# TO WHAT EXTENT CONTEMPORARY MATHEMATICAL SIENCE IS RELIABLE

**Valery Kh. Khakhanian**

Moscow State University of Railway Communications
*Obraztsova Str., 15, Moscow, Russia, 101475*

The crisis in foundation of mathematics at the end of 19th beginning of 20th centuries initiated a number of axiomatic set theoretical systems during the first half of the 20th century. These systems were the result of different philosophical approaches (in view of second Godel's Theorem) aimed at overcoming of the above crisis. But the way out of this situation has never been found.

In my article I offer a new approach to solve this problem using a basic axiomatic system of the set theory with intuitionistic logic. I will present a lot of mathematical results having been obtained during the last forty years. We will survey the development of the set theory with the intuitionistic logic underlining the main points and formulating unsolved problems and describe the basic system of the intuitionistic set theory.

**Key words:** arithmetic, mathematical analysis, axiomatic systems of set theory, intuitionism, constructivism, basic system, mathematics.

## 1. History of the problem

### 1.1. Crises in mathematics

During the development of mathematics there have been situations that required introducing of some new mathematical objects or explaining of some new results which seemed to be paradoxical. The crises in the foundation of mathematics mentioned below are the most important for the development of mathematics and philosophical views from which this development has been evaluated. The discovery of incommensurable segments and the appearance of a new class of irrational numbers was historically the first to appear. The discovery of the notion of the incommensurability and aporias formulated later by Zeno of Elea put quite a new question before mathematicians and philosophers of that time: about mathematics as an exact science. That gave a reason to speak about the crisis in Greek mathematics (see [1]).

Other important crisis which led to some new mathematical and philosophical views in mathematics consisted of proving the deducibility of the fifth postulate in Euclid's axiomatic geometry. Those unsuccessful attempts led mathematicians to discover some new geometrical systems and as a result to the question of what the real space of the

Universe is. At the end of the 18th — beginning of the 19th centuries Gauss, Lobachevsky and Bolyai made the first and a very timid attempts that nobody fully understood then, the step to the full formalization of the geometrical science. The step being brought to the logical end could give a possibility not only to prove that something exists, but also to know how to justify that something does not exist. The successive full formalization was made for the first time by D. Hilbert in 1899 (see [2]) also for Euclid's geometry, and than later was suggested by him as a substantial component of his programme of the justification of the whole mathematics (see [3]).

Here we should mention that one more important crisis in the foundations of mathematics was generated by contradictory results received in the calculus of infinitesimals in $17^{th}$ — $18^{th}$ centuries, that did not get a satisfactory justification during a long period of time (some mathematicians considered them as zeros, others — not). But we are more interested in the crisis related to the geometry.

The program of the foundations of mathematics became especially necessary when in the G. Cantor's teaching of sets ("naïve" set theory, see [4]) some contradictions were found. It was of course the most important crisis (third from those we are examining) in mathematics as that crisis touched the deep foundations of mathematics (contradiction opened by B. Russell in 1902 and not only it; we will cite it fully below). Without describing in details the appeared situation we should note that a number of possible ways out of that situation was suggested but unfortunately it was impossible to save the initial, "naïve" variant of G. Cantor's doctrine of sets. Systems of axiomatic set theory of Russell—Whitehead (simple theory of types TT), systems without types of E. Zermelo (Z), Zermelo—Fraenkel (ZF), W. Quine (NF), theory of classes of von Neumann—Bernays—Godel (NBG) and a number of other theories which did not become so well-known as the mentioned above and also new philosophical points of view concerning the question of mathematics justification by B. Russell and G. Frege (logicism), L. Brouwer (intuitionism), D. Hilbert (formalism and finitism), see [3; 5; 6], constructive tendencies in mathematics suggested by A. Markov and his school, see [7] and [8], J. Myhill, H. Friedman, M. Beeson and some other mathematicians (see, to review). It also concerns other contemporary views for the possible justification of mathematics by a number of mathematicians and philosophers (1) which do not lead us to the desirable and acceptable by everybody way out of the newest crisis in foundations of mathematics.

**1.2. Two trends in the development of mathematics**

We note two important trends in the development of mathematics, that as a noticeable line passing through the whole history of the latter. First it is a tendency to more formalized axiomatic presentation of different mathematical disciplines caused by the continually growing request to the strictness of mathematical proofs taking its origin from Euclid's geometrical axiomatic and being newly pushed later by Lobachevsky—Bolyai none-Euclid's geometry. Gradually such presentation gaining more and more degree of formality reaches the modern state of development in Hilbert's works (as well as in geometry), in Godel's works and works of other contemporary mathematicians, in first turn — mathematical logicians.

We should also note a second important tendency consisting in permanent struggle (particularly noticeable during the last two centuries) between two kinds of infinity: actual (one of the most bright representative is G. Cantor and his doctrine of sets) and potential (here we can cite L. Brouwer and A. Markov). Contradictions appeared just in connection with the adoption and development by Cantor of the non limited abstraction of actual infinity in the created by him "naïve" set theory (doctrine of sets). We should note the most famous of these contradictions. In the "naïve" set theory (with the intuitive meaning of the notion "set", see [4], page 173, first item § 1) the existence is contradictory: in the first place sets of all sets (contradiction with the fact that power of subsets set of the given set is strictly more than power of the initial set (theorem of Cantor who discovered this contradiction himself in 1899); secondly, the existence of all sets, which are not elements of themselves (the author is B. Russell, who published this contradiction in his letter to G. Frege June, 16, 1902).

It seems that appears a question of refusing to accept the notion of actual infinity, but the situation is not so simple. If we accept the conception of potential infinity or, and which is a more constructive supposition, the existence of very big and practically unreachable natural numbers (such situation in fact takes place in the modern cryptography, discrete mathematics and in a number of other branches of mathematics, largely basing on final mathematics and not requiring the appeal to the full abstraction of the potential infinity), then we shall not be able to receive without adding any "naturally acceptable" complementary principles, for example Church's Thesis and Markov's principle in the traditional constructivism of A. Markov (see, for example, [8]), even the most initial fragments of mathematical analysis and those mathematical disciplines which use results and consequences from mathematical analysis so as it is done with the help of not the most strong variants of axioms basing on the notion of actual infinity, as for example, limited form of the axiom of choice AC (see [10]). Just such difficulties face the intuitionists and the constructivists of any directions accepting any interpretation of the notion of the potential infinity.

### 1.3. D. Hilbert's formalism and finite point of view

An eminent German mathematician of that time D. Hilbert suggested a possible and very promising way out of the situation that appeared on the boarder of the 19th-20th centuries, situation in foundations of mathematics connected with contradictions found in Cantor's "naïve" set theory (see [11]). D. Hilbert's conception of the proof of the consistency of mathematics was in the initial division of all objects in mathematics as real and ideal. Not throwing away the latter, D. Hilbert suggested, firstly, formalizing completely the mathematics, in fact arithmetic or theory of natural numbers, as by that time it became clear that others branches of mathematical science could be brought to arithmetic (arithmetization of the whole mathematics). Dealing now with the fully formalized arithmetic, that is with syntax objects (finite constructions) we could try to prove the consistency of the obtained arithmetic calculus, that is to prove the impossibility of the conclusion of some syntax object (suggestion), semantically expressing the consistency of arithmetic in the constructed formalism. At the same time (condition of correctness) all the statements (theorems) deducible in the suggested formalism should be true in some natural semantics, for example, in the structure of natural numbers understood in the usual way. Metamathematical means used in this connection (D. Hilbert did not give their exact description) were recognized by all the mathematicians (so called finitism, or finite point of view). Nevertheless it turned out to exits the statements, true but not deducible, of the examined arithmetic formalism at the condition of the consistency of the latter (K. Godel, 1930, see [12]) and as consequence of this fact a principal impossibility was stated to prove the consistency of the arithmetic formalism by means of this formalism (second K. Godel's theorem, 1930, also see [12]). The analysis of the proof of the Godel's second theorem showed that the result stays true also for some more weak systems (for example, Robinson's arithmetic), without speaking of strong axiomatic systems, formalizing mathematical theory of the real numbers or set theory. We should note that up to now nobody could have constructed an axiomatic theory sufficiently rich in content where one could prove its own consistency. Thus, although the program to overcome the crisis risen in the end of the 19th — beginning of the 20th centuries in the foundations of mathematics, suggested by D. Hilbert, played a significant role in the development of mathematics (mainly mathematical logic). The principal goal of its creator was not achieved.

## 2. A possible approach to the justification of the set theory

### 2.1. Actual state of the foundations of mathematics (set theory)

So, the situation in the set theory could be characterized as follows. On the one hand it is quite clear that the return to the "naïve" variant of the set theory is not possible at the actual state of the problem to prove proof of the consistency of the formalized systems. On the other hand all the rational ways out of the situation seemed to be exhausted. All the mathematicians and philosophers working in the field of the set theory did not accept it unconditionally. In the book of A. Fraenkel and I. Bar-Hillel [5] — the most important specialists in the foundations of mathematics — we read [5. P. 347]: "The attitudes on how set theory might be given a satisfactory foundation are as yet widely divergent, and a host of problems connected herewith are far from being solved. Nevertheless, the great majority mathematicians refuse to accept the thesis that Cantor's ideas were pathological shaky, these mathematicians continue to apply successfully its concepts, methods, and results in most branches of analysis and geometry as well as in some parts of arithmetic and algebra, confident that future foundational research will converge towards a vindication of set theory to an extent that will be identical with, or at least close to, its classical one. This attitude is compatible with a readiness to interpret set theory in a way which might diverge considerably from customary ones, in line with the apparently existing need for a reinterpretation of logic and mathematics in general".

We should note that the cited lines were written almost a half-century ago and of course the authors could not take in to consideration all the modern results in the set theory (read: in the foundations of mathematics, as there appeared other mathematical disciplines pretending to the role that the set theory played and is still playing in the foundations of mathematics). Taking into account the above, we should note that the first part of the authors' conclusion concerning the rehabilitation of the set theory in its full (or at least almost full) classical content seems (and seemed before) to be too optimistic, as well as the possibility of the full revision of the interpretation of logic and mathematics in general, as no successful ideas of such revision (and directions of the revision) have not been even noticeable up to now. But nevertheless there emerged a direction (below we will give examples for arithmetic, theory of real numbers and set theory), consisting in local formalization that or another branch of mathematics and in studying of this branch from definite mathematical and philosophical points of view and metamathematics used there, depends as a rule, on philosophical view of the scientist who does not exclude the reasonableness and argumentability of the approach suggested by him to the foundations of mathematics. The correlation of different metamathematical divisions can also be studied from the formal point of view, so in this way the metamathematics of formalized systems of metamathematics appears.

One should note that "...problem of eliminating of paradoxes (better says *contradictions* — author) thus merges with the broader problem of the foundations of mathematics and logic. What is the nature of mathematical truth? What meaning do mathematical propositions have, and on what evidence do they rest? This broad problem, or complex of problems, exists for philosophy apart from the circumstance that paradoxes have arisen in the fringes of mathematics", see [6. P. 41—42].

Below we will try to suggest an approach to the justification of the set theory, but at first, as an example, for arithmetic (in spite of illusory simplicity of the latter and the readiness of almost all the mathematicians to believe in the consistency of some extension of arithmetic, such that the proof of the consistency of initial arithmetic formalism could be given in the limits of this extension), and then for the theory of the real numbers. Using this method, one could apply it for arithmetic with underlying intuitionistic logic, named Heyting's arithmetic HA. To apply it just for intuitionistic systems, justification of which proved less doubts at the beginning of the last century, than for classical systems (see, for examples, [13. P. 35], in italic upwards and [14]).

### 2.2. HA as a basic variant

HA — it is an axiomatic theory of Peano's arithmetic PA, but in the system of logical schemes of PA axioms there is no law of excluded middle or the law which equivalent to it of removal of double negation so largely used in mathematical proofs. Without entering in details we can describe the idea of the approach for the justification of the arithmetic HA and those additional principles which will be used at such approach. The suggested approach itself can also be used for the justification of much more powerful axiomatic theories such as mathematical analysis (theory of real numbers) and the set theory (which can be examined both with underlying intuitionistic logic and with classical logic, this will be done below). As a basic theory which from our point of view is clear, well understandable and not requiring a justification, it is just intuitionistic arithmetic HA. As additional principles we examine: CT — Church Thesis with choice (or its more weak variant CT!; both principles claim that there exist only effectively computable functions on natural numbers); principle P which is in the form $[\neg\varphi\rightarrow \exists x\psi(x)]\rightarrow\exists x[\neg\varphi\rightarrow\psi(x)]$ where the formula $\varphi$ does not content freely the variable $x$

(here we for the first time face the situation when for the constructive justification are taken not all the effectively computable functions as in the justification of CT, but only functions from the preassigned closed subset of the set of effectively computable functions; principle P expresses a restrictedly constructive point of view); ECT — is the Church's Thesis for effectively computable partial (that is everywhere not defined) functions of natural numbers; principle of constructive selection M (A.A. Markov's principle both in the strong form and in its weak variant $M^-$; this principle affirms that the search of the required natural number with the prescribed property, under the condition that such a number cannot not exist, will be completed without fail. All the exact mathematical formulations and detailed comments to them could be found in [15]. One can also find there all the proofs of the statements given below, which are to a certain way a justification of the suggested approach for axiomatic systems of arithmetic.

So, let us make some conclusions basing on mathematical results from [15]. Theory HA is a basic theory and it admits different extensions which have different semantics and often simply contradict each other. In the arithmetic HA one can interpret the following theories cited below. This is Peano's formal classical arithmetic PA, this is Markov's traditional constructivism HA+CT!+M and different variants of modifications of the traditional arithmetic constructivism, this is antitraditional constructivism HA+CT!+P, this is arithmetic of realizability HA+ECT. As it was mentioned above, for details see [15].

Let us note additionally that the basic system of arithmetic HA has a number of properties, that differs it from the classical arithmetic PA, that is to say HA has properties of disjunctivity (if the sentence $\varphi\vee\psi$ is deduced in HA, then the sentence $\varphi$ or the sentence $\psi$ will be deduced in HA; in the classical arithmetic PA this property is broken) and numeric extensionality (if the sentence $\exists x\varphi(x)$ is deduced in HA, then for some natural n formula $\varphi(n)$ is deduced in HA; in arithmetic PA this property is also broken). In HA the principle of the existence of the least element is broken, but in arithmetic PA this principle takes place. Also in the theories with the underlying intuitionistic logic principles with uniqueness in premise turn out to be deducibility weaker than with the same principles without uniqueness. For example, in HA+CT! the thesis CT is not deducible, and this is explained just by the absence of the principle of the existence of the least element.

We see that different extensions of the intuitionistic arithmetic HA described above are consistent relative to the basic system HA. That is why it is not possible to solve the problem of absolute consistency of any of these arithmetical theories. Therefore the phrase "HA is consistent (or PA, or any of the examined theories)" which is sometimes said, is not simply true.

Now let us pass to the examination of different formalizations of the theory of the real numbers.

### 2.3. Theories of the real numbers

For theories of the real numbers the situation is changing both by the choice of the basic theory established on the intuitionistic logic (really, why just intuitionistic logic? maybe one could take some other logic and maybe it would be possible to pass by Godel's theorem and to prove its own consistency in the limits of the arithmetic itself, but with another logic laying in the ground? this question is far from being investigated, but for the moment there is not enough of formal theories rich in content with such a property (proof of its own consistency)) and by the quantity of its different extensions. We will give far from all known extensions, even from [15]. That is why our description will be very short and we will give a summary of results for the analysis basing only on A. Dragalin's work [15].

One of the basic theories formalizing the intuitionistic approach to the theory of real numbers is the system BSK — the basic system of Kleene. He is also the author of the theory FIM (foundations of intuitionistic mathematics). We are not going to examine it here, but we can note that just the theory FIM is one of extensions of the system BSK. The system BSK also allows a classical model unlike the theory FIM. J. Myhill suggested another theory (theory MM) very well formalizing L. Brouwer's ideas for the theory of real numbers. This theory formalized very exactly L. Brouwer's original ideas concerning the notion of the real number with the help of Kripke's scheme (see [15] for more exact formulations).

The third basic variant, theory in the language with functional symbols and constructive operators — IDB (inductive definitions of L. Brouwer) may be extended by two (as a minimum) different ways: as the theory of sequences, given by law CS (G. Kreisel, see [15. P. 140] for the exact reference) and as the theory of sequences without law LS (G. Kreisel, the exact reference is in [15. P. 142]).

Completing here the very short description of formalized theories of the intuitionistic analysis, we can note that the variety of nuances in the interpretation of sequences of natural numbers represents more complicated picture just like spider's web (also see [15]) in comparison with the analogous picture emerging around the basic system of arithmetic HA. On the other hand, the strategic scheme of the investigation in the field of the formalized intuitionistic mathematical analysis is the same as for the formalized intuitionistic arithmetic: we find out a basic formalized calculus (for the analysis there are two such basic calculus, BSK and IDB) and around it we construct different extensions (including extensions of classical types), which sometimes simply contradict each other as in the case with intuitionistic arithmetic. Mutual relations of many different principles are investigated, consistency relative to the basic system of extensions received with the help of these principles. Its proof is accompanied by the construction of a big number of algebraic and topological models (and not only such models), allowing to achieve very sharp distinctions in the interpretations of properties of efficiency for the investigated calculus. While investigating formalized set theories, that is theories of higher range, this strategic scheme of investigation will be conserved in full power. We shall note also that a number of extensions of basic systems of formalized theory of the real numbers (excluding those founded on the underlying classical logic) has good effective properties (for example, the above mentioned properties of disjunctivity, numeric and full extensionality; the definition of the last property will be given below).

### 2.4. Typical and nontypical set theories

Let us notice at first that all the results which were given for HA and its extensions as well as for the theories of real numbers can be "lifted" on the level of Zermelo—Fraenkel's nontypical set theories with underlying intuitionistic logic (the theory ZFI or IZF). Which precisely results can be "lifted" and the exact mathematical as well as philosophical sense of these results for the set theory, it will be described later.

We will not examine the axiomatic systems of the typical set theory, as a number of results (practically all), received for intuitionistic formalized systems of the theory of real numbers (such theories can be represented as intuitionistic arithmetic of the second order) can be transfered to the arithmetic of any finite order, over the typical theory $HA^{\omega}$ (arithmetic of all finite orders) and to the simple type theory TT of Russell—Whitehead. Therefore we pass at once to the nontypical axiomatic set theory with intuiti-onistic logic ZFI.

The examined basic variant (BV) of the axiomatic set theory is $ZFI_R$ + DCS=BV included Heyting's arithmetic HA, Kleene's system BSK, standard axioms and schemes of axioms of set theory ZF and the axiom of double complement of sets (DCS), which states that for every set there is a set of its non-non-elements (for the classical logic it is a trivial fact; for the more precise informal description see [16] and [17]).

First most successful variants of the nontypical set theory appeared in [18] and [19]. It was proved that such systems can have the property of the extensionality: if it is proved that there exists a unique set with the property $\varphi$, then there will be found (effectively by $\varphi$) a formula $\psi_\varphi$ defining this set. These systems of the set theory assumed effective extensions of different kinds for arithmetic and the theory of real numbers contained in it, and also for the proper set theory, for example, the principle of uniformization UP which says that if for every set (but not a natural number) there exists a natural number with property $\varphi$, then there exists a unique natural number for all sets with the property $\varphi$. Thus, natural numbers are strictly defined objects, and sets are fuzzy objects, but it contradicts the classical understanding of the notion of set. For the first time this principle UP was introduced by A.S. Troelstra in 1973. There are also extensions for the systems of the set theory, conserving the mathematical analysis in the same form as it is present in classical theories (adding a limited form of the axiom of choice $AC_\omega$). The classical set theory of Zermelo-Fraenkel ZF is equiconsistent with BV. The full summary of existing results (received not only by the author, but also in the literature for the last 40 years) could be found in [16] and [17].

Let us give a very short resume of the most important results. So, for the set theory BV there were:

— investigated properties of the class of ordinals;

— investigated correlations of a number of additional postulates of intuitionistic, constructive and set theoretical nature;

— constructed generalized models of predicate realizability type;

— investigated a limited variant of the axiom of choice AC for problems of the consistency and independency with the set theory BV;

— constructed a class of functional algebraic models of the set theory BV and proved the theorem of correctness for this class of models.

### 3. The suggested justification

So, from one hand, as it was noticed before, the return to Cantor's "naïve" variant of the set theory is not possible. It is not possible in the framework both classical and intuitionistic, and, possibly, relevant logics but maybe not other logics (see also point 5 below about W.V. Quine's "New Foundations"). On the other hand it is clear that for the moment the way out of this situation that could be accepted by all the researchers does not exist (at least because there is no possibility to prove the consistency not only of the set theory (read: the whole mathematics) but also of any sufficiently rich in content axiomatic theory). What could be undertaken in such a situation? It seems natural to turn to such mathematical principles which cast doubts neither from the side of mathematicians nor from the side of philosophers-mathematicians. The similar situation was just advocated by D. Hilbert himself and they disputed about the principles and the methods used in mathematics to be considered as finite (read: reliable). Just such an approach, but only in the formalized and sufficiently effective (nonclassical) axiomatic variant, is suggested in the present report.

Concretely: all the external mathematics should be put in the framework of the axiomatic BV. The latter is not something congealed and can be extended (or contracted) depending on results in mathematics (first of all in the set theory with intuitionistic logic).

## 4. Some remarks connected with the justification of the classical set theory

In such a short paper it is not possible, of course, to review all variants of the justification of the set theory as such and mathematical knowledge as a whole. To my mind one will never succeed in giving a formal (mathematical) proof of the consistency (even if one succeeds to "pass over" the second Godel's theorem). To prove strictly relative consistency of any separate branch of mathematics (using axiomatic method and reducing it to another branch) is practically always possible. For example, as it was already noted, it is possible to bring mathematical discipline with classical logic to analogous discipline with intuitionistic logic. The techniques of such briging is quite well developed. First results in this field related to the investigation of logics were received by A.N. Kolmogorov [14] and V.I. Glivenko [20; 21]. The main method of bringing ($\neg\neg$ — interpretation) was suggested by K. Godel. In particular, we can consider the axiomatic system of the set theory ZF well justified, as it can be interpreted in the system of the set theory BV, described above, provided that the latter system is considered as well justified. Therefore, it is necessary to have a criterion of choice either of philosophical bases or of the mathematical knowledge (for example, philosophical principles of L. Brouwer or A.A. Markov) or of a number of classical principles which seem to us to be quite sure (of course, in axiomatic systems of underlying intuitionistic logic the choice of principles takes the main place).

Accepting the last point of view and surveying the results obtained by the present time (first of all in the field of the descriptive set theory, see, for example, [22] and [23]), one could try to choose the most acceptable principles — for example, the axiom of choice in the full volume or some statements about existence of so-called big cardinals), leading to very strange mathematical results which are badly imagined from the "naïve" point of view. For details see, for example, the work [10] in which for the classical system of the set theory ZF there is justified a statement that the replacement of the full variant AC (axiom of choice) for countable variant $AC_\omega$ and the addition to the obtained system of the set theory of the axiom of determinateness AD (which contradicts to the full form of the axiom of choice) represents a very natural variant of the axiomatic set theory not only for conserving of the mathematical analysis in the standard form without appearance of such unpleasant "monsters" as Vitali set (all sets are measurable according to Lebesgue), but also allows to obtain a number of results in the style of the "naïve" G. Cantor's set theory. We notice that a number of problems put at the end of the [10] by the author, is now resolved and on one hand, many hypothesis, formulated by the author of [10] are confirmed, and on the other hand, some doubts of the author are declined as well. The use of "good" non-effective and badly justified (from the mathematical and philosophical points of view) additional principles summoned during last time quite a number of works where authors try to prove the inconsistency of set theories extended with such principles (and sometimes they try to refuse even rather familiar principles). But for the moment all these attempts are unsuccessful. Proofs are either not sufficiently clear, or badly verified as they use the "naïve" techniques of proof in the spirit of the original G. Cantor's set theory and avoid exact mounting of the axiomatic method.

## 5. Remarks relating to the justification of W.V.Quine's typeless set theory "New Foundations"

W.V. Quine suggested in 1937 a system of the set theory NF ("New Foundations"), see [24], quite an original, sharply different from the standard set theory ZF, but much resembling to the Cantor's set theory and formulated axiomatically and technically connected with the type theory of Russell-Whitehead. Arithmetic that can be developed in this axiomatic system of the set theory on the base of the classical logic is practically standard. However, mathematical theories of higher order require a further investigation, as NF has a number of properties quite unfamiliar for the standard mathematical thought (that is in framework of ZF). Therefore, the justification of this theory requires a particular approach, as it has to take into consideration the philosophical and technical bases of NF. Here we have to restrict ourselves to these short remarks, as the question requires a further mathematical investigation. Let us note at last that Quine's theory NF is not unique, standing apart from others well studied set theories (see, for example, [25]).

### NOTES

(1) For example: *Meyer R.K., Mortensen C.* Inconsistent models for relevant arithmetic // The Journal of Symbolic Logic. — V. 49. — 1984. P. 917—929 or *Friedman H., Meyer R.K.* Whether relevant arithmetic // The Journal of Symbolic Logic. — V. 57. —1992. — P. 824—831.

## REFERENCES

[1] *Struik von Dirk J.* Abriss der Geschichte der Mathematik. — Berlin, 1963.
[2] *Hilbert D.* Grundlagen der Geometrie. — Leipzig und Berlin (Teubner), 1930.
[3] *Heyting A.* Mathematische Grundlagenforschung. Intuitionismus, Beweistheorie. (Ergebnisse der Mathematik und ihrer Grenzgebiete. Hrsg. v. d. Schriftleitung des „Zentralblatt der Mathematik". III, Bd., 4. Heft.) — Berlin, 1934. — XII.
[4] *Cantor G.* Proceeding on set theory. — M., 1985 (in Russian).
[5] *Fraenkel A.A., Bar-Hillel Y.* Foundations of set theory. — Amsterdam, 1958.
[6] *Kleene S.C.* Introduction to metamathematics. — New York, Toronto, 1952.
[7] *Markov A.A.* About constructive mathematics. Proceedings of Math. Institute of V.A. Steklov of Acad. of Science of USSR. — Vol. LXVII. — Moscow—Leningrad, 1962 (in Russian).

[8] *Kushner B.A.* Lectures on constructive mathematical analysis. — Moscow, 1973 (in Russian).
[9] *Beeson M.* Problematic Principles in Constructive Mathematics. — Preprint № 185. — Department of Mathematics. — University of Utrecht. — February 1981. — Part 2.
[10] *Kanovei V.G.* Axiom of choice and axiom of determinateness. — Moscow, 1984 (in Russian).
[11] *Hilbert D.* Uber dieGrundlagender Logik und der Arithmetik // Verhanndlungen des Dritfen Internationalen Mathematiker-Kongresses in Heidelberg vom. 8 bis 13 August 1904. — Leipzig, 1905 (application from VII of [2]).
[12] *Godel K.* Zur intuitionistishen Arithmetik und Zahlentheorie Ergebnisse eines math. — Koll., 1931—1932.
[13] *Kolmogorov A.N.* Jubilee edition in three books. Book 1.Truth is weal. Biobibliographia. — Phismathlit, 2003 (in Russian).
[14] *Kolmogorov A.N.* About principle tertium non datur. Mathem // Transections. — 1925. — V. 32. — № 4 (in Russian).
[15] *Dragalin A.G.* Mathematical Intuitionism. Introduction to the Proof Theory. Trans. of Math. Monographs. — 1988. — Vol. 67.
[16] *Khakhanian V.* Thesis “Models of intuitionistic set theory”. — Moscow, 1982 (in Russian).
[17] *Khakhanian V.* Thesis “Intuitionistic logic and set theory”. — Moscow, 2004 (in Russian).
[18] *Friedman H.* Some applications of Kleene’s method for intuitionistic systems // Lecture Notes in Mathematics. — № 337. — 1973.
[19] *Myhill J.* Constructive set theory // The Journal of Symbolic Logic. — V. 40. — № 3. — 1975.
[20] *Glivenko V.I.* Sur la logique de M. Brouwer // Academie Royale de Belgique, Bulletin de la classe des sciences. — Ser 5. — 4 (1928).
[21] *Glivenko V.I.* Sur quelques points de la logique de M. Brouwer // Academie Royale de Belgique, Bulletin de la classe des sciences. — Ser. 5, 15 (1929).
[22] *Jech T.* Set theory (english summary). The third millennium edition, revised and expanded Springer Monographs in Mathematics. — Springer-Verlag, Berlin, 2003.
[23] *Kanamori A*. The higher infinite. Large cardinals in set theory from their beginning. Second edition. Springer Monographs in Mathematics. — Springer-Verlag, Berlin, 2003.
[24] *Quine W.V.* New foundations for mathematical logic //Amer. Mathem. Monthly. — 44. — 1937.
[25] *Shestopal V.E.* A theory of elements and sets. Preprint of Institute of theoretical and experimental physics 90-8. — Moscow, 1990.

# В КАКОЙ СТЕПЕНИ СОВРЕМЕННЫЕ МАТЕМАТИЧЕСКИЕ НАУКИ ЯВЛЯЮТСЯ НАДЕЖНЫМИ

**В.Х. Хаханян**

Московский государственный университет путей сообщения
*Образцова ул., 15, Москва, Россия, 101475*

В статье предлагается философский подход к обоснованию математики (теории множеств как математической дисциплины, лежащей в основании математики), базирующийся на полученных к настоящему времени математических результатах в области неклассических аксиоматических формальных систем. Дается краткая историческая картина развития понятия строгости математических доказательств и обоснования математики от древних греков до наших дней. В статье приводится ряд новейших достижений в области формализованных теорий арифметики (теории чисел), теорий действительного числа (математического анализа) и аксиоматических теорий множеств, которые рассматриваются с подлежащей интуиционистской логикой, а также в области классических дескриптивной и аксиоматической теорий множеств.

**Ключевые слова:** арифметика, математический анализ, аксиоматические системы теории множеств, интуиционизм, конструктивизм, базисная система, математика.