THE REVIEW OF RELATIONS OF PRINCIPLES OF STRONG UNIFORMIZATION AND CHURCH THESIS IN INTUITIONISTIC SET THEORY

Valery Kh. Khakhanian

Chair of Mathematics
Moscow State University of railway communications
ul. Obraztsova, 9, stroenie 9, GSP-4, Moscow, Russia, 127994

We prove that the strong uniformization does not depend on Church thesis with choice in the set theory with intuitionistic logic and with axiom of extensionality.

Key words: intuitionistic set theory, Church thesis, uniformization, Markov principle.

In [1] was proved the independence of the principle of the strong uniformization on Church thesis with choice in the intuitionistic set theory with two kinds of variables, without extesionality plus the strong Markov principle and double complement of sets (DCS). The strong principle of uniformization is: $U=\forall x\exists n\varphi(x, n)\rightarrow\exists n\forall x\varphi(x, n)$.

This principle appeared for the first time in the work of A. Troelstra [2] for the second order arithmetic with variable on sets of natural numbers variables. Formula φ from U belongs to the language of the set theory with two kinds of variables (on naturals and on sets). Two predicates letters are $x\in y$ and $n = m$. An analogical principle U! contains the request of the uniqueness of natural n in the premise under quantifier of $\exists$. It is evident that $U \vdash U!$

Forms of Church thesis which are examined in the present work (with choice — strong thesis CT and with uniqueness — CT!) and strong Markov principle are well known and can be found in [3]. We note that the weak uniformization can be deduced from the weak Church thesis CT! (the proof is given in [4] and can be used in Zermelo-Fraenkel intuitionistic set theory). We give an additional formulation of the classical true principle DCS, which has in our double kinds language the following notation: $\forall a\exists x[\forall n(\neg\neg n\in a\rightarrow n\in x)\wedge\forall u(\neg\neg u\in a\rightarrow u\in x)]$. In [5] it was proved that all mentioned principles are consistent with full the intuitionistic set theory. In [3] we used a model in which all axiom schemes and axioms of the set theory and all above mentioned principles are realized excepting the uniformization that it was not possible to realize and its realization was not disproved.

The problem of investigations of relations of principles U, U!, CT, CT! on the level of the set theory is connected above all with the proof of nonderivability of principles of uniformization from principles of Church and determination of the role of extensionality in the second case. In the end of my work I will give the summary of results for the set theories of ZF-type with intuitionistic logic including the result which is obtained in the present work. We note also that for the intuitionistic type theory corresponding results were obtained by G. Scwartz in [4] and in the same work the role of extensionality was explained.

In the present work we announce the nonderivability of the strong uniformization in the intuitionistic set theory with two kinds of variables and extensionality (full set theory!) and additional Markov strong principle, DCS and strong Church thesis and in this way the result from [1] is reinforced and all open problems from [1] will be resolved. Furstly this present result was announced in [6]. We proved in [7] that in the model from [3] the uniformization is refused. Therefore models from [3] and [5] are different. In counterexample φ contains only one parameter.

Open problem is: to give a counterexample of the uniformization in which the same formula has no parameters.

We note also that while reading this work knowledge of [1; 3; 5] is useful and necessary though the full formulation of the set theory with intuitionistic logic and all additional principles will be given.

Remark 1. We choose the following notation: variables on natural numbers are n, m, p, l, k, h, s, variables on sets are a, b, x, y, z, u, v, w.

Let us give now an exact formulation of set theories ZFIR2 and ZFIC2. We start from ZFIR2. Logical axioms of this theory are axioms of intuitionistic logic of predicates with two mentioned kinds. The own axioms and scheme axioms include standard axioms HA and the following set theoretical axioms.

Let us give now the shortly formulation of set theories ZFIR2 and ZFIC2. We start from ZFIR2. Logical axioms of this theory are axioms of intuitionistic logic of predicates with two mentioned kinds. The own axioms and scheme axioms include standard axioms HA and the following set theoretical axioms.

1. Extensionality
2. Pair
3. Union
4. Power
5. Separation
6. ε-induction
7. Infinity
8. Replacement

Scheme of collection differs from scheme 8 only by the fact that in the premise the quantifier of existence does not contain a symbol of uniqueness.

Let us note also that the used abbreviations for the notation are standard and that the replacement is deduced from the collection. System ZFIR2 includes axioms 1—8, and system ZFIC2 includes axioms 1—7 and collection. In 1985 H. Friedman and A. Schedrov proved that the system with collection deductively is more strong that the system with replacement because ZFIR2 possesses the property of full existence but ZFIC2 has not the same property. In systems ZFIC2 and ZFIR2 Heyting's arithmetic HA is contained on the first level in the explicit form and it requires a kind of variables on naturals. In these systems the scheme of induction has a standard form and formula φ can admit parameters on all kinds of variables.

Theorem. ZFIC2 + M + DCS + CT $\nvdash$ U}

Lemma. For every prf $f(n, h, k, p, s)$ there exist a set x from V (universe of our model) such that the given function is not its function of extensionality.

Corollary. There is no prf such that it could be a function of extensionality for all sets from V. In particular the function $f(n, h, k, p, s)$ which for any h, k, p, s is not identical along the first argument is not the function of extensionality for all sets from V.

Notice 1. In the above Lemma instead of V we can take V_α (for example for $\alpha = \omega$).

Notice 2. We can try to use the given property of universum V for receiving a counterexample of the strong principle of uniformization to prove the independence of that principle on Church thesis with choice in the intuitionistic set theory. Formula φ will not contain any parameters in the non formal notation of this counterexample. But it is necessary to give a formal notation in the language of the set theory.

We present only meta notation the following example of strong uniformization (see also [1] or [7]):

$\forall x \exists n(\neg {<}n, x{>} \in a_r) \rightarrow \exists n \forall x(\neg {<}n, x{>} \in a_r)$. Here a_r is a free variable on sets with number r. We evaluate this variable on the set $y \in V$: $y = \{<0, w>: \exists x \in V_\alpha(x \neq \varnothing \wedge w = \{<n, h>, <m, v>: v \sim x$; ensembles m and n are not empty; the ensemble h is not empty and $\forall vh(\neg(vexth)$ (for example for numbers of the empty function $\}\} \cup \{<0, z>: z \sim w\}$. Here $\alpha \geq \omega$ (see *Notice 1*).

Now we give the summary of results for non type set theory with two kinds of variables. All principles have parameters of any kinds.

ZFIC2+M+DCS+CT $\nvdash$ U

ZFIC2+M+DCS+U! $\nvdash$ U

ZFIC2+M+DCS+CT!+U $\nvdash$ CT

ZFIR2+CT! $\vdash$ U!

ZFIR2+U $\vdash$ U!

ZFIR2 + CT $\vdash$ CT!

ZFIC2+M+DCS+U $\nvdash$ CT!

ZFIC2+M+DCS+U $\nvdash$ CT

The open problem is: ZFIC2 without extensionality+M+DCS+CT$\nvdash$ U!

REFERENCES

[1] *Khakhanian V.Kh.* The independence principle of uniformization from Church thesis in intuitionistic set theory // Mathematical Note 43, issue 5 (may). — 1988. — P. 685—691 (Russian).

[2] *Troelstra A.S.* Notes on the intuitionistic second order arithmetic // Lecture Notes in Mathematics. — 1973. — N. 337. — P. 171—205.

[3] *Khakhanian V.Kh.* Set theory and Church thesis // The investigations by non classical logics and formal systems. — M.: Nauka, 1983. — P. 198—208 (Russian).

[4] *Shwartz G.* Some applications of method of recursive realizability to intuitionistic set types // Questions of cybernetics. Non classical logics and its applications. — M., 1982. — P. 37—54 (Russian).

[5] *Khakhanian V.Kh.* The consistency of intuitionistic set theory with principles of Church thesis and uniformization // Vestnik of Moscow State University. Serie “Mathematics and Mechanics”. — 1980. — N 5. — P. 3—7 (Russian).
[6] *Khakhanian V.Kh*. The independence of strong uniformization from Church thesis in full set theory // Scientific mathematical readings in commemoration of M.Y. Suslin. Saratov, 16—21, October, 1989. Abstracts SGPI, 1989. — P. 91 (Russian).
[7] *Khakhanyan V.Kh.* Independence of the uniformity principle from Church's thesis in intuitionistic set theory // Izvestia RAN. Ser. Mat. — 2013. — 77:6. — P. 209—224.

ОБЗОР СООТНОШЕНИЙ ПРИНЦИПОВ УНИФОРМИЗАЦИИ И ТЕЗИСОВ ЧЁРЧА В ИНТУИЦИОНИСТСКОЙ ТЕОРИИ МНОЖЕСТВ

В.Х. Хаханян

Московский государственный университет путей сообщения
ул. Образцова, 15, Москва, Россия 101475

В работе рассмотрены доказанные ранее в статьях автора и приведен (без доказательства) последний из результатов о соотношениях разных видов принципа униформизации и тезисов Чёрча в теории множеств с интуиционистской логикой.

Ключевые слова: тезис Черчя, «сильная униформизация», интуиционистская логика, аксиома объемности.