2021. 17(6). 617–627

СТРОИТЕЛЬНАЯ МЕХАНИКА ИНЖЕНЕРНЫХ КОНСТРУКЦИЙ И СООРУЖЕНИЙ

STRUCTURAL MECHANICS OF ENGINEERING CONSTRUCTIONS AND BUILDINGS

ISSN 1815-5235 (Print), 2587-8700 (Online)

HTTP://JOURNALS.RUDN.RU/STRUCTURAL-MECHANICS

DOI 10.22363/1815-5235-2021-17-6-617-627
UDC 69.04

RESEARCH ARTICLE / НАУЧНАЯ СТАТЬЯ

Investigation of the accuracy and convergence of the results of thin shells analysis using the PRINS program

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Article history
Received: August 1, 2021
Revised: October 18, 2021
Accepted: October 30, 2021

For citation
Agapov V.P., Markovich A.S. Investigation of the accuracy and convergence of the results of thin shells analysis using the PRINS program. *Structural Mechanics of Engineering Constructions and Buildings.* 2021;17(6): 617–627. http://doi.org/10.22363/1815-5235-2021-17-6-617-627

Abstract. The theoretical foundations of compatible finite elements construction for static and dynamic analysis of single-layer and multilayer shells are discussed. These finite elements are implemented in the PRINS computer program. The paper presents verification tests to investigate the accuracy and convergence of the results of calculating various shells using these finite elements. Shell structures are widely used in various fields of technology – construction, mechanical engineering, aircraft construction, shipbuilding, etc. Specialists on the design and calculation of such structures need a reliable and accessible tool for the practical problems solving. Computer program PRINS can be one of such tools. It can be effectively used by engineers of design and scientific organizations to solve a wide class of engineering problems related to the calculations of shell structures. The paper describes the finite elements of the shells, implemented in the PRINS program. The results of verification calculations are presented, which confirm the high accuracy of this program.

Keywords: finite element method, PRINS program, calculation methods, shells, multilayer plates, multilayer shells, layered structures, mechanics of deformable bodies

Исследование точности и сходимости результатов расчета тонких оболочек с помощью программы ПРИНС

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История статьи
Поступила в редакцию: 1 августа 2021 г.
Доработана: 18 октября 2021 г.
Принята к публикации: 30 октября 2021 г.

Аннотация. Приводятся теоретические основы построения совместных конечных элементов для статического и динамического расчета как однослойных, так и многослойных оболочек. Данные конечные элементы реализованы в вычислительном комплексе ПРИНС. Представлены верифика-

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ционные тесты, на основании которых выполнено исследование точности и сходимости результатов расчета различных оболочек с использованием этих конечных элементов. Оболочечные конструкции находят широкое применение в различных областях техники – строительстве, машиностроении, самолетостроении, судостроении и т. д. Специалисты по проектированию и расчету таких конструкций нуждаются в надежном и доступном инструменте для решения практических задач. Вычислительный комплекс ПРИНС может быть одним из них. Описываются конечные элементы оболочек, реализованные в вычислительном комплексе ПРИНС. Получены результаты верификационных тестов, подтверждающие высокую точность и сходимость этих конечных элементов. Вычислительный комплекс ПРИНС может быть эффективно использован инженерами проектных и научных организаций для решения широкого класса инженерных задач, связанных с расчетами оболочечных конструкций.

Ключевые слова: метод конечных элементов, вычислительный комплекс ПРИНС, методы расчеты оболочек, многослойные пластины, многослойные оболочки, слоистые конструкции, механика деформируемых тел

Для цитирования
Агапов В.П., Маркович А.С. Исследование точности и сходимости результатов расчета тонких оболочек с помощью программы ПРИНС // Строительная механика инженерных конструкций и сооружений. 2021. Т. 17. № 6. С. 617–627. http://doi.org/10.22363/1815-5235-2021-17-6-671-627

Introduction

Shell structures are widely used in various fields of technology – construction engineering, machine-building, aircraft construction, shipbuilding and so on. Fundamental questions of the shell analysis theory have been studied in detail in the works of domestic and foreign authors [1–5]. In these works, equations were obtained that completely describe the stress-strain state of thin shells of arbitrary shape under arbitrary loading. However, these equations do not have a common analytical solution. Various authors have obtained particular solutions for shells of a relatively simple form. The search for such solutions is underway at the present time [6–9]. The most famous of them, tested by long-term practice of their use, are given in reference and educational literature [10; 11].

The finite element method, which appeared in 1956, became a universal tool for calculating shells of arbitrary shape [12]. The versatility of the method is provided by the fact that the shell surface is represented as a set of elements of a simple geometric shape, triangles and/or quadrangles, which can be both flat and curved. Attempts were made to construct a curvilinear finite element on the basis of the shell theory [13], but it was impossible to make such element universal. Therefore, at present, for the calculation of shells, either plane finite elements are used, built on the basis of the plate bending theory [14; 15], or curvilinear ones, built on the basis of the general theory of elasticity [16].

The finite element method has been implemented in various computer programs. Those, who have been thoroughly verified, use the confidence of calculators and designers. The subject of research in this article is created by Professor V.P. Agapov computer program PRINS, the development of which is carried out by Professor V.P. Agapov together with his followers. The theory and practical implementation of shell finite elements used in this program is briefly described, and numerous examples of calculation of shells of various shapes are given.

Method

In the PRINS program, plane triangle and quadrilateral finite elements, implemented in single-layer and multilayer versions, are used for the calculation of thin shells. Since PRINS is intended for calculations of both linear and nonlinear deformable structures, the fundamental position in the development of finite elements was to obtain the simplest mathematical formulations. This circumstance is explained by the need to use rather dense finite element (FE) grids in the calculations, on the one hand, and the need for multiple recalculation of the stiffness characteristics of elements in the process of nonlinear problems solving, on the other. Therefore, the simplest triangle (Figure 1, *a*) with linear approximating functions for membrane displacements and a function in the form of an incomplete cubic polynomial for deflections (1) was taken as the basis for the shell elements constructing.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y, \;\; v = \alpha_4 + \alpha_5 x + \alpha_6 y,$$

$$w = q_1 + q_2 x + q_3 y + q_4 x^2 + q_5 xy + q_6 y^2 + q_7 x^3 + q_8 xy^2 + q_9 y^3. \tag{1}$$

A finite element with such displacement functions has well known to specialists disadvantages [14], the main of which are non-invariance with respect to the local coordinate system and the lack of compatibility of rotations of the normal with adjacent elements at the boundaries that do not coincide with the local x_m axis. However, an advanced triangular finite element can be built on its basis. Two such elements are used in the PRINS program – multilayered EL34 and single-layered EL36. When developing multilayer element, the technique proposed by Professor Agapov [17] was used, and in the development of single-layer element the method described in the work of Clough and Tocher [14], was realised. The main idea in both cases is to use three so-called subtriangles with approximating displacement functions, taken in the form (1), to obtain the characteristics of a given triangular FE. Methodology of professor Agapov is illustrated in Figure 2, and the Clough and Tocher method is shown in Figure 3.

In both cases, the characteristics of subtriangles are initially formed in their local axes, then converted to axes common for a given triangle, summed and averaged.

a *b* *c*

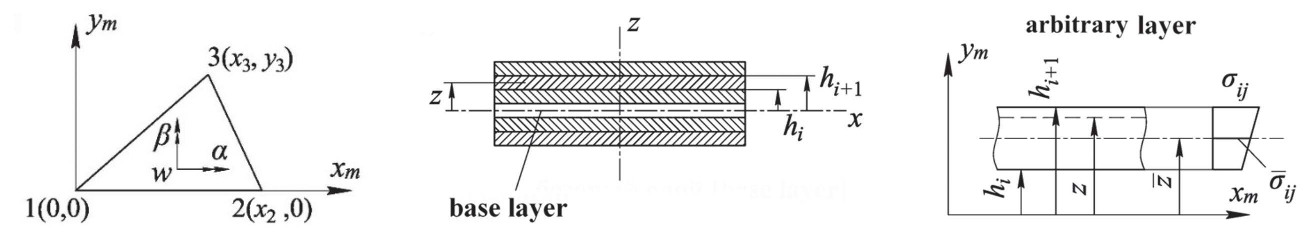

Figure 1. To create laminated shell FE:
a – triangular FE in local coordinates x_m–y_m; *b, c* – cross section

a *b*

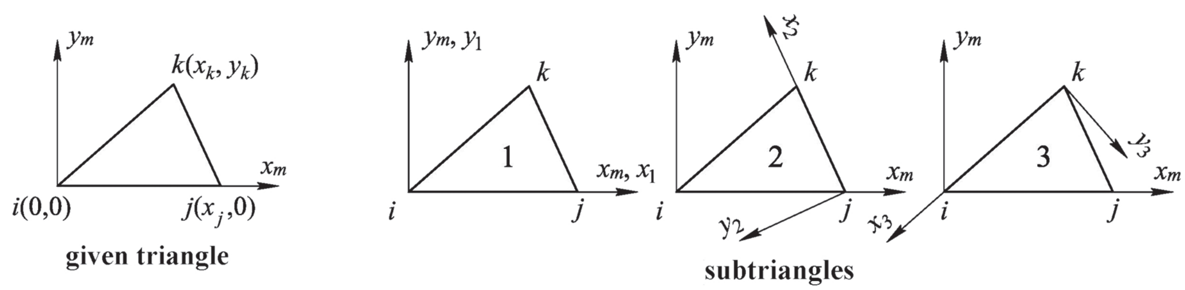

Figure 2. To calculate the bending stiffness matrix:
a – triangular FE with nodes *i*, *j*, *k*; *b* – subtriangles in local axes

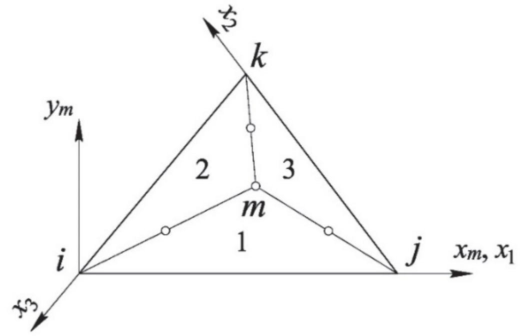

Figure 3. R.W. Clough & J.L. Tocher triangular finite element formation

As shown in [14; 17], the triangular elements thus improved have the property of invariance with respect to the coordinate axes and complete compatibility of displacements and rotations with adjacent elements at all boundaries.

The characteristics of a quadrangular FE are obtained by summing and averaging the characteristics of four triangles according to the scheme shown in Figure 4.

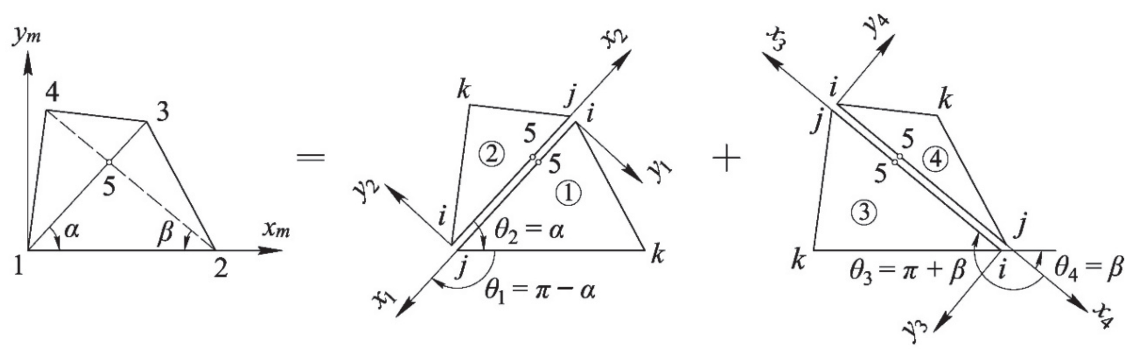

Figure 4. Forming a quadrangular finite element

A detailed description of the methods for obtaining all the characteristics of flat finite elements necessary for the calculation of shells can be found in [14; 15; 17]. The purpose of this work is a comparative analysis of shell FE implemented in PRINS program, as well as analysis of the accuracy and convergence of the results obtained with their help.

Results and discussion

To verify the above-described finite elements, we present a number of numerical calculations performed in PRINS program.

Single-layer shallow shell. We consider a shallow shell, the middle surface of which is an elliptical paraboloid (Figure 5, *a*) with the following initial data: $a = b = 10$ m, $h = 10$ cm, $f_1 = f_2 = 0.5$ m, $E = 3\times10^4$ MPa, $\nu = 0{,}2$, $q = 1$ kPa. The shell rests on transverse diaphragms that are rigid in its plane and flexible out of plane.

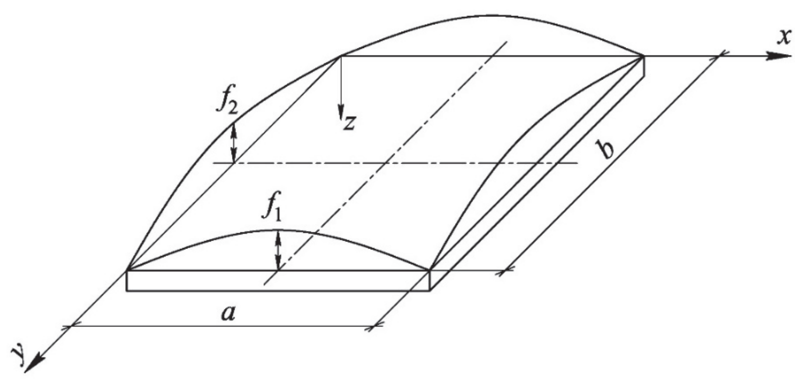

Figure 5. To the calculation of a shallow shell

The middle surface of the shell considered is described by equation:

$$z = f\left[\frac{f_1}{f}\left(2\frac{x}{a}-1\right)^2 + \frac{f_2}{f}\left(2\frac{y}{b}-1\right)^2 - 1\right]. \quad (2)$$

This surface is formed by moving a line $f_1(x)$ along a line $f_2(y)$.

The authors estimated the accuracy and analyzed the convergence of the calculation results obtained with using of triangular and quadrangular elements of a single-layer shell (type EL36). For these purposes, a total of twelve finite element schemes of the considered shell were built (Figure 6) with different mesh densities: 10×10, 14×14, 20×20, 30×30, 36×36, 40×40. An analytical solution to this problem is given in the manual [11].

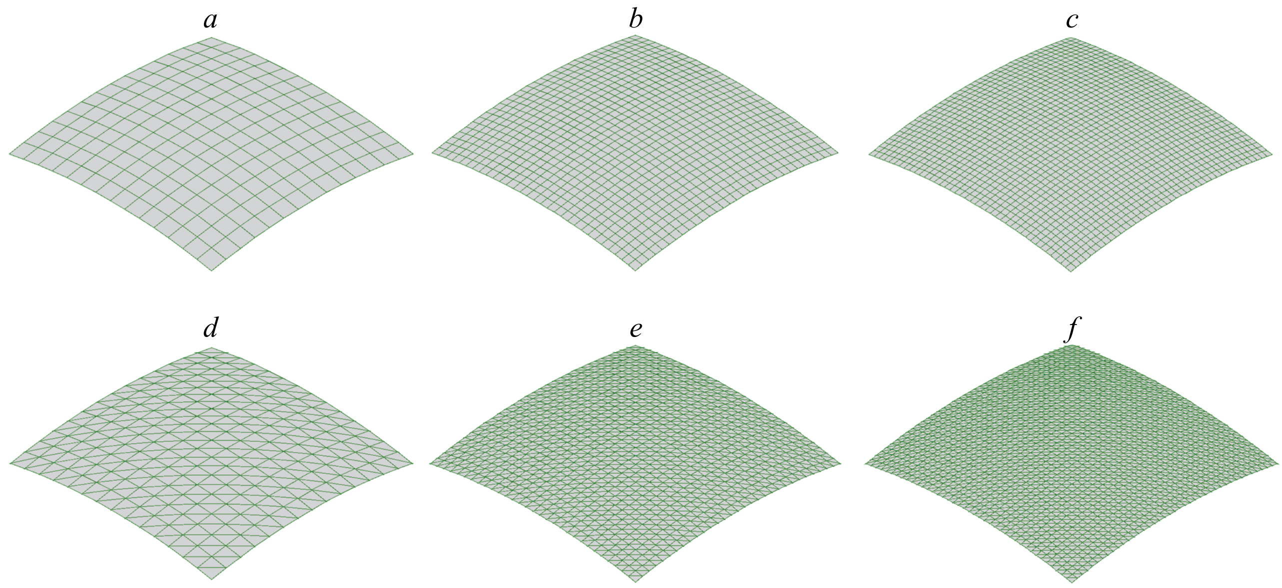

Figure 6. Finite element models of a shallow shell:
a, b, c – for calculating with rectangular FE; *d, e, f* – for calculating with triangular FE

Comparison of the results obtained in the PRINS program with the analytical calculation data was carried for the vertical displacement and for the total stresses at the upper (σ_u) and lower (σ_l) surfaces at the central point of the shell. Stresses were calculated by the formula

$$\sigma = \frac{N}{h} \pm \frac{6M}{h^2}, \tag{3}$$

where N and M are the lineal membrane force and the lineal bending moment in the shell section respectively.

The orientation of the shell surfaces is determined by the direction of the local z_m axis of finite elements, the positive direction of which at the central point coincides with the direction of the global z axis in Figure 5. The results of the numerical calculation of the shallow shell are presented in Table 1.

Table 1

Calculation results of the shell

FE mesh	Vertical displacement at the center point of the shell w, m			Stresses at the upper and lower surfaces at the center point of the shell		
				FEM solution		Analytical solution
	FEM solution	Analytical solution	Δ_w, %	$\sigma_{xx,u} = \sigma_{yy,u}$, kPa / $\sigma_{xx,l} = \sigma_{yy,l}$, kPa	Δ_σ, %	$\sigma_{xx,u} = \sigma_{yy,u}$, kPa / $\sigma_{xx,l} = \sigma_{yy,l}$, kPa
14×14 (3)	–0.000235	–0.000221	6.33	–146.24	0.58	–145.39/–113.6
				–122.01	7.9	
30×30 (3)	–0.00022		0.45	–145.8	0.28	
				–118.2	3.72	
14×14 (4)	–0.00023		4.07	–146.12	0.5	
				–121.88	7.29	
30×30 (4)	–0.000222		0.45	–145.82	0.29	
				–116.18	2.27	
40×40 (4)	–0.0002215		0.22	–145.6	0.15	
				–114.34	0.65	

The displacement and stress fields in the shell are shown in Figure 7.

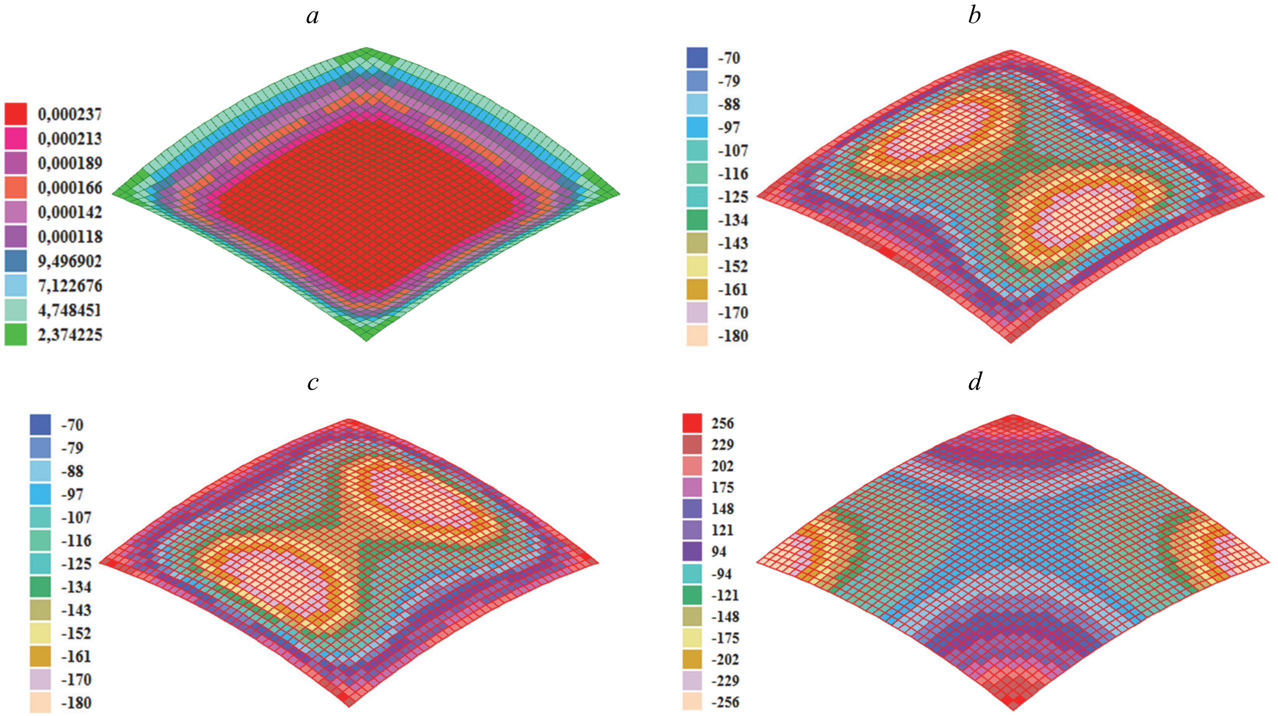

Figure 7. Calculating results of a shallow shell:
a – deformed shell scheme, total displacement fields, m; *b* – normal stress fields σ_{xx}, kPa; *c* – normal stress fields σ_{yy}, kPa ; *d* – shear stress fields τ_{xy}, kPa

As you can see from the Table 1, PRINS program provides an equally stable solution using both triangular and quadrangular FE. With a relatively coarse FE mesh, the calculation error is less than 6%. The displacement convergence graph is shown in Figure 8.

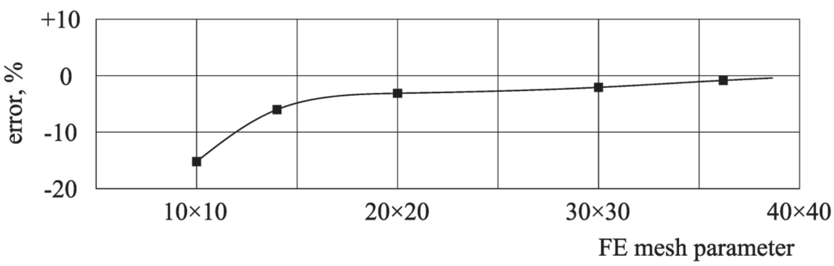

Figure 8. Convergence graph of calculation results for displacements

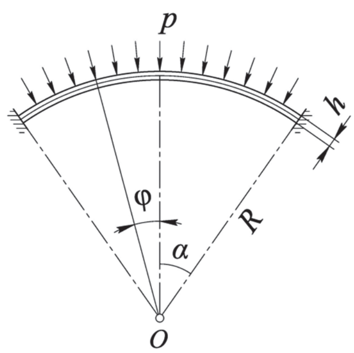

Figure 9. To the calculation of a spherical shell

Spherical shell under uniform pressure. The spherical shell clamped at the edges and loaded with the uniform load (Figure 9) is considered. The initial data are as follows: R = 2.28 m, α = 35°, h = 7.6 cm, $E = 3\times10^4$ MPa, $\nu = 0{,}167$.

An analytical solution of this problem by the Steuermann – Geckeler method is given in [11]. The shell verification calculation was performed by a shell finite element (type EL36) at different FE mesh densities: 8×32, 12×48, 16×64, 32×128 (Figure 10).

The results of the spherical shell analysis, obtained with the aid of PRINS program, are shown in Figure 11.

The graphs of the convergence of the calculation results for meridional bending moments and circumferential normal stresses are shown in Figures 12 and 13 respectively.

The data presented in these figures show the high accuracy of the finite elements used for calculations.

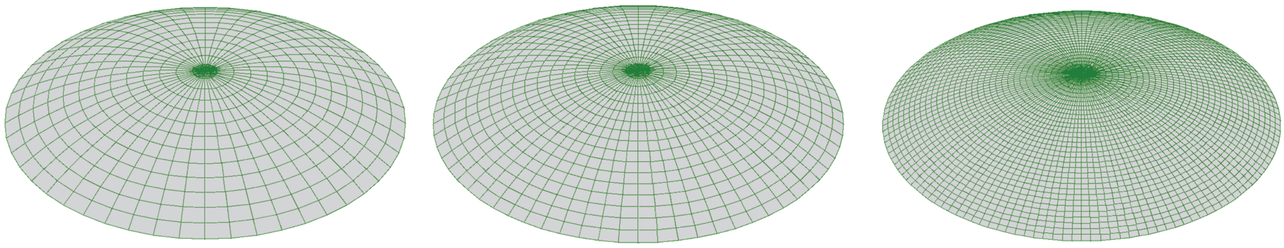

Figure 10. Finite element models of a spherical shell

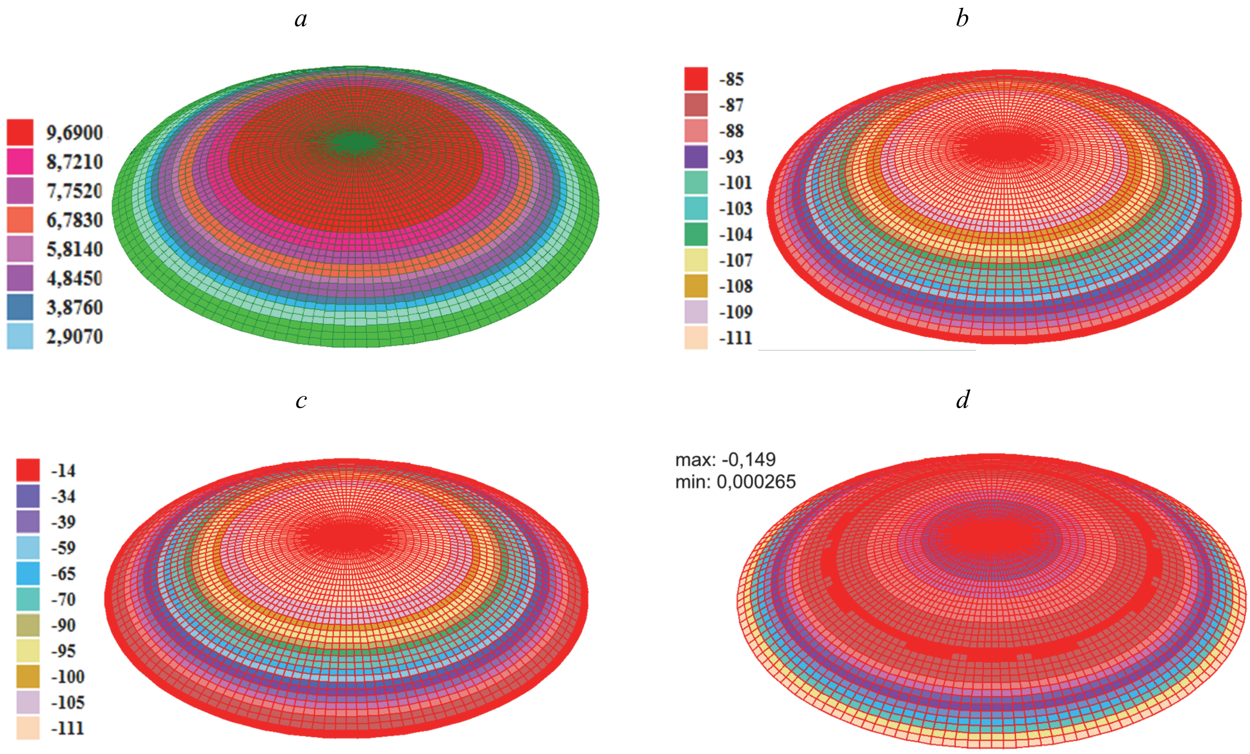

Figure 11. Calculating results of a spherical shell:
a – deformed shell scheme, total displacement fields $\times 10^6$, m; *b* – normal stress fields σ_{xx}, kPa;
c – normal stress fields σ_{yy}, kPa; *d* – bending moments fields M_x, kNm/m

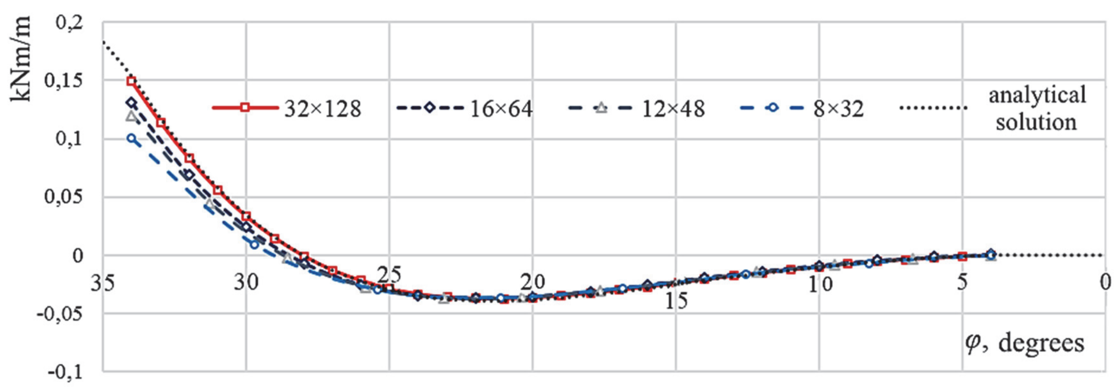

Figure 12. Convergence graph of the calculation results by the meridional bending moments M_x

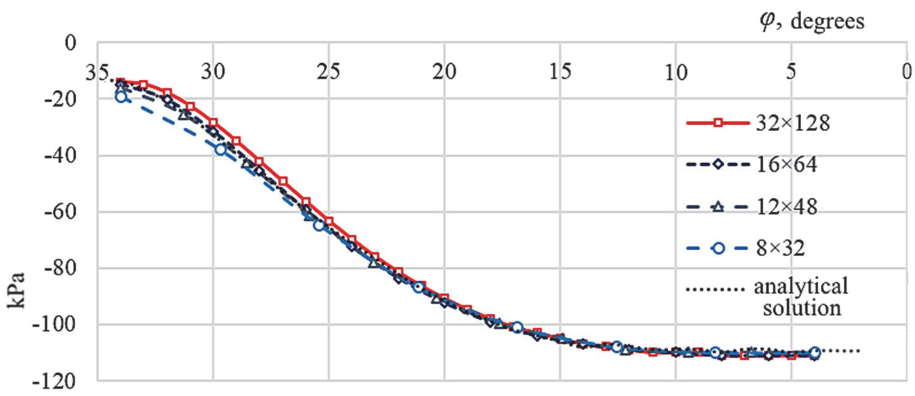

Figure 13. Convergence graph of calculation results for circumferential stresses σ_y

Tank consisting of a cylindrical part and a spherical dome. Let us consider a dome-shaped shell, turning into a cylindrical one, under the action of uniform pressure (Figure 14). The initial data are as follows: $R_m = R_t = 20$ m, $h = 50$ cm, $E = 3\times10^4$ MPa, $\nu = 0,2$. The tank is rigidly fixed in the base.

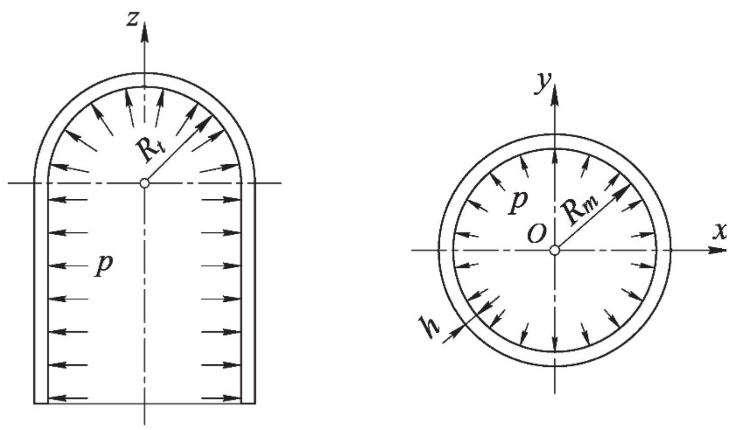

Figure 14. To the calculation of a tank

The finite element schemes of the reservoir were constructed using triangular and quadrangular shell elements (type EL36) and had the following parameters: 12×24, 18×36, 24×48, 30×60 (Figure 15).

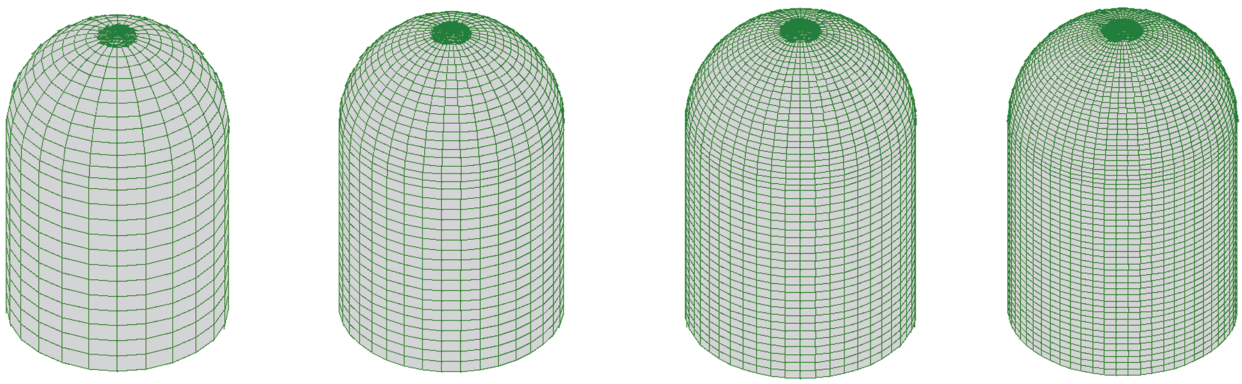

Figure 15. Tank finite element models

The stress state of the reservoir under consideration can be determined using the membrane theory of shells (the edge effect is not considered).

Normal stresses in the cylindrical part are equal:

$$\sigma_m = \frac{pR}{2h}, \quad \sigma_t = \frac{pR}{h}. \tag{3}$$

The stresses in a spherical dome are determined by the equation:

$$\sigma_t = \sigma_m = \frac{pR}{2h}. \tag{4}$$

The results of the reservoir analysis with the aid of PRINS program are presented in Table. 2.

Table 2

Tank calculating results

FE mesh	Cylindrical shell				Spherical shell			
	FEM solution		Analytical solution		FEM solution		Analytical solution	
	σ_m, kPa	σ_t, kPa	σ_m, kPa	σ_t, kPa	σ_m, kPa	σ_t, kPa	σ_m, kPa	σ_t, kPa
12×24	9900	19 800	10 000	20 000	9950	9990	10 000	10 000
18×36	9960	19 900			9960	10 000		
24×48	9980	20 000			9980	10 000		
30×60	9990	20 000			9990	10 000		

The displacement and stress fields in the tank are shown in Figure 16.

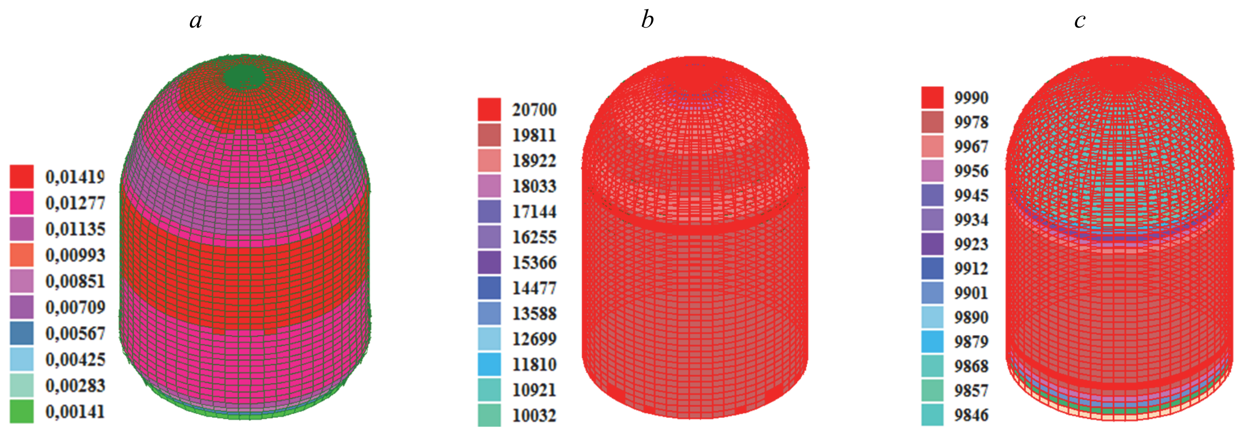

Figure 16. Tank calculating results:
a – deformed tank scheme, total displacement fields, m; *b* – normal stress fields in the meridional direction σ_m, kPa;
c – normal stress fields in the circumferential direction σ_t, kPa

As you can see from the Table 2, the results obtained using the PRINS practically coincide with the analytical solution according to the membrane theory.

Flat layered cylindrical panel. The calculation of a flat layered cylindrical panel rested on transverse diaphragms that are rigid in its plane and flexible out of plane (Figure 17, *a*) is presented. Panel dimensions: a_1 = 1 m, a_2 = 2 m, R_1 = 3 m. The cross-section of the panel consists of five layers symmetrically located relative to the middle surface (Figure 17, *b*).

The characteristics of the layers are as follows: h_1 = 0.5 cm, h_2 = 1.5 cm, h_3 = 1.6 cm, $E_1 = 7 \times 10^4$ MPa, $E_2 = 2.6 \times 10^4$ MPa, E_3 = 195 MPa, $\nu_1 = 0.3$, $\nu_2 = 0.13$, $\nu_3 = 0.4$, q = 35 kPa.

The calculations were carried out using a triangular multilayer FE (EL34). Four different finite element meshes were used: 6×12, 9×18, 12×24, 15×30 (Figure 18).

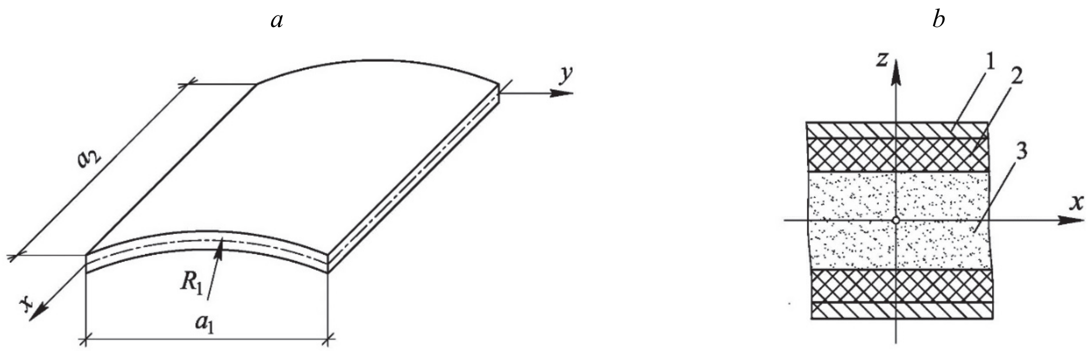

Figure 17. To the calculation of a laminated cylindrical panel:
a – general view; *b* – cross-section of the panel

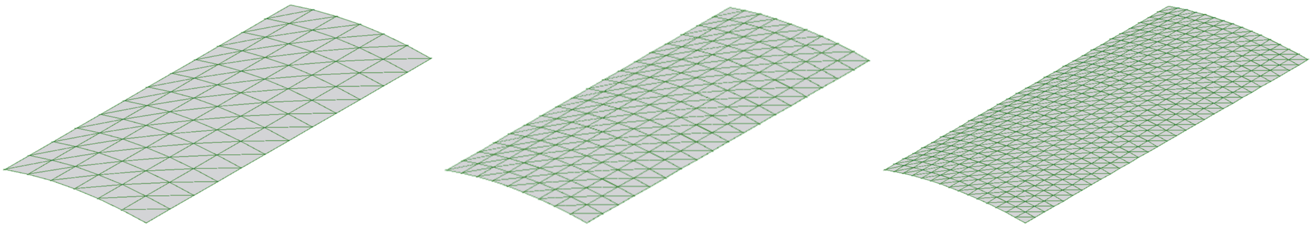

Figure 18. Finite element models for calculating a laminated panel

Table 3

Calculation results of laminated cylindrical panel

FE mesh	Vertical displacement at the center point of the shell *w*, m		Forces in the middle of the panel				
			FEM solution		Analytical solution		Error, Δ_w, %
	FEM solution	Analytical solution	M_y, kNm/m	N_y, kN/m	M_y, kNm/m	N_y, kN/m	
6×12	0.000459	0.000455	3.086	–15.8	3.05	–15.29	0.88
9×18	0.000458		3.08	–15.5			0.65
12×24	0.000456		3.075	–15.4			0.2

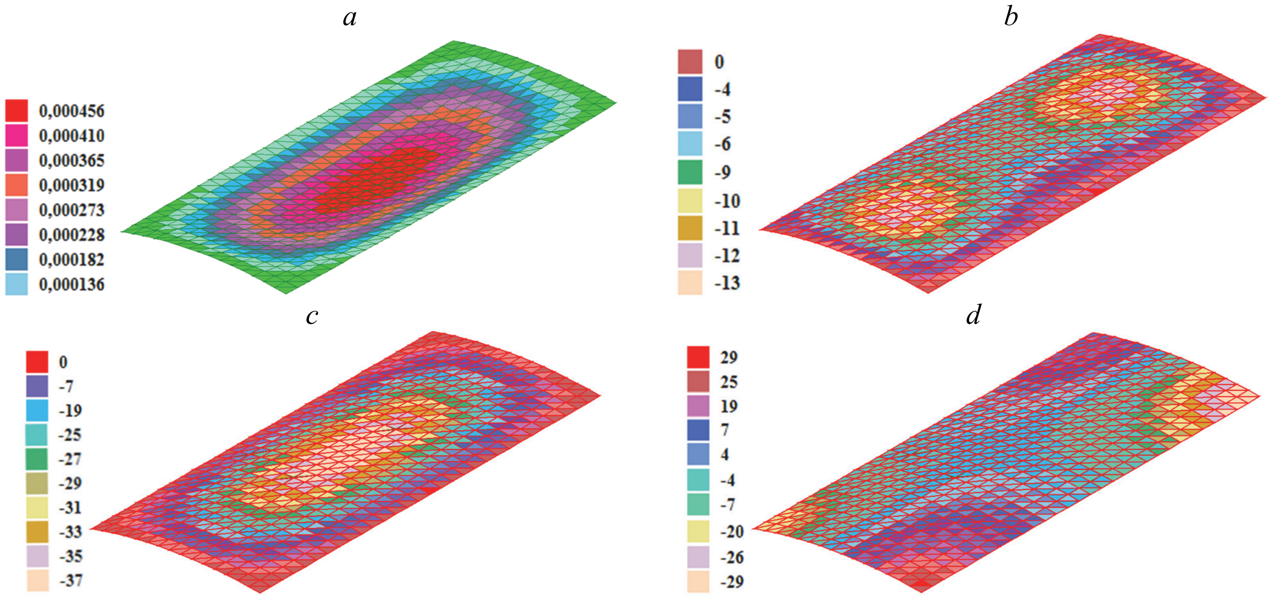

Figure 19. Calculating results of a laminated cylindrical panel:
a – deformed shell scheme, total displacement fields, m; *b* – normal force fields N_x, kN/m;
c – normal force fields N_y, kN/m; *d* – shear force fields N_{xy}, kN/m

An analytical calculation of the problem under consideration is given in the book [18].

The results of numerical calculations are presented in Table 3.

The displacement and stress fields in a laminated panel are shown in Figure 19.

The calculation error ranges from 3.5 to 0.2%, depending on the dimension of the FE mesh.

Conclusion

The principles of the shell finite elements constructing described in this article were implemented in PRINS program.

On the basis of numerous verification tests, it has been established that the finite elements (type EL36 and type EL34) used for single-layer and multilayer shell analysis have a fast convergence and have a sufficiently high accuracy. For rectangular planar shallow shells with side length *l*, the optimal size of the finite element that provides the required solution accuracy with significant savings in computing resources is $\left(\frac{1}{24} \div \frac{1}{36}\right) l$. For the calculation of cylindrical and spherical shells, the size of the finite element is recommended to be taken within $\left(\frac{1}{16} \div \frac{1}{24}\right) d$.

PRINS program can be effectively used by specialists from design and scientific organizations to solve a wide class of engineering problems.

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