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Mathematical Model of Deformation of an Orthotropic Shell Under Blast Loading

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Abstract. This paper proposes a mathematical model of the deformation of a thin-walled shell structure under dynamic loading, specifically, blast loading. To account for the damping of the resulting vibrations, the author's previously proposed model was modified by adding a Rayleigh dissipation function to the Euler — Lagrange equations. The mathematical model also accounts for geometric nonlinearity, transverse shear, and material orthotropy. The software implementation performed in Maple. To demonstrate the applicability of the developed model, examples of calculations of shallow doubly curved shells under blast loading of varying intensities and with different damping coefficients in the Rayleigh dissipation function are provided.

Keywords: shells, blast loading, geometric nonlinearity, damping, Rayleigh dissipation function

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Математическая модель деформирования ортотропной оболочки при действии взрывной нагрузки

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Аннотация. Предложена математическая модель деформирования тонкостенной оболочечной конструкции при динамическом воздействии, в частности — взрывной нагрузки. Для учета затухания возникающих колебаний была модифицирована предложенная автором ранее модель путем добавления в уравнения Эйлера — Лагранжа функции диссипации Рэлея. Также математическая модель учитывает геометрическую нелинейность, поперечные сдвиги и ортотропию материала. Программная реализация выполнена в ПО Maple. Для демонстрации применимости разработанной модели приведены примеры расчетов пологих оболочек двоякой кривизны при действии взрывной нагрузки разной интенсивности и при выборе разного коэффициента демпфирования в функции диссипации Рэлея.

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Ключевые слова: оболочки, взрывная нагрузка, геометрическая нелинейность, демпфирование, функция диссипации Рэлея

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1. Introduction

Thin-walled shells deform in a significantly nonlinear manner, and special methods and algorithms must be developed to calculate them [1–5]. One important task in the study of thin-walled structures is the analysis of their deformation under dynamic loads.

Dynamic impacts on shells cause vibrations, and one of the important factors in performing calculations is taking damping into account [6; 7]. It is especially important to consider damping when the load is applied for a short time, as in the case of explosive impacts, and further behavior of the structure can only be accurately described by taking into account the attenuation of vibrations. In relation to the calculation of shell structures, explosive loads were considered in [8–13]. For example, Godoy and Ameijeiras [12] investigate the deformation of vertical steel oil storage tanks with flat roofs during an explosion close to the structure. The energy values are analyzed at changing peak pressure and buckling shape. In [9], calculations of spherical shells made of FGM are performed, and the calculation algorithm and results are presented in the form of dynamic responses, phase portraits, and natural frequency values.

Mechanics uses the variational principles of Lagrange and Hamilton, which solve time-dependent problems based on the law of conservation of energy and are therefore not applicable to dissipative systems [14]. A number of attempts to overcome this problem can be found in literature. One of the first papers devoted to accounting for dissipation in the Lagrangian formulation was published by Leech [14] in 1958. The Lagrangian function was extended by the Rayleigh dissipation function (1877). This formulation was called the modified Hamilton principle [15] (or extended [16]). Effectively, this approach allows the “classical” Lagrange equations to be extended to non-conservative (i.e., dissipative) systems [14; 17; 18].

The approach based on adding Rayleigh dissipation function to the Euler — Lagrange equations [14; 19–23] was also used in [25–27].

Thus, [24] investigates forced nonlinear vibration of double-curved shells in accordance with Koiter's theory. Various types of bifurcations are analyzed.

M. Amabili [26] investigates high-amplitude (geometrically nonlinear) vibration of circular cylindrical shells. The equations of motion are obtained using the energy approach that takes into account damping via the Rayleigh dissipative function. The results for four different nonlinear theories of thin shells are compared.

The study of E.P. Detina [6] is also worth noting, in which the Rayleigh dissipative function is modified, called the Kelvin — Voigt dissipative function. The proposed function is proportional to the square of the material strain rate, in contrast to the Rayleigh dissipative function, which is proportional to the square of the displacement velocity.

There is also an approach that takes into account energy dissipation by adding to the functional the ratio of damping energy, dissipated per vibration cycle, to the maximum deformation energy [27–31]. However, implementing this approach is computationally more complex.

The aim of this study is to extend the mathematical model and the algorithm previously developed by the author [32; 33] to the problems of calculating shell structures under blast load taking into account damping.

2. Theory and Methods

To obtain the principal relations of the mathematical model, the total energy functional is used (dissipation is not taken into account at this stage):

$$I = \int_{t_0}^{t_1} \left(E_k - E_s \right) dt, \tag{1}$$

where E_k is the kinetic energy; t is time; $E_s = E_p - \mathrm{A}$ is the difference between the potential energy of deformation of the system and the work of external forces

$$E_s = \frac{1}{2} \int_{a_1}^{a} \int_{0}^{b} \Bigg(N_x \varepsilon_x + N_y \varepsilon_y + \frac{1}{2}\left(N_{xy} + N_{yx} \right) \gamma_{xy} + M_x \chi_1 + M_y \chi_2 + \left(M_{xy} + M_{yx} \right) \chi_{12} +$$

$$+ Q_x \left(\Psi_x - \theta_1 \right) + Q_y \left(\Psi_y - \theta_2 \right) - 2 \left(P_x U + P_y V + qW \right) \Bigg) AB dx dy. \tag{2}$$

Geometric relations taking into account nonlinearity will take the following form:

$$\varepsilon_x = \frac{1}{A}\frac{\partial U}{\partial x} + \frac{1}{AB} V \frac{\partial A}{\partial y} - k_x W + \frac{1}{2}\theta_1^2;$$

$$\varepsilon_y = \frac{1}{B}\frac{\partial V}{\partial y} + \frac{1}{AB} U \frac{\partial B}{\partial x} - k_y W + \frac{1}{2}\theta_2^2;$$

$$\gamma_{xy} = \frac{1}{A}\frac{\partial V}{\partial x} + \frac{1}{B}\frac{\partial U}{\partial y} - \frac{1}{AB} U \frac{\partial A}{\partial y} - \frac{1}{AB} V \frac{\partial B}{\partial x} + \theta_1 \theta_2; \tag{3}$$

$$\theta_1 = -\left(\frac{1}{A}\frac{\partial W}{\partial x} + k_x U \right), \quad \theta_2 = -\left(\frac{1}{B}\frac{\partial W}{\partial y} + k_y V \right), \quad k_x = \frac{1}{R_1}, \quad k_y = \frac{1}{R_2}.$$

Curvatures χ_1, χ_2 and twist χ_{12} functions for this model become the following:

$$\chi_1 = \frac{1}{A}\frac{\partial \Psi_x}{\partial x} + \frac{1}{AB}\frac{\partial A}{\partial y}\Psi_y, \quad \chi_2 = \frac{1}{B}\frac{\partial \Psi_y}{\partial y} + \frac{1}{AB}\frac{\partial B}{\partial x}\Psi_x;$$

$$\chi_{12} = \frac{1}{2}\left(\frac{1}{A}\frac{\partial \Psi_y}{\partial x} + \frac{1}{B}\frac{\partial \Psi_x}{\partial y} - \frac{1}{AB}\left(\frac{\partial A}{\partial y}\Psi_x + \frac{\partial B}{\partial x}\Psi_y \right) \right). \tag{4}$$

The geometry of the shell structure is defined by the Lamé parameters and the values of the principal radii of curvature.

Also, expressions for the forces and moments reduced to the mid-surface of the shell and per unit length of the cross-section are required for the use in functional (2):

$$N_x = \frac{E_1 h}{1 - \mu_{12}\mu_{21}} \left(\varepsilon_x + \mu_{21} \varepsilon_y \right);$$

$$N_y = \frac{E_2 h}{1 - \mu_{12}\mu_{21}} \left(\varepsilon_y + \mu_{12} \varepsilon_x \right);$$

$$N_{xy} = N_{yx} = G_{12} h \gamma_{xy};$$

$$M_x = \frac{E_1}{1 - \mu_{12}\mu_{21}} \left(\frac{h^3}{12} \right) \left(\chi_1 + \mu_{21} \chi_2 \right); \tag{5}$$

$$M_y = \frac{E_2}{1-\mu_{12}\mu_{21}}\left(\frac{h^3}{12}\right)(\chi_2 + \mu_{12}\chi_1);$$

$$M_{xy} = M_{yx} = 2G_{12}\left(\frac{h^3}{12}\right)\chi_{12};$$

$$Q_x = G_{13}kh(\Psi_x - \theta_1);$$

$$Q_y = G_{23}kh(\Psi_y - \theta_2),$$

where N_x, N_y, N_{xy}, N_{yx} are the normal forces along axes x, y and membrane shear forces in the xOy plane; M_x, M_y, M_{xy}, M_{yx} are the bending and twisting moments; Q_x, Q_y are the shear forces in the xOz and yOz planes; E_1, E_2 are the elasticity moduli; G_{12}, G_{13}, G_{23} are the shear moduli; μ_{12}, μ_{21} are the Poisson's ratios.

The proposed mathematical model is based on the hypotheses of the Timoshenko model (Reissner–Mindlin, FSDT) and allows for the consideration of rotational inertia and transverse shear. Then the kinetic energy [32; 33]

$$E_k = \frac{\rho}{2}\int_{a_1}^{a}\int_{0}^{b}\int_{-h/2}^{h/2}\left(\left(\frac{\partial U^z}{\partial t}\right)^2 + \left(\frac{\partial V^z}{\partial t}\right)^2 + \left(\frac{\partial W^z}{\partial t}\right)^2\right)ABdxdydz, \tag{6}$$

$$U^z = U + z\Psi_x, \quad V^z = V + z\Psi_y, \quad W^z = W.$$

By evaluating the integral with respect to variable z in (6), one obtains

$$E_k = \frac{\rho}{2}\int_{a_1}^{a}\int_{0}^{b}\left(h\left(\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial t}\right)^2\right) + \frac{h^3}{12}\left(\left(\frac{\partial \Psi_x}{\partial t}\right)^2 + \left(\frac{\partial \Psi_y}{\partial t}\right)^2\right)\right)ABdxdy. \tag{7}$$

The approximating functions (in accordance with the L.V. Kantorovich's method) are substituted into functional (1). After evaluating the integrals with respect to variables x and y in terms of known functions, functional I represents a one-dimensional functional in terms of functions $U_{ij}(t) - \Psi_{yij}(t)$. Next, the well-known Euler — Lagrange equation [32; 33] is used:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{X}_j(t)} + \frac{\partial E_s}{\partial X_j(t)} = 0, \quad j = 1, 2, ..., 5N, \tag{8}$$

where $X(t) = \left(U_{ij}(t), V_{ij}(t), W_{ij}(t), \Psi_{xij}(t), \Psi_{yij}(t)\right)^T$, $i, j = 1,.., \sqrt{N}$, and the dot represents the derivative with respect to time.

Next, the Kantorovich method and the Rosenbrock method (for numerical solution of rigid ODE systems) are used to perform the calculations. The Kantorovich method is used to reduce a multidimensional functional to a one-dimensional one. For this, the unknown displacement functions and deflection angles are represented as follows [33]:

$$U(x,y,t) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} U_{kl}(t)X_1^k Y_1^l, \; V(x,y,t) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} V_{kl}(t)X_2^k Y_2^l,$$

$$W(x,y,t)=\sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}W_{kl}(t)X_3^kY_3^l, \tag{9}$$

$$\Psi_x(x,y,t)=\sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\Psi_{xkl}(t)X_4^kY_4^l,\quad \Psi_y(x,y,t)=\sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}}\Psi_{ykl}(t)X_5^kY_5^l,$$

where $U_{kl}-\Psi_{ykl}$ are the unknown functions of *t*; $X_1^k,...,X_5^k$, $Y_1^l,...,Y_5^l$ are the known approximation functions.

The Euler — Lagrange equations (8) are supplemented with a term that takes into account damping based on the Rayleigh dissipation function. In well-known studies, the Rayleigh dissipation function written for the model of structural deformation does not take into account transverse shear (Kirchhoff — Love, Koiter, CSDT models) and the membrane thickness

$$F=\frac{c}{2}\int_{a_1}^{a}\int_{0}^{b}\left(\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2+\left(\frac{\partial W}{\partial t}\right)^2\right)ABdxdy. \tag{10}$$

At the same time, how exactly coefficient *c* is defined, as well as its dimension and order, depends on accounting for the membrane thickness.

In this study, similar to the expression for kinetic energy, the Rayleigh dissipation function for the Timoshenko — Reissner model is written:

$$F=\frac{c}{2}\int_{a_1}^{a}\int_{0}^{b}\int_{-h/2}^{h/2}\left(\left(\frac{\partial U^z}{\partial t}\right)^2+\left(\frac{\partial V^z}{\partial t}\right)^2+\left(\frac{\partial W^z}{\partial t}\right)^2\right)ABdxdydz. \tag{11}$$

After integrating (11) with respect to variable *z*, one obtains

$$F=\frac{c}{2}\int_{a_1}^{a}\int_{0}^{b}\left(h\left(\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2+\left(\frac{\partial W}{\partial t}\right)^2\right)+\frac{h^3}{12}\left(\left(\frac{\partial \Psi_x}{\partial t}\right)^2+\left(\frac{\partial \Psi_y}{\partial t}\right)^2\right)\right)ABdxdy. \tag{12}$$

Now, a term containing the Rayleigh dissipation function (taking into account the proposed refinements) is added to the Euler — Lagrange equation, as it is done, for example, in [19; 24; 26]

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot X_j(t)}+\frac{\partial E_s}{\partial X_j(t)}-\frac{\partial F}{\partial \dot X_j(t)}=0,\quad j=1,2,...,5N. \tag{13}$$

System of equations (13) is complemented with initial conditions at $t = 0$

$$U_{ij}=V_{ij}=W_{ij}=\Psi_{xij}=\Psi_{yij}=0,\ \dot U_{ij}=\dot V_{ij}=\dot W_{ij}=\dot\Psi_{xij}=\dot\Psi_{yij}=0,\quad i,j=1,2,..,\sqrt{N}, \tag{14}$$

or

$$X_j=0,\quad \dot X_j=0,\quad j=1,2,..,5N.$$

The system of differential equations (13), (14) is further solved using one of the numerical methods; in this study, the Rosenbrock method is applied to the problem.

3. Analysis

To demonstrate the applicability of the above approach, a thin-walled shallow shell of double curvature with thickness $h = 0.09$ m, linear dimensions $a = b = 10.8$ m and principal curvature radii $R_1 = R_2 = 40.05$ m is analyzed. The material parameters correspond to fiberglass T10/UPE22-27 (elasticity moduli $E_1 = 0.294\times10^5$ MPa, $E_2 = 0.178\cdot10^5$ MPa, shear moduli $G_{12} = G_{13} = G_{23} = 0.0301\times10^5$ MPa, Poisson's ratio $\mu = 0.123$, density $\rho = 1800$ kg/m^3), the edges of the structure are simply supported. The load is explosive, directed perpendicular to the surface, and depends on time as follows: $q = q_0 \exp\left(-\frac{t}{t_0}\right) + q_{sv}$, $q_0 = 1$ MPa, $t_0 = 0.01$ s.

Self-weight is also taken into account. The analysis is performed with $N = 4$ in the Kantorovich method. Using a program developed by the author in Maple software, the dynamic response of the system at different coefficients $c = 100$ N·s/m^3 = 0.0001 MPa·s/m, $c = 0.001$ MPa·s/m, $c = 0.002$ MPa·s/m is shown. For comparison, the results without considering damping, when $c = 0$ N·s/m^3, are presented (Figure 1). Hereinafter in the figures it is shown that the curve with a larger amplitude corresponds to the central part of the structure ($x = a / 2$, $y = b / 2$), while the one with a smaller amplitude corresponds to the quarter ($x = a / 4$, $y = b / 4$). Figure 2 shows the same data for $q_0 = 10$ MPa.

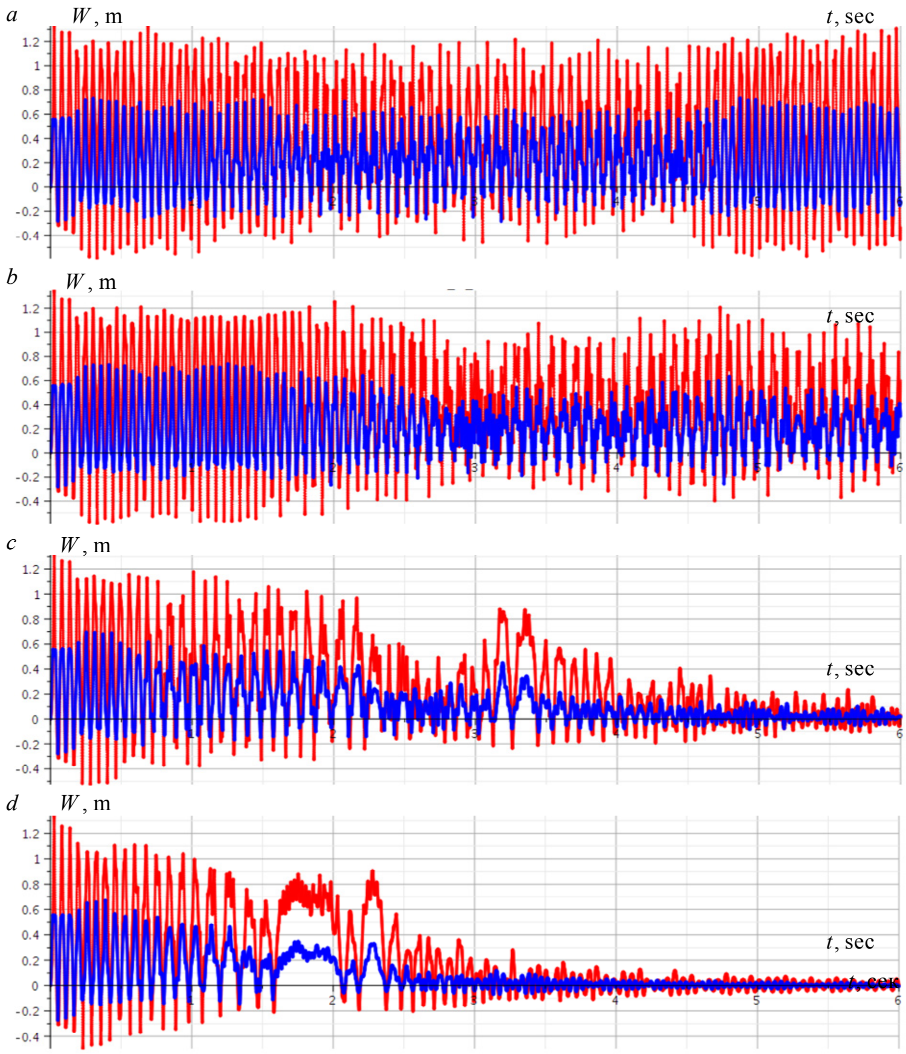

Figure 1. Dynamic response under blast loading ($q_0 = 1$ MPa):
a — $c = 0$ MPa·s / m; *b* — $c = 0.0001$ MPa·s / m; *c* — $c = 0.001$ MPa·s / m; *d* — $c = 0.002$ MPa·s / m
Source: made by A.A. Semenov.

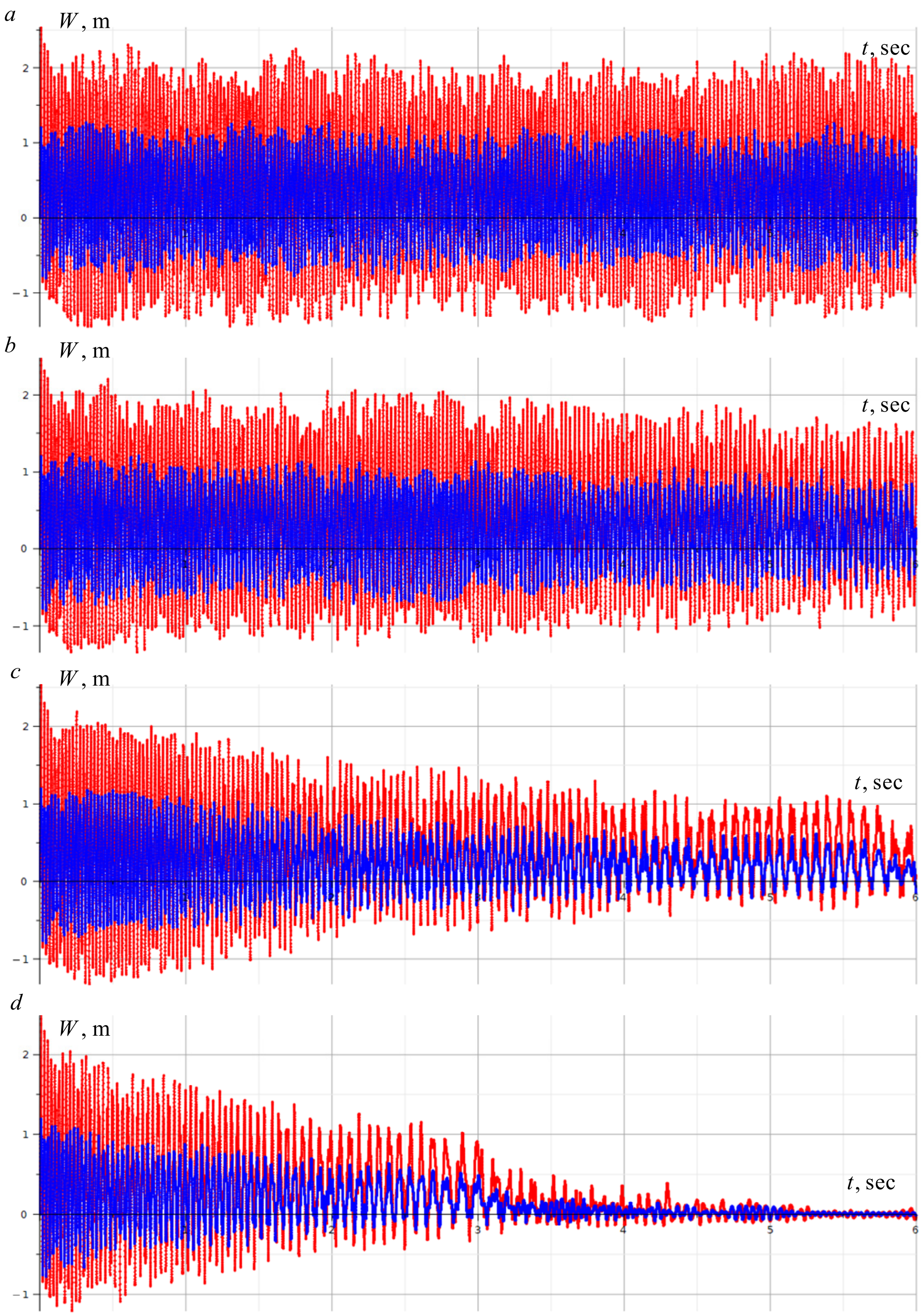

Figure 2. Dynamic response under blast loading (q_0 = 10 MPa):
a — *c* = 0 MPa·s / m; *b* — *c* = 0.0001 MPa·s / m; *c* — *c* = 0.001 MPa·s / m; *d* — *c* = 0.002 MPa·s / m
Source: made by A.A. Semenov.

It is evident that with a higher value of coefficient *c*, the damping of vibration occurs more rapidly. The search and analysis of its possible values close to the real data for the materials under consideration will be the subject of further research.

To assess how destructive the impact of an explosive load is, the graphs of normal stresses are also constructed for $q_0 = 1$ MPa and $c = 0.001$ MPa·s/m (Figure 3), and further — for $q_0 = 10$ MPa and $c = 0.001$ MPa·s/m (Figure 4). It can be seen from the graphs that at $q_0 = 10$ MPa the values of stress exceed the ultimate stress values for this material by several times, and at $q_0 = 1$ MPa they are close to the ultimate stress values, and at certain moments they surpass them.

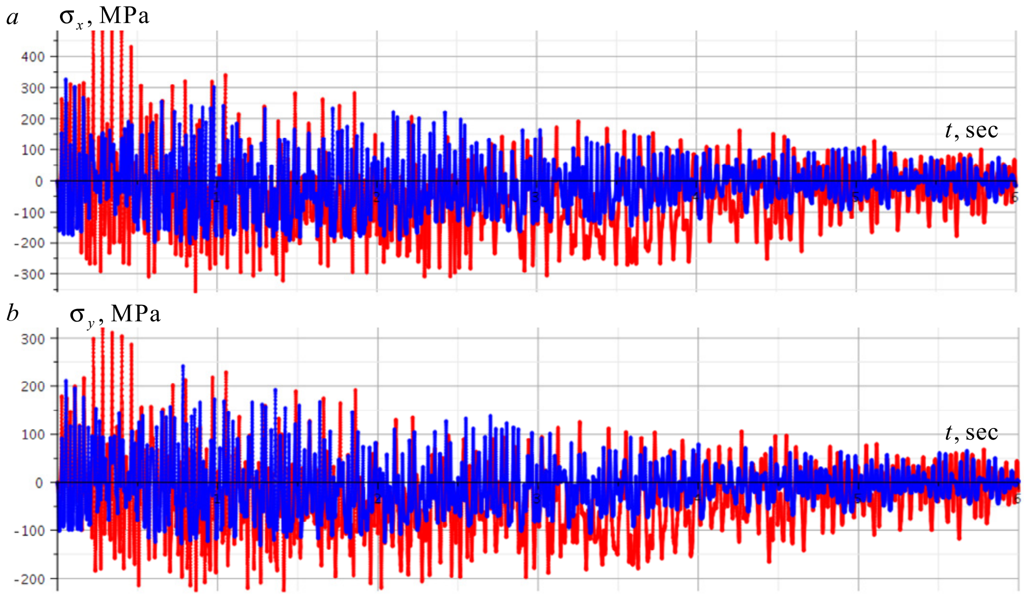

Figure 3. Normal stress values under blast loading ($q_0 = 1$ MPa), $c = 0.001$ MPa·s / m

Source: made by A.A. Semenov.

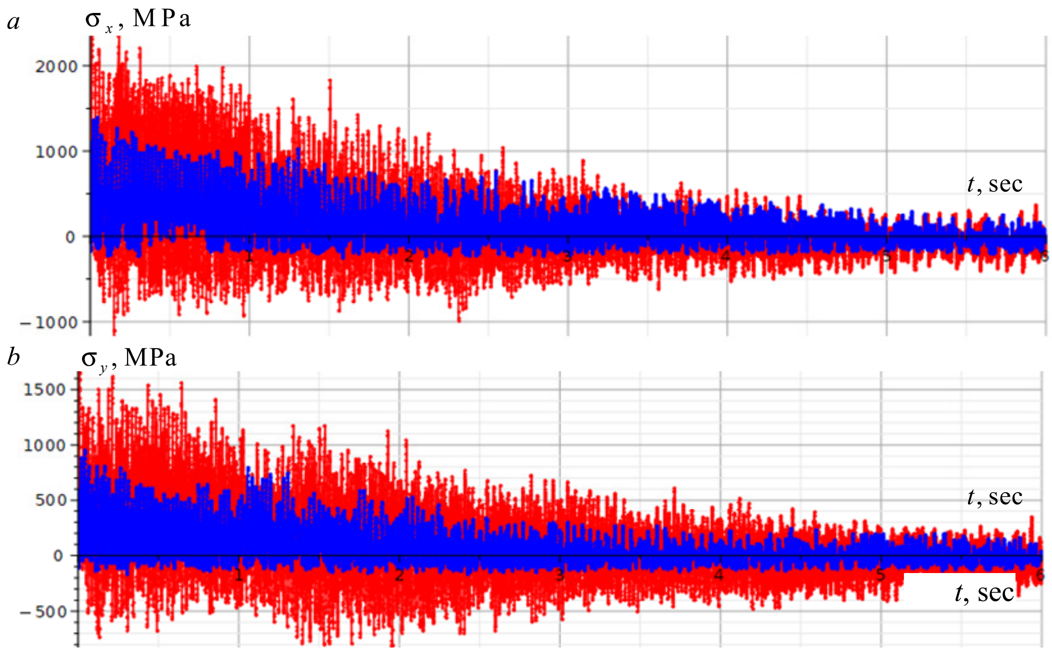

Figure 4. Normal stress values under blast loading ($q_0 = 10$ MPa), $c = 0.001$ MPa·s / m

Source: made by A.A. Semenov.

4. Conclusion

Computer modeling technologies allow to study thin-walled structures taking into account nonlinear effects. The proposed mathematical model using the Rayleigh dissipation function allows to extend the applicability of the models and calculation algorithms previously developed by the author to a wider class of problems. This includes simulating the dynamic response of a structure to an explosive load when the load application time is short and the vibration process involves damping. The data obtained on the stress values during vibrations are also of interest, as they may exceed the ultimate values.

Thus, a new mathematical model of the deformation of an orthotropic shell under the action of an explosive load has been obtained.

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