Buckling analysis

THEORETICAL APPROACH FOR THE GEOMETRICALLY NONLINEAR BUCKLING ANALYSIS OF SINUSOIDAL VELAROIDAL SHELLS

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The main subjects of the article are theoretical strength and buckling analyses of a sinusoidal velaroidal shell subjected to its self-weight and uniformly distributed load in geometric nonlinearity. A short history of finite element applications to shell buckling is given. The finite element method is used in its matrix formulation. The elastic stiffness matrixes in the local coordinate system of the membrane element are defined in their general form. The out-of-plane geometric stiffness matrix for the plate along the same lines is derived.
KEY WORDS: sinusoidal velaroidal surfaces, nonlinear buckling analysis, linear elastic stiffness matrices, normal stiffness, geometric stiffness matrix, chain rule differentiation.

Velaroidal surface is a surface of translation on the flat rectangular plan with a generating curve of variable curvature [1],[2]. Thus, the surface is limited by four mutually orthogonal contour straight lines ($k_x = k_y = 0$) lying in the same plane.

A sinusoidal velaroid generates by two families of half waves of the sinusoids lying in mutually perpendicular planes and facing by convexities into the same side [3]. Each set of sinusoids has the identical period. Sinusoidal velaroid is limited by a flat rectangular contour.

The history of finite element applications to shell buckling is extensive going back to the work of Clough and Johnson (1968). The natural mode contribution of Argyris *et al.* (1977) was a major addition to shell theory. It was recently modified to include elastoplastic effects (Argyris *et al.* 2000). Horrigmoe and Bergan (1978) used the co-rotational method for nonlinear analysis and Bathe and Ho (1980), Hsiao (1987), Mohan and Kapania (1997), Peng and Crisfield (1992) improved element performance along those lines. The 3-D elasticity "degenerate" element of Ahmad *et al.* (1970) was followed by, among others, Bathe and Balourchi (1980), Hughes and Lui (1981), Dvorkin and Bathe (1984), and Buechter and Ramm (1992). In an excellent review, Ibrahimbegovic (1997) addresses the various approaches and the complex issues involved.

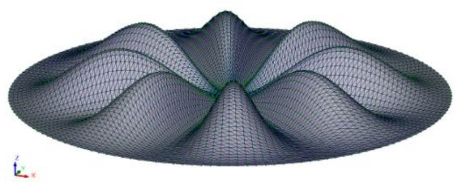

Fig.1. A sinusoidal velaroidal shell

Here the derivation of the geometric stiffness matrix is somewhat different but consistent with the approach used throughout this text. The linear equilibrium equations for a flat triangular shell element in its local coordinates system are first perturbed to yield the in-plane geometric stiffness matrix. Then out-of-plane considerations that involve the effect of rigid body rotations on member forces yield an out-of-plane geometric stiffness matrix. The shell element that was chosen for that purpose combines the constant stress triangle (CST) flat triangular membrane element (Zienkeiwicz (1977)) and of the discrete Kirchhoff theory (DKT) flat triangular plate element (Batoz *et al. (1980))*.

Let's consider a sinusoidal velaroidal shell [4],[5] with the inner radius $r_0 = 0$, the outer variables radii from *10m* to *20m* and the number of waves $n = 8$ (Fig. 1).

The Geometric Stiffness Matrix of Triangular Element Shells.

The local geometric stiffness matrix of the shell element is split into three distinct matrices:

$$[K_G^e]_{TOTAL}^{shell} = [K_G^e]_{IP}^{mem} + [K_G^e]_{IP}^{plate} + [K_G^e]_{OP}^{shell}, \tag{1}$$

where the first, second and third terms on the R.H.S. of Eq.1. represent the in-plane geometric stiffness matrix of the membrane, the in-plane geometric stiffness matrix of the plate and the out-of-plane geometric stiffness matrix of the shell element respectively. The total, 'tangential' stiffness matrix for use in nonlinear analysis will include, in addition, the linear elastic stiffness matrices of a plane stress triangular element (membrane) and that of a triangular plate element.

The geometrically nonlinear triangular shell element has eighteen local degrees of freedom (DOF's): 3 displacements and 3 rotations at each node. The membrane element contributes to nine displacement DOF's only. The basic three noded constant stress triangular flat element has only six local displacement DOF's that are shown in Figure 2. The out-of-plane contribution (the normal stiffness) of the membrane element to the basic local shell element is a displacement DOF in the direction normal to the plane of the element.

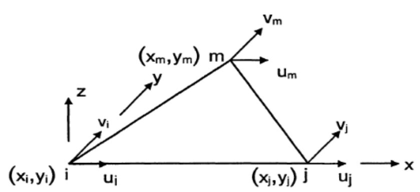

Fig. 2. Triangular membrane element

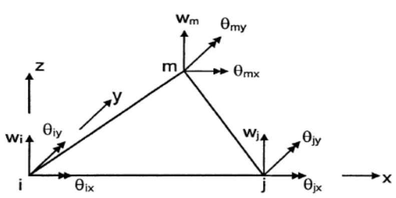

Fig.3. Triangular plate element

The flexural element contributes to eighteen local DOF's. The basic three noded plate triangular flat element has only nine DOF's that are shown in Figure 3. The in-plane contribution to the basic local element adds two displacement DOF's in the plane of the element. The out-of-plane contribution adds a rotation DOF in the direction normal to the plane of the element.

In-Plane Contribution of the Triangular Membrane Element.

The elastic stiffness matrix in the local coordinate system of the membrane element has the general form:

$$[K_E^e]^{mem} = \int_A B_{mem}^T D B_{mem}\, t\, dx\, dy. \tag{2}$$

The local in-plane contribution of the membrane element to the geometric stiffness matrix is the gradient of the nodal force vector:

$$[K_G^e]_{IP}^{mem} = \nabla F_{mem}^e. \tag{3}$$

In-Plane Contribution of the Triangular Plate Bending Element.

The general form of the elastic stiffness matrix of the DKT triangular plate element may be written as:

$$[K_E^e]^{plate} = 2A \int_0^1 \int_0^{1-\eta} B_{plate}^T \left(\frac{t^3}{12} D\right) B_{plate}\, d\xi\, d\eta, \tag{4}$$

where ξ and η are the usual area coordinates and B_{plate}, which appears explicitly in Batoz *et al.* (1980) as a function of η is slightly more complex than the B_{mem} of the membrane. Here again the local in-plane contribution of the plate element to the geometric stiffness matrix will be derived as the gradient for fixed M, of the element nodal force vector which is given as:

$$F^e_{plate} = \left(2A\int_0^1\int_0^{1-\eta}\frac{t^3}{12}B^T_{plate}DB_{plate}d\xi d\eta\right)q^e =$$

$$= 2A\int_0^1\int_0^{1-\eta}B^T_{plate}\left(\frac{t^3}{12}DB_{plate}q^e\right)d\xi d\eta = 2A\int_0^1\int_0^{1-\eta}B^T_{plate}Md\xi d\eta \tag{5}$$

where

$$M = \{M_{xx}, M_{yy}, M_{xy}\}^T, \tag{6}$$

$$F^e_{plate} = \{F^e_i, F^e_j, F^e_m\}^T;$$
$$F^e_r = \{F_{rz}, M_{rx}, M_{ry}\}^T \; r = i, j, m, \tag{7}$$

$$q_e = \{w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad w_3 \quad \theta_{x3} \quad \theta_{y3}\}. \tag{8}$$

The in-plane geometric stiffness matrix may be written symbolically as:

$$[K^e_G]^{plate}_{IP} = \nabla F^e_{plate} = \begin{bmatrix} (A_{ii})^{plate}_{IP} & (A_{ij})^{plate}_{IP} & (A_{im})^{plate}_{IP} \\ (A_{ji})^{plate}_{IP} & (A_{jj})^{plate}_{IP} & (A_{jm})^{plate}_{IP} \\ (A_{mi})^{plate}_{IP} & (A_{mi})^{plate}_{IP} & (A_{mm})^{plate}_{IP} \end{bmatrix}, \tag{9}$$

where
$$(A_{rs})^{plate}_{IP} = \begin{bmatrix} \frac{\partial F_{rz}}{\partial x_s} & \frac{\partial F_{rz}}{\partial y_s} & 0 \\ \frac{\partial M_{rx}}{\partial x_s} & \frac{\partial M_{rx}}{\partial y_s} & 0 \\ \frac{\partial M_{ry}}{\partial x_s} & \frac{\partial M_{ry}}{\partial y_s} & 0 \end{bmatrix}. \tag{10}$$

The expressions for the individual terms of the geometric stiffness matrix were obtained in closed form using symbolic algebra.

When applied to the plate, chain rule differentiation with respect to the coordinates yields

$$\begin{aligned} dF^e_{plate} = &\sum_{r=i,j,m}\left[\frac{\partial}{\partial x_r}\left(2A\int_0^1\int_0^{1-\eta}\left(B^T_{plate}\right)_{fixed}Md\xi d\eta\right)dx_r\right. \\ &\left.+\frac{\partial}{\partial y_r}\left(2A\int_0^1\int_0^{1-\eta}\left(B^T_{plate}\right)_{fixed}Md\xi d\eta\right)dy_r\right] \\ &+\sum_{r=i,j,m}\left[\frac{\partial}{\partial x_r}\left(2A\int_0^1\int_0^{1-\eta}B^T_{plate}M_{fixed}\,d\xi d\eta\right)dx_r\right. \\ &\left.+\frac{\partial}{\partial y_r}\left(2A\int_0^1\int_0^{1-\eta}B^T_{plate}M_{fixed}\,d\xi d\eta\right)dy_r\right] \end{aligned} \tag{11}$$

where $dx_r = u_r$; $dy_r = v_r$; $r = i,j,m$.

The first expression on the R.H.S. of Eq. 11 returns the elastic stiffness matrix whereas the second expression becomes the geometric stiffness matrix. It is left to clarify what is meant by M_{fixed} in Eq. 11. The moment vector M, of Eq. 6 contains three components that are functions of ξ and η and defined as:

$$M_{xx}(\xi,\eta) = \left(M^j_{xx} - M^i_{xx}\right)\xi + \left(M^m_{xx} - M^i_{xx}\right)\eta + M^i_{xx}, \tag{12}$$

$$M_{yy}(\xi,\eta) = (M_{yy}^{j} - M_{yy}^{i})\xi + (M_{yy}^{m} - M_{yy}^{i})\eta + M_{yy}^{i}, \tag{13}$$

$$M_{xy}(\xi,\eta) = (M_{xy}^{j} - M_{xy}^{i})\xi + (M_{xy}^{m} - M_{xy}^{i})\eta + M_{xy}^{i}, \text{ where} \tag{14}$$

$$\{M_{xx}^{i} \quad M_{yy}^{i} \quad M_{xy}^{i}\}^T = M(\xi = 0, \eta = 0), \tag{15}$$

$$\{M_{xx}^{j} \quad M_{yy}^{j} \quad M_{xy}^{j}\}^T = M(\xi = 1, \eta = 0), \tag{16}$$

$$\{M_{xx}^{m} \quad M_{yy}^{m} \quad M_{xy}^{m}\}^T = M(\xi = 0, \eta = 1). \tag{17}$$

When the components of M (Eqs. 12-14) are inserted into Eq. 11 it is the values of these components at the nodes (Eqs. 15-17) that are held fixed.

Out-of-Plane Contribution to the Shell Geometric Stiffness Matrix.

This section will derive the out-of-plane geometric stiffness matrix for the plate along the same lines and subsequently present a combined out-of-plane contribution. The derivation starts with the change in a vector G, due to a small rotation that is given by Goldstein (1950) as

$$dG = \omega \times G, \tag{18}$$

where ω is the rigid body rotation vector due to changes in the geometry. In terms of joint displacements with respect to the local coordinate system, two components θ and -φ, of the rigid body rotation are obtained from Figure 4. The third component is chosen for the plate arbitrarily, as the local z-rotation of node i. Recall that for the membrane this component is included in the in-plane contribution:

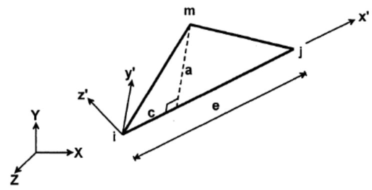

Fig. 4. A triangular finite element in its coordinate system

$$\omega_{x'} = \frac{c-e}{ea}(\delta_i)_{z'} - \frac{c}{ea}(\delta_j)_{z'} + \frac{1}{a}(\delta_m)_{z'};\ \omega_{y'} = \frac{a}{ea}(\delta_i)_{z'} + \frac{a}{ea}(\delta_j)_{z'}\ \omega_{z'} = \theta_{iz}, \tag{19}$$

where $(\delta_r)_{z'}$ is the displacement in the local (primed) z-direction of node r.

At each node, the forces, moments and rotations may be written as

$$F_r = F_{rz}k';\ M_r = M_{rx}i' + M_{ry}j' + M_{rz}k';\ \omega = \omega_{x'}i' + \omega_{y'}j' + \omega_{z'}k'$$

for $r = i, j, m$ and the changes in the force and moment vectors are given in more detail as

$$dF_r = -F_r \times \omega = \begin{bmatrix} 0 & F_{rz} & 0 \\ -F_{rz} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{Bmatrix}, \tag{20}$$

$$dM_r = -M_r \times \omega = \begin{bmatrix} 0 & M_{rz} & -M_{ry} \\ -M_{rz} & 0 & M_{rx} \\ M_{ry} & -M_{rx} & 0 \end{bmatrix} \begin{Bmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{Bmatrix}. \tag{21}$$

The out-of-plane stiffness contribution of the plate is now obtained as:

$$(dF^e)_{plate} = [K_G]_{OP}^{plate} \cdot \delta^e = -F^e \times \omega = \begin{bmatrix} F^i \\ F^j \\ F^m \end{bmatrix} [A_i \quad A_j \quad A_m] \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}, \tag{22}$$

where

$$F^e = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \\ F_m \\ M_m \end{Bmatrix};\quad F^r = \begin{bmatrix} 0 & F_{rz} & 0 \\ -F_{rz} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & M_{rz} & -M_{ry} \\ -M_{rz} & 0 & M_{rx} \\ M_{ry} & -M_{rx} & 0 \end{bmatrix}. \tag{23}$$

Now ω can be described in terms of the displacement vector as:

$$\omega = A\delta^e = [A_i \quad A_j \quad A_m]\begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}, \tag{24}$$

where

$$(A_i) = \begin{bmatrix} 0 & 0 & -\frac{e-c}{ae} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

$$(A_j) = \begin{bmatrix} 0 & 0 & -\frac{c}{ae} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{e} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{26}$$

$$(A_m) = \begin{bmatrix} 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

and $$\{d_i\} = \begin{Bmatrix} (\delta_i)_{x'} \\ (\delta_i)_{y'} \\ (\delta_i)_{z'} \\ (\theta_i)_{x'} \\ (\theta_i)_{y'} \\ (\theta_i)_{z'} \end{Bmatrix}; \ \{d_j\} = \begin{Bmatrix} (\delta_j)_{x'} \\ (\delta_j)_{y'} \\ (\delta_j)_{z'} \\ (\theta_j)_{x'} \\ (\theta_j)_{y'} \\ (\theta_j)_{z'} \end{Bmatrix}; \ \{d_m\} = \begin{Bmatrix} (\delta_m)_{x'} \\ (\delta_m)_{y'} \\ (\delta_m)_{z'} \\ (\theta_m)_{x'} \\ (\theta_m)_{y'} \\ (\theta_m)_{z'} \end{Bmatrix}. \tag{28}$$

Carrying out Eq. 22 for the plate and adding the out-of-plane contribution to the geometric stiffness matrix, in the local coordinate system more explicitly (adjusted to 18 DOF's) for the membrane, results in the following out-of-plane shell geometric stiffness matrix with respect to the local coordinate system:

$$[K_G^e]_{OP}^{shell} = \begin{bmatrix} (A_i)_1 & (A_i)_2 & (A_i)_3 \\ (A_j)_1 & (A_j)_2 & (A_j)_3 \\ (A_m)_1 & (A_m)_2 & (A_m)_3 \end{bmatrix} \tag{29}$$

where

$$(A_r)_1 = \begin{bmatrix} 0 & 0 & a_r & 0 & 0 & 0 \\ 0 & 0 & b_r & 0 & 0 & 0 \\ 0 & 0 & \alpha_r & 0 & 0 & 0 \\ 0 & 0 & c_r & 0 & 0 & -M_{ry} \\ 0 & 0 & d_r & 0 & 0 & M_{rx} \\ 0 & 0 & e_r & 0 & 0 & 0 \end{bmatrix} \ (30); \quad (A_r)_2 = \begin{bmatrix} 0 & 0 & f_r & 0 & 0 & 0 \\ 0 & 0 & g_r & 0 & 0 & 0 \\ 0 & 0 & \beta_r & 0 & 0 & 0 \\ 0 & 0 & h_r & 0 & 0 & 0 \\ 0 & 0 & i_r & 0 & 0 & 0 \\ 0 & 0 & j_r & 0 & 0 & 0 \end{bmatrix}, \tag{31}$$

$$(A_r)_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_r & 0 & 0 & 0 \\ 0 & 0 & \lambda_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_r & 0 & 0 & 0 \\ 0 & 0 & m_r & 0 & 0 & 0 \end{bmatrix}, \tag{32}$$

$$\alpha_r = -(F_r)_{y'}\frac{e-c}{ae} - \frac{(F_r)_{x'}}{e};\ \beta_r = -(F_r)_{y'}\frac{c}{ae} - \frac{(F_r)_{x'}}{e};\ \lambda_r = \frac{(F_r)_y}{a};\ \ a_r = \frac{F_{rz}}{e};$$

$$b_r = F_{rz}\frac{e-c}{ae};\ c_r = \frac{M_{rz}}{e};\ d_r = M_{rz}\frac{e-c}{ae};\ m_r = \frac{M_{ry}}{a};\ e_r - M_{ry}\frac{e-c}{ae} - \frac{M_{rx}}{e};$$

$$f_r = -\frac{F_{rz}}{e};\ g_r = F_{rz}\frac{c}{ae};\ \ h_r = -\frac{M_{rz}}{e};\ \ i_r = M_{rz}\frac{c}{ae};$$

$$j_r = -M_{rz}\frac{c}{ae};\ k_r = -\frac{F_{rz}}{a};\ l_r = -\frac{M_{rz}}{a}.$$

Conclusions: The theoretical approach of the buckling and strength analyses of the sinusoidal velaroidal shell is worked out using the finite element method. The in-plane and the out-of-plane shell geometric stiffness matrices with respect to the local coordinate system are obtained. This result gives a possibility of further numerical strength and buckling analyses of the sinusoidal velaroidal shells.

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К РАСЧЕТУ НА УСТОЙЧИВОСТЬ В ГЕОМЕТРИЧЕСКИ НЕЛИНЕЙНОЙ ПОСТАНОВКЕ ОБОЛОЧЕК В ВИДЕ СИНУСОИДАЛЬНОГО ВЕЛАРОИДА

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В статье рассматривается расчет на прочность и устойчивость в геометрической нелинейной постановке для синусоидальных велароидальных оболочек под действием собственного веса и равномерно распределенной нагрузки. Приводится краткая история применения метода конечного элемента для изучения потери устойчивости оболочки. Метод конечного элемента используется в матричной формулировке. Эластичная матрица жесткости в локальной системе координат мембранного элемента определяется в общем виде. Получена матрица геометрической жесткости вне плоскости для пластины вдоль тех же линий.

КЛЮЧЕВЫЕ СЛОВА: синусоидальные велароидальные оболочки, расчет на устойчивость в нелинейной постановке, линейная матрица упругой жесткости, нормальная жесткость, геометрическая матрица жесткости.