}}}; \tag{16}$$

$$M = \frac{R^2T\left(1-u\right)}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{1}{2}}}; \tag{17}$$

$$N = 0; \tag{18}$$

$$K = -M^2 / \left(A^2B^2-F^2\right) < 0. \tag{19}$$

In expressions (13)–(19), the coefficient of the first fundamental form $F \neq 0$ shows that coordinate lines u, v are non-orthogonal. The coefficient of the second fundamental form $N = 0$ shows that coordinate lines v coincide with the straight generators of the surface. The coefficient of the second fundamental form $M \neq 0$ shows that the coordinate grid u, v is non-conjugate. The ruled surface presented in Figure 1, *a* is a surface of negative Gaussian curvature, since $K < 0$.

The coefficients of the fundamental quadratic forms of the surface shown in Figure 1, *b* are also determined by expressions (13)–(19). Since the ruled surfaces presented in Figure 1, *a* and 1, *b* have the same coefficients of the fundamental forms, they are identical surfaces. They are both cylindroids [25].

The coefficients of the fundamental quadratic forms of the surface in Figure 1, *c* are expressed as follows:

$$E = A^2 = \boldsymbol{r}_u^2 = T^2 + R^2; \tag{20}$$

$$F = \boldsymbol{r}_u \boldsymbol{r}_v = 0; \tag{21}$$

$$G = B^2 = \boldsymbol{r}_v^2 = R^2 (1-u)^2 / \left(1-v^2\right); \tag{22}$$

$$L = 0; \tag{23}$$

$$M = 0; \tag{24}$$

$$N = \frac{(\boldsymbol{r}_{vv} \boldsymbol{r}_u \boldsymbol{r}_v)}{\sqrt{A^2 B^2 - F^2}} = \frac{-TR(1-u)}{\left(T^2 + R^2\right)^{\frac{1}{2}} \left(1-v^2\right)}; \tag{25}$$

$$K = \frac{LN - M^2}{A^2 B^2 - F^2} = 0. \tag{26}$$

In expressions (20)–(26), the coefficient of the second fundamental form $L = 0$ shows that the curved coordinate lines u coincide with the straight generators of the surface. The coefficient of the first fundamental form $F = 0$ shows that coordinate lines u, v are orthogonal and the coefficient of the second fundamental form $M = 0$ shows that the coordinate grid u, v is conjugate. Therefore, the introduced curvilinear system of coordinates u, v is defined in lines of principal curvatures. The ruled surface shown in Figure 1, *c* is a surface of zero Gaussian curvature, since $K = 0$.

This ruled surface is a right circular cone. Differentials of the corresponding arclengths of coordinate lines u and v can be determined using the expressions

$$ds_u = A du, \quad ds_v = B dv.$$

3.1.2. Second Group of Three Surfaces

The coefficients of the fundamental quadratic forms of the surface shown in Figure 2, *a* have the following form:

$$E = A^2 = r_u^2 = u^2 v^2 R^2 / \left(1-u^2\right) + u^2 T^2 (1-v)^2 / \left(1-u^2\right) + R^2; \tag{27}$$

$$G = B^2 = \boldsymbol{r}_v^2 = \left(T^2 + R^2\right)\left(1-u^2\right) = B^2(u); \tag{28}$$

$$F = \boldsymbol{r}_u \boldsymbol{r}_v = -vR^2 u + T^2 u (1-v); \tag{29}$$

$$L = -\frac{R^2 T}{\sqrt{A^2 B^2 - F^2} \left(1-u^2\right)}; \tag{30}$$

$$M = 0; \tag{31}$$

$$N = 0; \tag{32}$$

$$K = 0. \tag{33}$$

Expressions (27)–(33) indicate that the system of curvilinear coordinates u, v is non-orthogonal ($F \neq 0$), but conjugate ($M = 0$). Coordinate lines v coincide with the straight generators ($N = 0$) of the cylindrical surface ($K = 0$) shown in Figure 2, *a*.

The coefficients of the fundamental quadratic forms of the surface shown in Figure 2, *b* have the following form:

$$E = A^2 = \boldsymbol{r}_u^2 = R^2 + R^2 v^2 u^2 / \left(1-u^2\right) + T^2\left(1-v^2\right); \quad (34)$$

$$G = B^2 = \boldsymbol{r}_v^2 = R^2\left(1-u^2\right) + T^2 v^2 \left(1-u\right)^2 / \left(1-v^2\right); \quad (35)$$

$$F = \boldsymbol{r}_u \boldsymbol{r}_v = -R^2 uv + vT^2\left(1-u\right); \quad (36)$$

$$L = \frac{R^2 T\left(1-u\right)v^2}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{3}{2}}\left(1-v^2\right)^{\frac{1}{2}}}; \quad (37)$$

$$M = -\frac{R^2 T\left(1-u\right)v}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{1}{2}}\left(1-v^2\right)^{\frac{1}{2}}}; \quad (38)$$

$$N = \frac{R^2 T\left(1-u\right)\left(1-u^2\right)^{\frac{1}{2}}}{\sqrt{A^2B^2-F^2}\left(1-v^2\right)^{3/2}}; \quad (39)$$

$$K = \frac{R^4 T^2\left(1-u\right)^2 v^4}{\left(A^2B^2-F^2\right)^2\left(1-u^2\right)\left(1-v^2\right)^2} > 0. \quad (40)$$

The corresponding coefficients of the fundamental quadratic forms of the surface shown in Figure 2, *c* have the following form:

$$E = A^2 = \boldsymbol{r}_u^2 = T^2 + R^2 v^2 u^2 / \left(1-u^2\right) + R^2\left(1-v^2\right); \quad (41)$$

$$G = B^2 = \boldsymbol{r}_v^2 = R^2\left(1-u^2\right) + R^2 v^2 \left(1-u\right)^2 / \left(1-v^2\right); \quad (42)$$

$$F = \boldsymbol{r}_u \boldsymbol{r}_v = vR^2\left(1-2u\right); \quad (43)$$

$$K = -\frac{R^2 T\left(1-u\right)v^2}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{3}{2}}\left(1-v^2\right)^{\frac{1}{2}}}; \quad (44)$$

$$M = \frac{R^2 T\left(1-u\right)v}{\sqrt{A^2B^2-F^2}\left(1-u^2\right)^{\frac{1}{2}}\left(1-v^2\right)^{\frac{1}{2}}}; \quad (45)$$

$$N = -\frac{R^2 T\left(1-u\right)\left(1-u^2\right)^{\frac{1}{2}}}{\sqrt{A^2B^2-F^2}\left(1-v^2\right)^{\frac{3}{2}}}; \quad (46)$$

$$K = \frac{R^4 T^2 (1-u)^2 v^4}{\left(A^2 B^2 - F^2\right)^2 \left(1-u^2\right)\left(1-v^2\right)^2} > 0. \quad (47)$$

By comparing equations (34)–(40) and (41)–(47), it can be observed that the surfaces presented in Figures 2, *b* and 2, *c* have the same values of the coefficients of the second fundamental form (L, M, N), only with the opposite signs, and the same positive Gaussian curvature ($K > 0$).

The geometry of the remaining two groups of three surfaces (Figures 3 and 4) can be investigated in a similar manner.

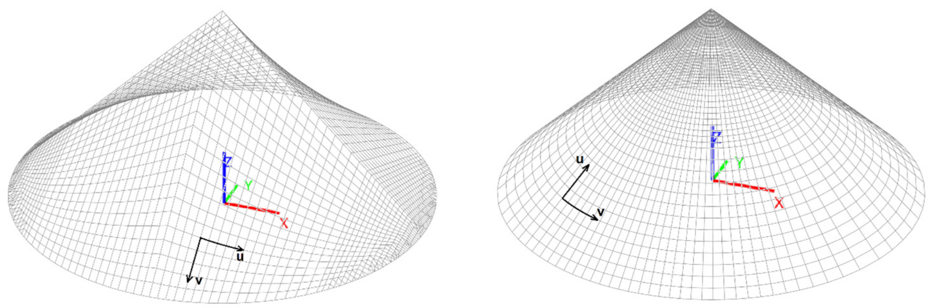
3.2. Static Analysis

The shells with the middle surfaces shown in Figure 1 are subjected to a uniformly distributed load $q = 1$ kN/m^2. The load acts in the opposite direction to the fixed axis Oz.

It is assumed that $T = R = 5$ m, constant shell thickness $h = 7$ cm, elastic modulus of the shell material $E_b = 32500$ MPa and Poisson's ratio $v = 0.17$. The shell is fixed at the base along the contour $z = 0$.

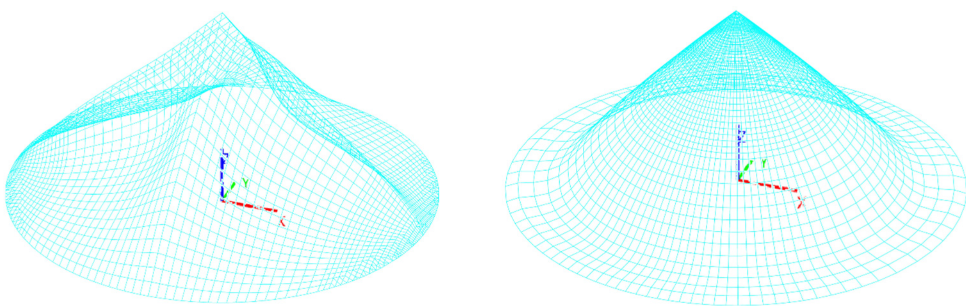
It was previously established that the surfaces in Figure 1, *a* and 1b are identical, despite being constructed differently by the process of moving the straight generators within the geometric frame. Thus, the static analysis is performed for two cases of the middle surface: cylindroid (Figure 1, *a*) and cone (Figure 1, *c*). The finite element models are developed in SCAD v21 software for the two cases of shells and are depicted in Figure 5, including the directions of curvilinear coordinates u and v. The geometry of the models is defined by parametric equations (10) and (12) respectively. The meshes of FE-models consist of plane shell elements.

Figure 6 shows the exaggerated deformed shapes of the analyzed shells under the applied vertical load.



a *b*

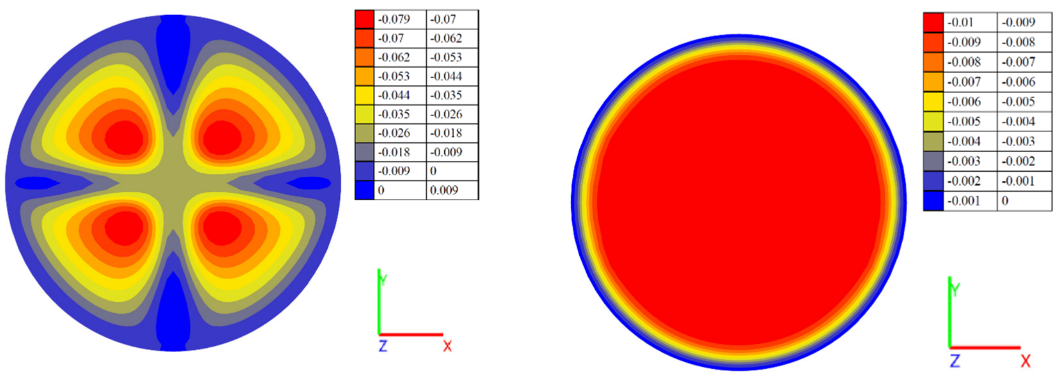
Figure 5. Finite element model:
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



a *b*

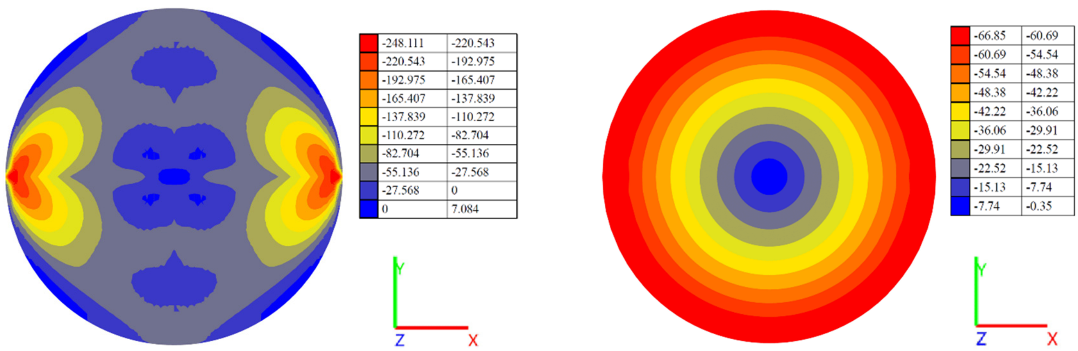
Figure 6. Deformed shape:
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.

The left-hand sides (a) of Figures 7–12 demonstrate the computed strength parameters of the shell with the middle surface shown in Figure 1, *a*. Correspondingly, the right-hand sides (b) of Figures 7–12 show the computed stress-strain state parameters of the shell with the middle surface shown in Figure 1, *c*. Vertical displacements (Figure 7) are positive in the upwards direction. Normal stresses N_u and N_v (Figures 8–9) are directed along coordinate lines *u* and *v* respectively; positive values of normal stress indicate tension. M_u and M_v (Figures 10–11) represent bending moments, which act in the sections orthogonal to coordinate lines *u* and *v* respectively and are calculated as moment per unit length of these lines. Equivalent compressive stress (Figure 12) is computed as von Mises stress.



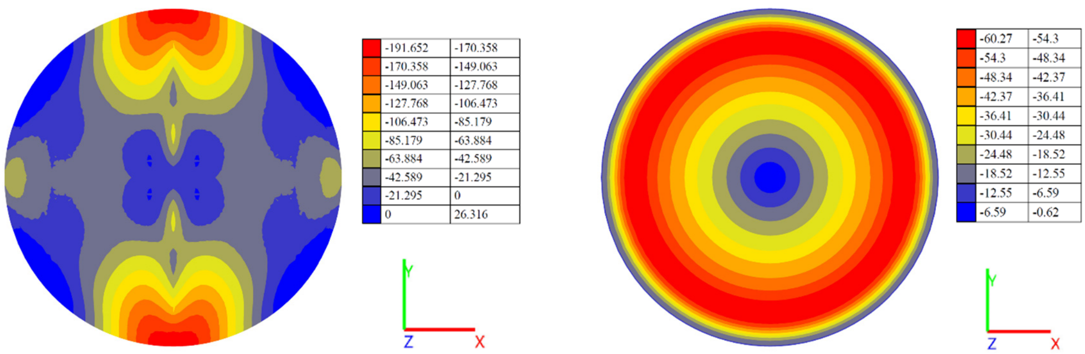
a *b*

Figure 7. Distribution of displacements along *z*-axis (mm):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



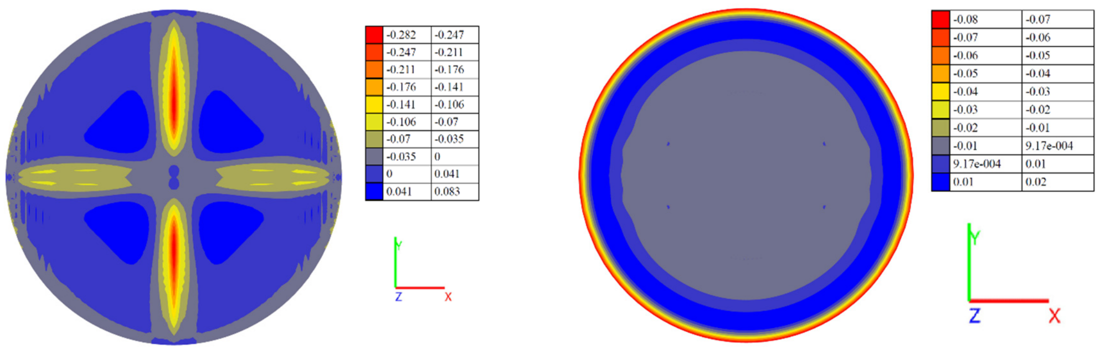
a *b*

Figure 8. Distribution of normal stresses N_u (kN/m^2):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



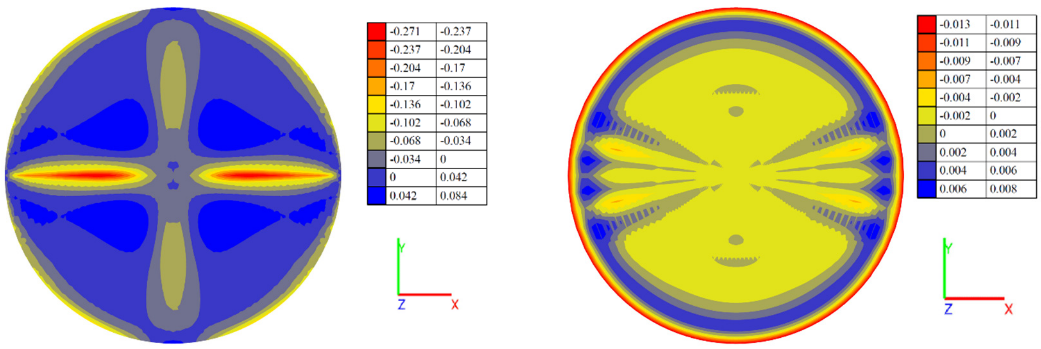
a *b*

Figure 9. Distribution of normal stresses N_v (kN/m^2):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



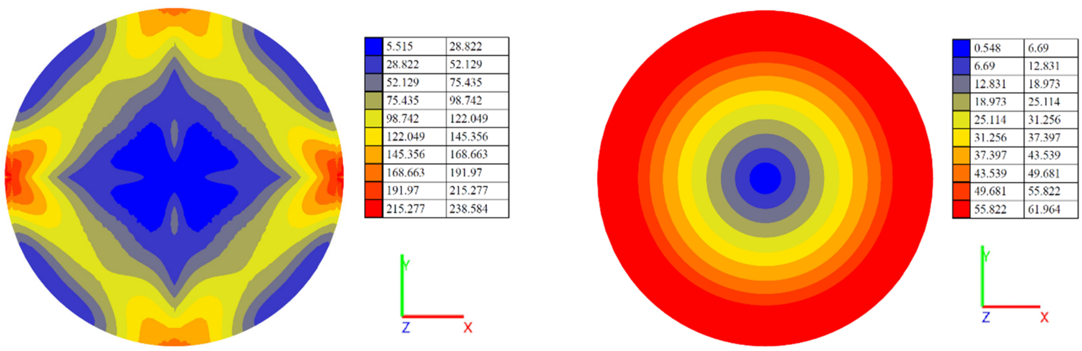
a *b*

Figure 10. Distribution of bending moments M_u (kN·m/m):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



a *b*

Figure 11. Distribution of bending moments M_v (kN·m/m):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.



a *b*

Figure 12. Distribution of equivalent von Mises compressive stress at the middle surface (kN/m^2):
a — shell with cylindroidal middle surface; *b* — shell with conical middle surface
S o u r c e: compiled by Valery Karnevich.

4. Discussion

This paper shows the construction of 4 groups of three surfaces, based on the previously obtained analytical and parametric equations of surfaces with the geometric frame of three superellipses. All 12 surfaces contain a circle as one of the plane curves of the frame. The presented surfaces are visualized graphically (see Figures 1–4) for better perception by architects and engineers. Using the methods of differential geometry, the detailed analysis of 6 algebraic middle surfaces of shells was performed for the first time. As a result of the geometric analysis, two surfaces in one group of three surfaces (see Figure 1) came out identical, and in the case of the other group (see Figure 2) all three surfaces are geometrically different. In the opinion of the authors, these surfaces can be taken as a basis for the shapes of civil and mechanical engineering structures. At least, these surfaces can be in the reserve of surfaces waiting for their implementation [26] within the framework of one of the modern architectural styles. The number of new forms of thin shells can be significantly expanded by taking fragments of different superellipses as the plane curves of the geometric frame [27].

The comparative static analysis of two thin shells (see Figure 5), the middle surfaces of which belong to one group of three surfaces with identical frames, was undertaken to provide insight into the structural differences. It is clear from the deformed shapes (Figure 6), displacement distributions (Figure 7), stress and moment distributions (Figures 8–12) that the behavior of the two shells with the same dimensions, material and applied static load differs drastically. All distributions of the strength factors in the circular cone are rotationally symmetric. In the cylindroid, these distributions are symmetric about the radial edges of the shell, which lie along the x and y axes. The maximum vertical displacement of the cylindroid is about 8 times higher than that of the cone (see Figure 7). The maximum stresses and moments (see Figures 8–12) are about 3–4 times greater in the cylindroid. The greatest normal stresses along curvilinear coordinates u, v in the cylindroid concentrate at the bottom of the radial edges (see Figure 8). The normal stresses in the circular cone are more linearly distributed and are larger near the circular base (see Figure 9). Moreover, the cylindroid shell has areas of tensile stress, whereas the cone exhibits pure compression. The maximum bending moments in the cylindroid concentrate along the radial edges (Figure 10). The bending moments in the circular cone are slightly greater near the base (see Figure 11), but are very small overall. It should be noted that the values of the strength factors along curvilinear coordinates u, v cannot be compared directly for the two shells, since their curvilinear coordinate grids are different (Figure 5). Hence, the distributions of von Mises compressive stress were obtained for the two shells (see Figure 12). These equivalent stress distributions roughly locate the dangerous areas of the shells.

5. Conclusion

Developments in mechanical and civil engineering require new more efficient solutions. One possible method of improving the load-bearing capacity of shell structures is modification of their geometry. This paper examines thin shells, the middle surfaces of which are defined by three plane curves of the geometric frame: a circle in the horizontal plane and two superellipses in the two vertical planes. It is shown that by varying the values of the exponents of the superellipses, it is possible to obtain a variety of outstanding shapes.

1. The method of defining the geometry of surfaces by using the curves of their frames allows to obtain a group of three surfaces — one for each curve of the frame. Further geometric analysis is required to determine the differences within the group. Some surfaces within a group may be identical, and in the other group some may share particular geometric characteristics, but be different overall, as confirmed by the findings in this paper.

2. It is shown that shells with geometrically different middle surfaces, but defined by the same frame, exhibit completely dissimilar behavior under static load. The presented static analysis of the two shells formed by the same main frame shows advantage of the circular cone over the cylindroid. However, a more detailed analysis is required for selecting the optimal shell, by testing for different dimensions, material properties and constraints. In some cases, material consumption, the simplest method of shell fabrication, or enclosed volume may be taken as the optimality criterion, which can be potentially satisfied by particular shell shapes demonstrated in this paper.

The analysis of available sources allowed to conclude that in the beginning of the 21st century the period of decline of interest for shell structures and thin-walled shells was over. This happened owing to the appearance of new structural materials, expansion of the inventory of analytical, point, spline and frame surfaces suitable for use as middle surfaces of shells, the development of more accurate calculation methods and computer software on their basis, and most importantly there was an increased demand for the creation of curvilinear large-span shell structures. These conclusions are confirmed by appearance of new architectural styles, directions, and style flows in the recent decades. Most architects and designers believe that curvilinear structures can become an alternative to traditional forms of buildings, while others, on the contrary, believe that the curvilinearity of buildings will quickly bore the inhabitants.

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