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Verification of a labor market domain using an academic crowdsourcing system

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Students desiring to become a valuable good in the labor market are willing to pay a considerable monetary cost to obtain knowledge about their prospective job opportunities, nowadays with a diminishing interest in the obtainment of a diploma. Considering the behavior of the labor market as a domain theory under uncertainty, it is straightforward to expect the presence of contradictions, in the form of salaries unable to be classified due to high inconsistency and variation. We provide an algorithm to verify a labor market domain theory based on a crowdsourcing academic system, in which feedback about possible contradictions is generated as a result of consultations with experts inside of the market and clustered into different contexts. We found that the verification process can be repeated iteratively as long as the students' overall tuition is equal or greater than a quantity partially defined by the number of different profiles of the students.

Key words and phrases: labor, market, crowdsourcing, curriculum, multiagent, verification, contradiction

1. Introduction

Assessment of the job market is an important practice in the field of academic curriculum development and in the election of a career path for prospective professionals before becoming part of the educational system [1]. Nowadays, however, students are more than ever concerned about correctly selecting a professional path in which they won't lose time in learning skills that they are not likely going to apply, even if it means the possibility of not obtaining a diploma [2].

As one of the popular topics in today's sharing economy, crowdsourcing [3], [4] has become a feasible tool for people to obtain knowledge at reasonable

costs from a set of experts in a field. Such experts, with varying levels of expertise, can be found in platforms like Amazon's Mechanical Turk [5]. In an academic environment, the roles of teacher and student (expert and requester) can also be integrated in a crowdsourcing system, and if a single agent is allowed to perform both roles interchangeably, then the agent can be further benefitted by an academic platform such as VUZ [6].

Entering an environment like VUZ requires a certain level of previous knowledge of the labor market [7], either compiled by the student or requested to experts. A platform like this, however, can also aid in gathering feedback [8] from the job market in order to actualize itself and optimize the benefits for the requesters and workers. Such feedback provides a way to verify [9] the previous knowledge about the market and points to possible improvements to the system.

In particular, the verification of a domain theory [10], in this case the labor market, allows to tackle all the possible issues that could cause a failure of a forecast, which in this work we take as the projected salary of a new professional in its field of expertise in the job market [11].

We structure this paper as follows: the basic concepts are detailed in the second section, before we explain the concept of labor market in the next part. The fourth section deepens into the concepts of verification and contradiction, while the fifth section focuses on the crowdsourcing system's ability to generate feedback and sustain itself in time.

1.1. Labor market

The labor market can be thought of as an open multiagent system (MAS) with limited resources [12] where agents seek to maximize their revenues by performing a balance between exploration [13] and working in exchange for a retribution. In the simplest case, requesters can be thought of as being separate from the workers [14] in the sense that their interrelations are not necessary evident to the workers.

1.2. Academic crowdsourcing

Agents can enter the open MAS freely, but often the process is channeled by an academic institution. In such cases, students agree to pay the institution more depending on its level of prestige in certain field of study [15]. Nowadays, however, prospective professionals are growingly looking for ways to land in the desired position in the market in a simplified way, often bypassing the obtainment of a diploma [16]. Personalized courses designed by a set of experts conform the core functioning of the academic platform VUZ and they correlate and complement the skills already mastered by the student, so that all excessive costs are avoided.

1.3. Verification

We take into account a formal system $G =< T, F, X, I >$ [17], where T is a finite set of basic characters known as the alphabet of the theory, F corresponds to a collection of formulas, X is referred to as a set of axioms and I is a set of inference rules. Then, the objective of the verification process is

to find all the discrepancies [18] among the elements of the formal system, starting with the set $T' \subseteq T$ that contains the object types, the properties and values of the types and the logical and functional relations between them.

1.4. Contradictions

Contradictions [19] arise in the formal system when the process of recognition of objects in the real environment [20] results in discrepancies with the entities and relations in the respective domain theory under which the recognition is performed. In particular, contradictions are reflected by the presence of 'impossible' elements of the alphabet [21] in the real environment as well as the absence of expected elements. On the other hand, they point to conflicts in the axioms [22] or contrasting states of the system caused by defects in the inference rules.

1.5. Retribution

A key point in the VUZ framework is the treatment of students as experts, along with the teachers [23]. This means that every level of expertise from the agents of the system is taken into account, and is utilized in order to teach other students with a lower level of knowledge in a determined field of study. As such, a student is required to pay a certain cost to obtain a crowdsourced assessment of his/her skills, as well as a personalized curriculum [24] for the obtainment of the desired new skills. The cost, however, is mitigated by giving the student the opportunity to share his/her knowledge with others in exchange for a retribution [25], [26]. This is an internal benefit from the academic crowdsourcing system, obtained prior to entering the labor market. Once in the market, the new professionals will obtain a real salary which will become the source of feedback for the system.

1.6. Profiles

The profiles of the agents in the crowdsourced system consist in a set of skills that gradually grows while the agents develops themselves as workers in the system [27]. The description of a skill is maximally general in order for the skill to be used as a variable in different kinds of tasks. In particular, the agent performs a task by adopting a certain role where a set of key skills are necessary and actively used.

On the other hand, roles and task performing are valid and can be evaluated and priced only when observed from the point of view of a certain context [28]. This implies that the same skills can have a different monetary value in different situations: contexts induce different partitions of the set of skills [29]. Like roles, contexts are described by means of a profile, albeit not of skills but of locators: they function as dimensions and can take different values ordered in a scale, giving the context a defined structure to answer to questions involving the place, time and other situational aspects of task performing.

1.7. Feedback

Gaining direct knowledge from the labor market is crucial in the development and functioning of the academic crowdsourcing system. Once a student leaves the system and joins the job market, he/she is nevertheless asked to communicate his/her salary and to provide a picture of the context in which he/she works, which can include a set of indicators such as weighted coefficients associated to quantitative and qualitative attributes [30]. Such picture is then introduced into the system and helps determine possible contradictions.

2. Model description

Let $(M, <)$ be a set of salaries ordered in ascending order, R a set of roles, C a set of contexts and $rt : R \times C \to M' \subseteq M$ a function assigning a collection of retributions to agent performing a certain role. Then the system is in presence of a contradiction when $\max(M') - \min(M') > \xi$, where ξ is an arbitrarily defined constant that we call the classification threshold [31].

It is assumed that further refinement of the set of skills and locators for the corresponding role and context will lead to a reduction in the constant ξ, ultimately pointing to the skills and location of a unique agent. This, however, is implausible in real environments, and for this reason we introduced the arbitrary constant into the model.

Such contradictions can be traced back to a single source, provided sufficient resources for analysis.

2.1. Incomplete role profile, correct context profile

The set of skills or locators associated to a role or context corresponds to a subset of the total collection of skills or locators known to the system, which in turn makes possible the overall classification of roles and contexts. Contradictions can be mitigated and the classification threshold attained by adding skills to a role known or suspected to be incomplete. This additional dimension can be partitioned into different values according to its scale, and a reduced role can be obtained by selecting the corresponding element from the partition.

2.2. Complete role profile, incorrect context profile

The same procedure can be applied in the case of incorrectly constructed context profiles. In particular, dimensions can be added to a locator in the context in order to refine the description of the time, place and other situational aspects.

2.3. Complete role profile, correct context profile

Contradictions can also arise when both the role profile and the context profile are guaranteed to be correctly constructed.

Assuming that every possible skill and locator is known to the system and available to be used, this scenario points to issues in the definition of the

parameters of the system. In other words, the assessment of the agent's skill levels and/or the locator value in the scale is not accurate and needs to be revised. On the other hand, there is the possibility that not all the skills or locators are known to the system, case in which it is necessary to engage in exploration of the entirety of the state space [32] in order to add the missing elements to the system.

3. Algorithmic implementation

3.1. A verification algorithm

We now propose an algorithm destined to fix the profiles after the detection of a contradiction. Let P be a set of (role and context) profiles and $d : P \times P \rightarrow [0, +\infty[$ a semimetric [33] among the profiles. On the other hand, let $S \subseteq E$ and $L \subseteq E$ be a set of skills and locators respectively, both belonging to a set of elements and yielding $P = \{(S', L')\}_{S' \subseteq S, L' \subseteq L}$. Then a set can be constructed such that each skill or locator will be labelled with the distance of its profile to a determined central profile.

$$D(p) = \{(e, d(p, q))|e \in E,\ e \notin S',\ e \notin L' \forall p = (S', L') \wedge p \neq q\}. \qquad (1)$$

Equation (1) allows to build an algorithm for gradually adding elements to profiles.

Algorithm 1 The addElements function

Input: a profile set P, a set E of profile elements, and a chosen $p \in P$
Output: a modified profile p

```
O_r ← D(p); a set of tuples of elements with distances
O*_r ← order(O_r); the set O_r in ascending order
S' ← skills(p); the set of skills of profile p
L' ← locators(p); the set of locators of profile p
for all o ∈ O_r do
  if o ∉ S' and o ∉ L' then
    p.add(o)
    if checkVerify(p) = true then
      return p
    end if
  end if
end for
```

The importance of the function `addElements` (Algorithm 1) is due to the property of the academic crowdsourcing system of continually receiving feedback from new professionals. In particular, each added element in the function corresponds to a new skill or locator which needs to be asked to experts in the labor market in order to refine the system. Their continual contribution will allow to perform the algorithm repeatedly, not forgetting to check for the validity of the profile on each iteration by means of the function

`checkVerify` (which yields $True$ if the desired classification threshold is attained).

3.2. An optimization algorithm

An optional but recommended continuation of `addElements` in Algorithm 1 corresponds to a function to clean (optimize) the resulting profiles. Due to the continual addition of elements to a specific profile in order to mitigate a contradiction, an excess of skills or locators is generated, which in turn calls for an optimization.

Algorithm 2 The cleanOptimize function

Input: a profile set P, a set E of profile elements, and a chosen $p \in P$
Output: a modified profile p

```
E' ← elements(p); the set of elements of profile p
C* ← combinations(E'); the set of combinations of elements of profile p
C ← combinations(E); the set of combinations of elements of all profiles
for all c ∈ C* do
  if c ∉ C − C' then
    k = 0
    for all e ∈ C do
      p.remove(e)
      if checkVerify(p) = false then
        p.add(e)
        k = k + 1
      end if
    end for
    if k = |c| then
      set c.isKey = true
    end if
  end if
end for
return p
```

Concretely, the proposed function `cleanOptimize` (Algorithm 2) takes all the skills and/or locators and subdivides them in groups different from the element sets of other profiles. Afterwards, the function checks by subtraction (by means of the function `checkVerify`) if the eliminated elements are necessary skills or locators, leaving only the key elements in the profile.

4. Discussion

4.1. A cost balance equation

We calculate the cost of performing the verification algorithm for a determined number of profiles in a time lapse, and compare it to the expected revenue

from students' tuition. Let $D_c = |S'| + |L'| + (|E| - (|S'| + |L'|)) * |P - 1|$ the cost of obtaining the set $D(p)$ and $O_c = \gamma(|D(p)| \log(|D(p)|))$ the cost of ordering the set

$$H_{\text{stud}} >= k * \beta * (D_c + O_c + 4 * |D(p)| + 2|E'| * 2|E'| - |E| * (2 + 2|c|)). \quad (2)$$

Then the verification algorithm and its subsequent optimization can be performed along k iterations as long as students' tuition (H_{stud}) is equal or greater than the expression in Equation (2), where γ corresponds to a cost constant for ordering and β is a defined constant for performing common operations.

4.2. Implementation outlook

This paper is incorporated into a body of work related to the processes of recognition and verification under uncertainty. We plan to integrate such concepts into an expanded notion of academic life cycle management, where crowdsourcing will be utilized not only to individually assist students in achieving a certain goal, but also coordinate them collectively so that a certain institution can attain its own objectives with their help.

We also study the implementation of means to store data and multimedia related to the academic platform, so that every bit of information needed by the students could be one day provided by the system without delay. In addition, we have also studied new ways to incorporate knowledge related to Big Data to correctly manage uncertainty from the recollection of the primary data.

On the other hand, we plan to extend the knowledge gained in the academic crowdsourcing environment to other fields such as the legal one. In particular, we hope to implement a crowdsourcing management system capable of registering normative events and linking them in a network of judicial decisions assessed by a set of experts with varying degrees of expertise.

5. Conclusion

In this work we presented an example of verification using a labor market as a domain theory. We outlined our verification approach based on the application of an academic crowdsourcing system, which serves as an introduction of new professionals into the open MAS of the job market. Students in the crowdsourcing system not only receive knowledge from experts, but also become experts according to their level of experience and knowledge. Once outside of the system, students become professionals and are expected to collaborate with the functioning of the system by providing feedback from the labor market.

In particular, information about the work salary is recollected from professionals in different contexts of the job market, and later processed in the system in order to identify contradictions in the understanding of the domain. Such contradictions arise in the form of salaries unable to be classified due to high inconsistency and variation. In our model, we characterized agent roles by means of skills and context by means of locators. We proposed a method to tackle salarial contradictions, which is based in the gradual addition and/or

removal of skills or locators in order to achieve a reduced salarial gap and attain the desired classification threshold.

Based on our proposed algorithms, we found that the verification process can be repeated iteratively as long as the students' overall tuition is equal or greater than a quantity defined by the sizes of the profile set and the element set, as well as the values of certain operational constants.

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Верификация рынка труда с помощью академической краудсорсинговой системы

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В настоящее время студенты, желающие иметь преимущество на рынке труда, готовы платить значительные деньги за информацию о потенциальных возможностях трудоустройства, тогда как получение диплома волнует их все в меньшей степени. Рассматривая поведение этого рынка труда в качестве теории предметной области в условиях неопределённости, ожидаются некоторые противоречия в виде уровней заработной платы, которые невозможно классифицировать из-за высокой противоречивости и изменчивости. Нами представлен алгоритм верификации теории предметной области рынка труда на основе краудсорсинговой академической системы, в которой обратная связь о возможных противоречиях формируется в результате консультаций с экспертами на рынке и группируется в различных контекстах. Нами обнаружено, что процесс проверки может повторяться итеративно, если общая стоимость обучения студентов равна или превышает количество, частично определяемое числом различных профилей студентов.

Ключевые слова: рынок труда, краудсорсинг, учебный план, мультиагент, верификация, противоречие

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Numerical determination of the singularity order of a system of differential equations

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We consider moving singular points of systems of ordinary differential equations. A review of Painlevé's results on the algebraicity of these points and their relation to the Marchuk problem of determining the position and order of moving singularities by means of finite difference method is carried out. We present an implementation of a numerical method for solving this problem, proposed by N. N. Kalitkin and E. A. Al'shina (2005) based on the Rosenbrock complex scheme in the Sage computer algebra system, the package CROS for Sage. The main functions of this package are described and numerical examples of usage are presented for each of them. To verify the method, computer experiments are executed (1) with equations possessing the Painlevé property, for which the orders are expected to be integer; (2) dynamic Calogero system.

This system, well-known as a nontrivial example of a completely integrable Hamiltonian system, in the present context is interesting due to the fact that coordinates and momenta are algebraic functions of time, and the orders of moving branching points can be calculated explicitly. Numerical experiments revealed that the applicability conditions of the method require additional stipulations related to the elimination of superconvergence points.

Key words and phrases: CROS, Finite-difference methods, sage, Calogero system, Painlevé property

1. Introduction

One of the main problems that arise in numerical analysis of systems of nonlinear ordinary differential equations is the appearance of moving singular points. It should be recalled that linear systems do not have such features and, therefore, the region of existence of the solution is always known in advance. In nonlinear problems, on the contrary, it is never clear beforehand whether a solution is defined for all considered values of the independent variable or not. Euler also noted that when approaching moving singular points of the solution of the Riccati equation, the approximate solution is increasingly deflecting from the exact one, and it has only recently been shown that, despite this, the finite difference method allows searching for the position and orders of moving singularities. In the present paper we report the implementation of this method in the computer algebra system [1], preceded by the necessary theoretical introduction.

2. Moving singularities of the solutions of ordinary differential equations

If a singular point of the general solution to an ordinary differential equation or a system of such equations depends on the integration constant, it is called a moving singular point [2]. The behavior of the solution in the vicinity of a moving singular point was studied by Painlevé in the very end of the 19-th century and was presented in his famous 'Stockholm Lectures' [3].

Theorem 1 (Painlevé, 1897; [2]). *A moving singular point of a solution of an ordinary first-order differential equation*

$$F(\dot{x}, x, t) = 0, \quad F \in \mathbb{Q}[v, x, t], \tag{1}$$

is always algebraic, i.e., in the vicinity of the singular point such a solution can be expanded in Puiseux series

$$x = C(t-a)^p + \dots, \quad p \in \mathbb{Q}. \tag{2}$$

Remark 1. Here it is assumed that the Puiseux series converges absolutely and uniformly in a certain vicinity of point $t = a$.

Remark 2. In some papers on numerical solution of ordinary differential equations such a singularity is referred to as a pole of the order $|p|$ even when $p \notin \mathbb{Z}$.

A generalization of this result to the case of a system having the form

$$\frac{dx_1}{f_1} = \dots = \frac{dx_n}{f_n} = \frac{dt}{f_0} \tag{3}$$

requires some stipulations. Let $f_0, \dots, f_n$ be polynomials from $\mathbb{Q}[t, x_1, \dots x_n]$. Generally the hypersurfaces

$$f_0(x_1, \dots, x_n, t) = 0, \dots, f_n(x_1, \dots, x_n, t) = 0 \tag{4}$$

have a finite number of crossing points in the projective space $\mathbb{P}^{n+1}$. Fixed singular points of the system's solution can be calculated as projections of these points on the t-axis.

Definition 1. System (3) is called singular, if the system of algebraic equations (4) has an infinite number of solutions.

Theorem 2 (Painlevé, 1897; [3]). *A moving singular point of a non-singular system* (3) *is always algebraic, i.e., in its vicinity the solution can be expanded in a Puiseux series:*

$$x_1 = C_1(t-a)^p + \dots, \quad p \in \mathbb{Q}. \tag{5}$$

Remark 3. All systems, in which $f_1, \dots$ do not contain t explicitly, i.e., autonomous systems, are singular in the sense of Definition 1. However, a similar theorem can be proved for the most important autonomous systems, e.g., the problem of n bodies [4].

3. Evaluation of the order of an algebraic singularity using the finite difference method

If the solution is unknown, it can be found using the finite difference method. In this regard, the following problem naturally arises

Problem [G.I. Marchuk, 2003]. For a given Cauchy problem and interval $a < x < b$ determine the position of moving singular points in this interval and their orders by analyzing one or several approximate solutions of the Cauchy problem.

Probably, many authors believed that this problem has no solution, since, as Euler noted, the finite difference method describes the solution the worse, the closer we get to a singular point. Nevertheless, G. I. Marchuk's problem was solved by N. N. Kalitkin and E. A. Al'shina in 2005 [5]. The proposed method was then approved and developed at Moscow State University [6]–[11]; the paper by A. A. Belov [6] contains historical information of great interest never published earlier.

The solution proposed by E. A. Al'shina can be described as follows.

1. There exist such schemes, e.g., the complex Rosenbrock scheme of the first order (CROS), for which the approximate solution tends to a finite value, when the exact solution has a pole.
2. At regular points the approximate solution $x(t_n, \Delta t)$ can be expanded in an asymptotic series

$$x(t_n, \Delta t) = \varphi(t_n) + r(t_n)\Delta t^r,$$

   where $x = \phi(t)$ is the exact solution and r is the order of approximation.
3. Therefore, at regular points the ratio

$$\frac{x(x_n, \Delta t) - x(t_n, \Delta t/2)}{x(t_n, \Delta t/2) - x(t_n, \Delta t/2^2)} \simeq \frac{1 - \frac{1}{2^r}}{\frac{1}{2^r} - \frac{1}{2^{2r}}} = 2^r.$$

A remarkable fact is that after the moving singularity it equals

$$\frac{x(t_n, \Delta t) - x(t_n, \Delta t/2)}{x(t_n, \Delta t/2) - x(t_n, \Delta t/2^2)} = 2^p,$$

where p is the order of algebraic singularity.

The complex first-order Rosenbrock scheme for autonomous system of differential equations

$$\frac{du}{dt} = F(u) \tag{6}$$

is written as

$$\begin{cases} \hat{u} = u + \tau \mathrm{Re}\, k, \\ \left(E - \dfrac{1+i}{2}\tau \dfrac{\partial F}{\partial u}\right) k = F(u). \end{cases} \tag{7}$$

Not perfectly rigorous, but convincing substantiation of the present approach for the CROS scheme: let in the vicinity of a singular point the behavior of the function be described by expression

$$u(t) \sim (t_0 - t)^{-\beta}.$$

Then the behavior of the derivative has the form

$$u'(t) \sim \beta(t_0 - t)^{-\beta-1} = \beta u^{1+\frac{1}{\beta}}.$$

Let us denote this "effective" right-hand side as $F(u)$. In this case

$$F_u = \beta \cdot \frac{\beta+1}{\beta} u^{\frac{1}{\beta}} = (\beta+1)u^{\frac{1}{\beta}}.$$

The Rosenbrock schemes have the form

$$\begin{cases} (1 - \alpha\tau F_u(u))\, k = F(u), \\ \hat{u} - u = \tau \mathrm{Re}\, k, \end{cases}$$

where τ is the time step of the grid.

In application to our right-hand side we get

$$\left(1 - \alpha\tau(\beta+1)u^{\frac{1}{\beta}}\right) k = \beta u^{1+\frac{1}{\beta}} \quad \Rightarrow \quad k = \frac{\beta u^{1+\frac{1}{\beta}}}{1 - \alpha\tau(\beta+1)u^{\frac{1}{\beta}}},$$

$$\hat{u} - u = \tau \mathrm{Re}\, \frac{\beta u^{1+\frac{1}{\beta}}}{1 - \alpha\tau(\beta+1)u^{\frac{1}{\beta}}},$$

or, for the complex Rosenbrock scheme, which is of interest for us, i.e., for $\alpha = (1+i)/2$,

$$k = \frac{\beta u^{1+\frac{1}{\beta}}}{1 - \frac{1+i}{2}\tau(\beta+1)u^{\frac{1}{\beta}}} = \beta u^{1+\frac{1}{\beta}} \frac{1}{1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}} - \frac{i}{2}\tau(\beta+1)u^{\frac{1}{\beta}}} =$$

$$= \beta u^{1+\frac{1}{\beta}} \frac{1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}} + \frac{i}{2}\tau(\beta+1)u^{\frac{1}{\beta}}}{\left(1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2 + \left(\frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2},$$

$$\operatorname{Re} k = \beta u^{1+\frac{1}{\beta}} \frac{1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}}{\left(1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2 + \left(\frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2}.$$

Thus,

$$\widehat{u} - u = \tau\beta u^{1+\frac{1}{\beta}} \frac{1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}}{\left(1 - \frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2 + \left(\frac{1}{2}\tau(\beta+1)u^{\frac{1}{\beta}}\right)^2},$$

from which it follows that the fixed point of the scheme is

$$u_0 = \left(\frac{2}{\tau(\beta+1)}\right)^{\beta}. \tag{8}$$

Now let us apply the Richardson estimate of the effective accuracy order, which has the form

$$p_{\text{eff}} = \log_r \frac{u^{(\tau/r)} - u^{(\tau)}}{u^{(\tau/r^2)} - u^{(\tau/r)}}, \tag{9}$$

where r is the grid densening factor (we used 3 grids with the step τ, τ/r and τ/r^2). From (8) and (9) we get

$$p_{\text{eff}} = \log_r \frac{\left(\frac{2r}{\tau(\beta+1)}\right)^{\beta} - \left(\frac{2}{\tau(\beta+1)}\right)^{\beta}}{\left(\frac{2r^2}{\tau(\beta+1)}\right)^{\beta} - \left(\frac{2r}{\tau(\beta+1)}\right)^{\beta}} = \log_r \frac{1}{r^{\beta}} = -\beta,$$

which demonstrates that the Richardson estimate of the effective accuracy order using the Rosenbrock complex scheme yields the order of the algebraic singularity.

4. The package CROS for Sage

The computer algebra system Sage [1] is perfectly suitable for computer experiments with both symbolic and numerical methods. Therefore, we decided to implement the above method of determining the singularity order in this system by means of a small package CROS [12], which allows the calculation of the position and orders of the moving singular points.

4.1. CROS

Function `cros(F,ics)` for the Cauchy problem described in terms of its arguments

$$\dot{u} = F(u), \quad u\big|_{t=a} = u_0, \tag{10}$$

considered on segment $a \leqslant t \leqslant b$, calculates the values of column u at $NS+1$ points of the grid, covering segment $[a,b]$ uniformly with the step $\Delta t = (b-a)/(NS)$. The function returns a list of values of column u, calculated at points

$$a + \frac{b-a}{N} j, \quad j = 0, 1, \ldots N.$$

Natural number S indicates by how many times the grid formed by division of the initial segment into N parts becomes denser.

Necessary arguments:

— F is a list of right-hand sides of the ODE, its element type is symbolic expression,
— ics is a list of initial data, its element type is equality of the form variable == value. The value can be any number or expression, arithmetic manipulations are supported by Sage.

Optional arguments:

— a is the initial value of variable t, by default $a = 0$
— b is the finite value of variable t, by default $b = 1$,
— N is the number of grid nodes before densening, by default $N = 10$,
— S is the densening index; ultimately the scheme uses the step $\Delta t = (b-a)/(NS)$, by default $S = 1$.

As initial conditions, it is recommended to take the numbers from $\mathbb{R}$, otherwise the calculations become extremely time-consuming, which is a specific feature of the CROS scheme.

Example 1. Consider the Cauchy problem

$$y'' = y^2, \quad y\big|_{x=0} = 1, \quad y'\big|_{x=0} = 1$$

on the segment $0 < x < 3$. Let us rewrite it in the form (10)

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = z \\ \dot{z} = y^2, \\ x = 0, \quad y = z = 1, \quad \text{t=0}. \end{cases}$$

When specifying the initial conditions let us use decimal fractions, which will be perceived by the system as numbers from $\mathbb{R}$.

```
sage: load('cros.sage')
None
sage: var('x,y,z')
(x, y, z)
sage: cros([1,z,y^2],[x==0, y==1.0, z==1.0], b=3.0)
[[0, 1.00000000000000, 1.00000000000000], [0.300000000000000,
1.33821049499058, 1.37883146513243], [0.600000000000000,
1.81581265489380, 2.05236796585663], [0.900000000000000,
2.53640347402030, 3.28908311525028], [1.20000000000000,
3.68660561742990, 5.67422154586290], [1.50000000000000,
```

```
5.58820279611495, 10.4804480613621], [1.80000000000000,
8.65508112242668, 20.0486307394803], [2.10000000000000,
12.8620917982321, 36.1835389093226], [2.40000000000000,
16.9996351976843, 54.5720610523068], [2.70000000000000,
19.8371868069049, 67.2778475556264], [3.00000000000000,
21.3360253982491, 72.9430480385885]]
```

For plotting (Figure 1) it is possible to use a standard procedure:

```
sage: sage: point([ [xx,yy] for [xx,yy,zz] in
  cros([1,z,y^2],[x==0, y==1.0, z==1.0], b=3.0)])
Graphics object consisting of 1 graphics primitive
```

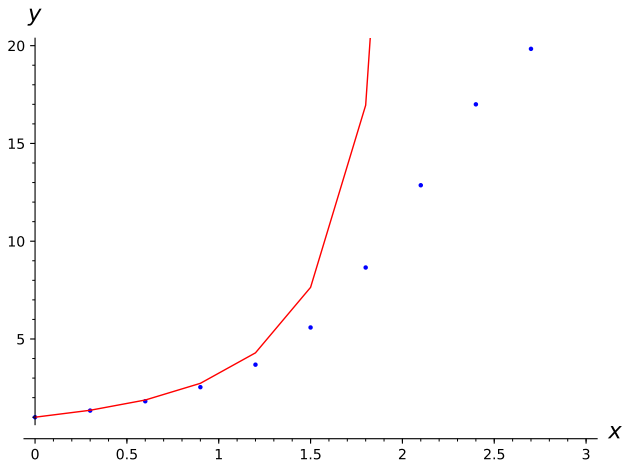

Figure 1. Solution of the initial problem from example 1 using the CROS scheme at $N = 10$, $S = 1$ (points) and $S = 4$ (line)

4.2. The order

Let $u = (x, y, \dots)$ and let u', u'', u''' be three solutions to problem (10) calculated by the CROS scheme. The first solution is obtained with the division of the segment by N points, the second by $2N$, and the third by 2^2N points. Then the expression

$$s_j = \log_2 \left| \frac{y''_j - y'_j}{y'''_j - y''_j} \right|$$

before a singular point asymptotically in Δt equals 2, and after the singular point it equals the order of the passed singularity. This expression is referred to as effective order of the scheme at the j-th node of the grid.

The function `eff_order(F,ics)` computes the effective order for the solutions of the Cauchy problem

$$\dot{u} = F(u), \quad u\big|_{t=a} = u_0,$$

using the CROS scheme on the segment $a \leqslant t \leqslant b$. The function returns a list of points (x'_j, s_j).

Necessary arguments:

— F is a list of right-hand sides of the ODE, its element type is symbolic expression,
— ics is a list of initial data, its element type is equality of the form variable == value. The value may be any number or expression, arithmetic manipulations are supported by Sage.

Optional arguments:

— a is the initial value of variable t, by default $a = 0$,
— b is the final value of variable t, by default $b = 1$,
— N is the number of grid nodes, by default $N = 10$.

As initial conditions it is recommended to take the numbers from $\mathbb{R}$, otherwise the computations appear to be extremely time-consuming (a specific feature of the CROS scheme).

Example 2. Let us calculate the effective order for the solution of the Cauchy problem

$$y'' = y^2, \quad y\big|_{x=0} = 1, \quad y'\big|_{x=0} = 1$$

from example 10 on the segment $0 < x < 3$.

```
sage: eff_order([1,z,y^2],[x==0, y==1.0, z==1.0], b=3.0, N=30)
[[0.100000000000000, 2.00989109022581], [0.200000000000000,
2.01268790858706], [0.300000000000000, 2.01483966814792],
[0.400000000000000, 2.01644106174018], [0.500000000000000,
2.01754741718014], [0.600000000000000, 2.01818124385354],
[0.700000000000000, 2.01833386808129], [0.800000000000000,
2.01796281442251], [0.900000000000000, 2.01698449712759],
[1.00000000000000, 2.01526061186748], [1.10000000000000,
2.01257491160427], [1.20000000000000, 2.00859410681839],
[1.30000000000000, 2.00280107081003], [1.40000000000000,
1.99437737792691], [1.50000000000000, 1.98198860002381],
[1.60000000000000, 1.96337299865009], [1.70000000000000,
1.93450889341422], [1.80000000000000, 1.88781839346059],
[1.90000000000000, 1.80800548867119], [2.00000000000000,
1.66166416882944], [2.10000000000000, 1.36965695426710],
[2.20000000000000, 0.734022444594345], [2.30000000000000,
-0.678328975943153], [2.40000000000000, -2.12961258492029],
[2.50000000000000, -2.03173073133296], [2.60000000000000,
-1.98022752373820], [2.70000000000000, -1.99096481400289],
[2.80000000000000, -1.99866158256879], [2.90000000000000,
-2.00066044407114]]
```

It is well seen that the singularity is somewhere near $x = 2.2$ and has an order of -2, see also Figure 2.

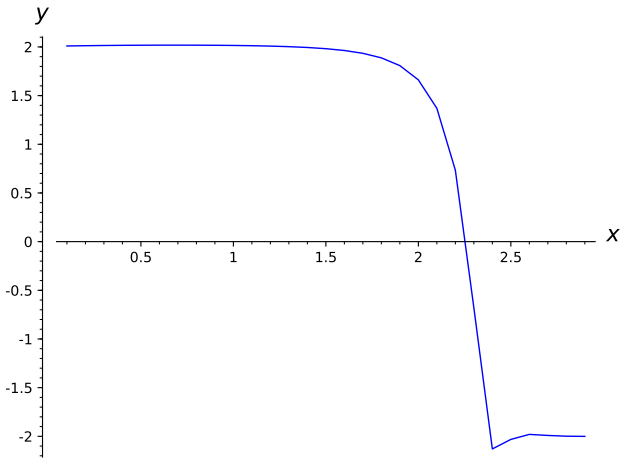

Figure 2. Effective order of the solution to the initial problem from example 1 using the CROS scheme

5. Testing the package

5.1. Examples with integer orders

An ODE is said to possess Painlevé property if the order p of all moving singularities is a negative integer [2]. We know all ODEs of the second order possessing this property, and this provides us with appropriate material for testing the developed package.

Example 3.

$$\begin{cases} y'' = 6y^2 + t, \\ y(0) = y(1) = 1. \end{cases} \tag{11}$$

Applying the CROS package we see that y has a pole of the second order. This result agrees well with V. V. Golubev's theorem (1912) [2, p. 199].

Example 4.

$$\begin{cases} y'' = 2y^3 + ty + 1, \\ y(0) = y(1) = 1. \end{cases} \tag{12}$$

We see that y has a pole of the first order. This contradicts V. V. Golubev's theorem (1912) [2, p. 200]. However, at present a number of solutions of the second Painlevé equation are known in rational functions, which have poles of the first order [13], [14, §32.8]. Therefore, the statement of the above theorem should be considered erroneous.

Example 5.

$$\begin{cases} y'' = \dfrac{(y')^2}{y} + e^t(z^2+1) + e^{2t}\left(y^3 + \dfrac{1}{y}\right), \\ y(0) = y(1) = 1. \end{cases} \tag{13}$$

We see that y has a pole of the first order.

5.2. Calogero system

Among mechanical systems, the Calogero system [15] is the most suitable for testing. Let us consider N material points of unit mass on a straight line, attracting or repulsing each other with the force inversely proportional to the cube of the distance. Let q_n be the position of the n-th point, then

$$\ddot{q}_n = -\frac{\partial U}{\partial q_n}, \tag{14}$$

where

$$U = \sum_{i<j} V(q_i - q_j), \quad V(x) = \frac{b}{|x|^2}.$$

This system is Hamiltonian with

$$H = \frac{1}{2}\sum p_n^2 + U(q)$$

and, what is most essential for the subsequent analysis, it can be written in the form

$$\frac{d\hat{L}}{dt} = [\hat{A}, \hat{L}], \tag{15}$$

where

$$\hat{L}(p,q) = \operatorname{diag}(p_1, p_2, \dots, p_N) + i\left(\frac{1-\delta_{jk}}{q_j - q_k}\right)$$

and

$$\hat{A}(p,q) = i\operatorname{diag}(d_1, d_2, \dots, d_N) - i\left(\frac{1-\delta_{jk}}{(q_j - q_k)^2}\right), \quad d_k = \sum_j \frac{1-\delta_{jk}}{(q_j - q_k)^2}.$$

This representation was guessed and can be checked by direct substitution.

Equation (15) means that the eigenvalues of matrix $\hat{L}(p(t), q(t))$ are independent of t. Indeed, let us introduce the matrix $\hat{U}(t)$ as a solution to the Cauchy problem

$$\frac{d\hat{U}}{dt} = \hat{A}\hat{U}, \quad \hat{U}(0) = E.$$

A conjugate matrix satisfies the equation

$$\frac{d\hat{U}^*}{dt} = -\hat{U}^*\hat{A}$$

and, therefore,

$$\frac{d\hat{U}\hat{U}^*}{dt} = \hat{A}\hat{U}\hat{U}^* - \hat{U}\hat{U}^*\hat{A} = -[\hat{A}, (\hat{U}\hat{U}^*)], \quad \hat{U}\hat{U}^*(0) = E.$$

The only solution to this Cauchy problem is $\hat{U}\hat{U}^* = E$, therefore, $\hat{U}$ is a unitary matrix. It remains to note that

$$\frac{d\hat{U}^*\hat{L}\hat{U}}{dt} = -\hat{U}^*\hat{A}\hat{L}\hat{U} + \hat{U}^*[\hat{L},\hat{A}]\hat{U} + \hat{U}^*\hat{L}\hat{A}\hat{U} = 0,$$

so that

$$\hat{U}^*(t)\hat{L}\big|_t\hat{U}(t) = \hat{L}\big|_0,$$

i.e., the eigenvalues of the matrices $\hat{L}\,(p(t),q(t))$ and $\hat{L}\,(p(0),q(0))$ on any solution of (14) coincide, which was to be proved.

The proved statement means that the eigenvalues of matrix $\hat{L}(p,q)$ are integrals of motion for system (14). It is convenient to use them for constructing symmetric functions

$$F_k(p,q) = \operatorname{Sp}\hat{L}(p,q)^k, \quad (k = 1,\dots N),$$

which will be rational integrals of motion. These integrals are in involution. It is most simply seen from the fact that the repulsing particles at $t \to +\infty$ will spread (i.e., $|q_j - q_k| \to \infty$). Therefore

$$F_k = \sum p_j^k + \dots$$

and (F_k, F_r) tends to zero on any trajectory. On the other hand, the Poisson bracket is an integral of motion. Therefore, this expression must be identically zero, i.e., $(F_k, F_r) = 0$. Hence, *the Calogero system is completely integrable and has N rational integrals of motion.*

The description of solution used above can be derived as follows: for the matrix

$$\hat{Q}(q) = \operatorname{diag}(q_1,\dots q_N)$$

it is valid that

$$\frac{d}{dt}\hat{U}^*\hat{Q}\hat{U} = -\hat{U}^*\hat{A}\hat{Q}\hat{U} + \hat{U}^*\operatorname{diag}(p_1,\dots)\hat{U} + \hat{U}^*\hat{Q}\hat{A}\hat{U} =$$
$$= \hat{U}^*([\hat{A},\hat{Q}] + \operatorname{diag}(p_1,\dots))\hat{U} = \hat{U}^*\hat{L}\hat{U} = \hat{L}\big|_{t=0},$$

from where using the Taylor formula we immediately have

$$\hat{U}^*\hat{Q}\hat{U} = \hat{Q}\big|_{t=0} + t\hat{L}\big|_{t=0}.$$

Therefore, *general solution $q_1 = q_1(t),\dots$ is a set of eigenvalues of the matrix*

$$\hat{Q}\big|_{t=0} + t\hat{L}\big|_{t=0}$$

and is given by algebraic functions of t and initial data.

Thus, the Calogero system is an example of a system with rational Hamiltonian $H(p,q)$, which has N rational integrals in involution, and whose general

solution is provided by algebraic, but not rational functions of t and initial data.

Two bodies attract, the order of singularity 1/2 (Figure 3):

```
sage: var('t, q1,q2,p1,p2')
(t, q1, q2, p1, p2)
sage: L=cros([1, p1,p2,  diff(1/(q1-q2)^2,q1),
diff(1/(q1-q2)^2,q2)],
[t==0, q1==0.0, q2==1.0, p1==0.0, p2==0.0], N=10^2)
```

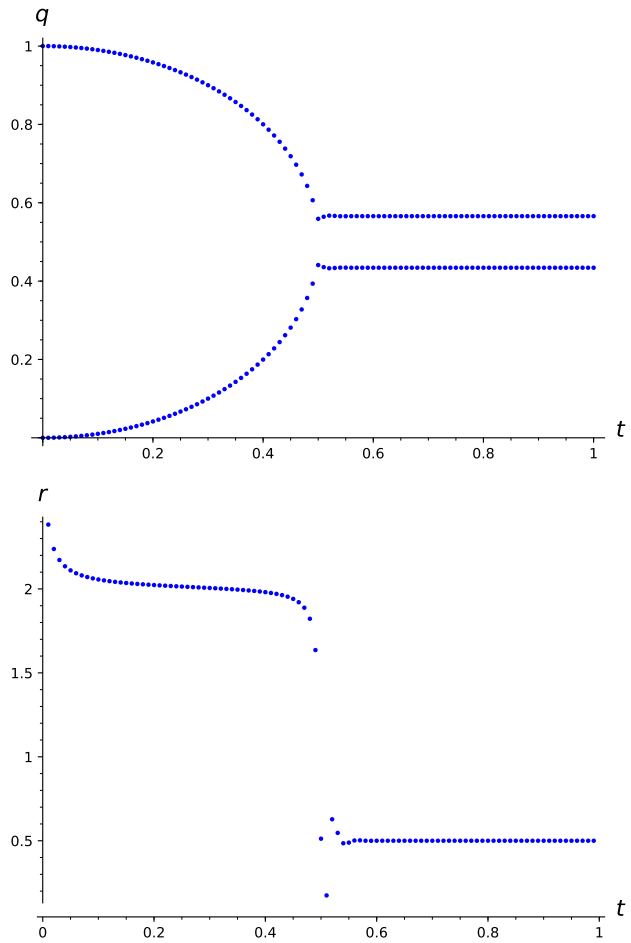

Figure 3. Collision in the two-body problem

Three bodies attract, the order of singularity at the collision points 1/2 (Figure 4):

```
sage: var('t, q1,q2, q3, p1, p2, p3')
(t, q1, q2, q3, p1, p2, p3)
sage: V=(1/(q1-q2)^2 + 1/(q1-q3)^2 + 1/(q2-q3)^2)
sage: L=cros([1, p1,p2,p3, diff(V,q1),diff(V,q2),diff(V,q3)],
[t==0, q1==0.0, q2==1.0, q3==3.0, p1==0.0, p2==0.0, p3==0],
b=1, N=10^2)
```

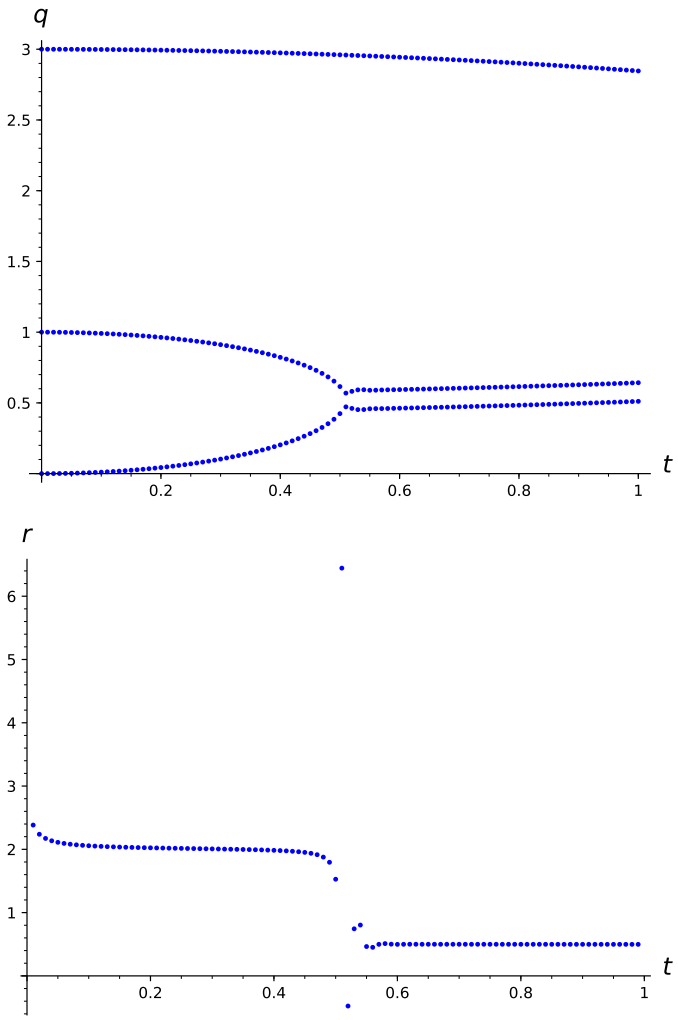

Figure 4. Collision in a three-body problem

5.3. Points of superconvergence

Experimentally, it was found that in some cases the use of the proposed technique leads to false operation. Thus, e.g. the Cauchy problem

$$\begin{cases} y'' = y - \dfrac{y^3}{6}, \\ y\big|_{x=0} = y'\big|_{x=0} = 0 \end{cases}$$

on the segment $0 < x < 1$ using the CROS method is solved facing no singularities. For sure, in Figure 5 we compare solutions found using the CROS and rk4 schemes. However, the plot of effective orders (Figure 6) demonstrates strange peculiarities.

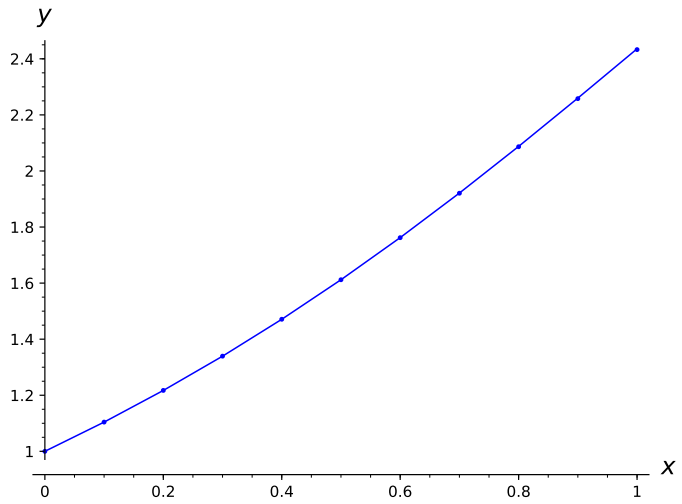

Figure 5. Solutions found using the CROS scheme (line) and rk4 (dots)

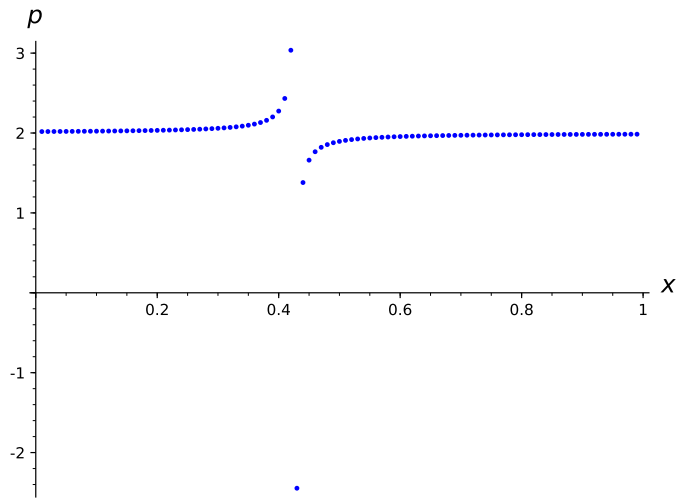

Figure 6. Effective order

The efforts of reducing the step and minimizing the rounding error did not lead to any changes, which suggests that this artifact is not a defect of the program. We explain it as follows. In theory the formula

$$y_n - \phi(x_n) = g_n \Delta x^2 + O(\Delta x^3)$$

is used assuming without any justification that $g_n \neq 0$. In the present case we deal with a single node, in which $g_n = 0$. This fact is well seen in Figure 7. Thus, the diagram of effective order indicates not only moving singular points, but also "removable" singular points, at which a superconvergence of the scheme takes place.

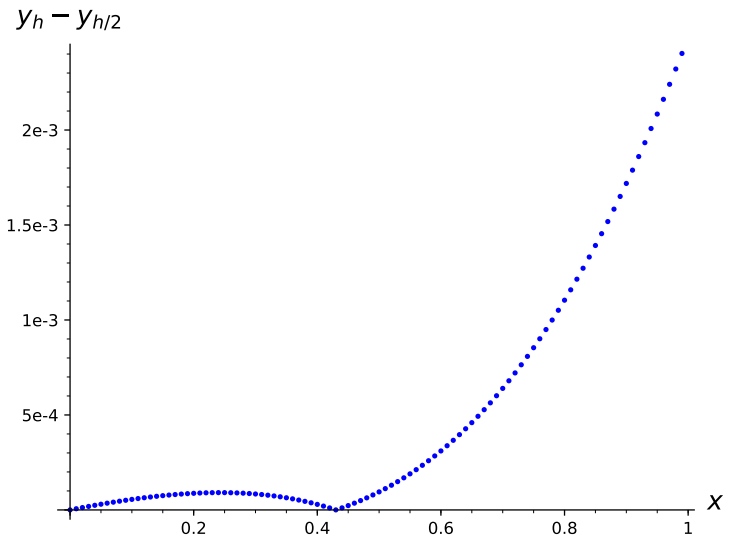

Figure 7. Plot of difference between two approximate solutions calculated on grids with the step $\Delta x = 0.01$ and 0.005

Remark 4. Note that the point $x = 0$ is always such a point of superconvergence, due to which at the beginning of the plot strange fluctuations are always observed.

The revealed effect means that a rigorous substantiation of the method for determining the order of singularities requires elimination of a certain number of special cases, including superconvergence.

6. Conclusion

Numerical experiments convincingly verify the numerical method for determining the position and order of moving singular points of ordinary differential equations, based on the Rosenbrock difference scheme. Moreover, this method allows easy correction of errors in the order of singular points determined in the course of theoretical studies of equations that possess the Painlevé property. On the other hand, we revealed a phenomenon of false operation of the algorithm of moving singular point recognition at the points of superconvergence.

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The authors are grateful to A. A. Belov (Moscow State University) for discussing the topics covered in this article. All calculations presented in the paper are executed in the computer algebra system Sage [1].

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Численное определение порядка особенности системы дифференциальных уравнений

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В статье рассматриваются подвижные особые точки систем обыкновенных дифференциальных уравнений. Дан обзор результатов Пенлеве об алгебраичности этих точек и их связи с задачей Г. И. Марчука об определении положения и порядка подвижных особых точек по методу конечных разностей. Представлена реализация численного метода решения этой задачи, предложенная Н. Н. Калиткиным и Е. А. Альшиной (2005) на основе комплексной схемы Розенброка, в системе компьютерной алгебры Sage — пакет CROS for Sage. Описаны основные функции этого пакета, приведены численные примеры использования каждой из них. В целях верификации метода проведены компьютерные эксперименты: (1) с уравнениями, обладающими свойством Пенлеве, для которых порядки должны получаться целыми числами; (2) с динамической системой Калоджеро.

Эта система, хорошо известная как нетривиальный пример вполне интегрируемой гамильтоновой системы, в данном контексте интересна тем, что координаты и импульсы являются алгебраическими функциями времени, причём порядки подвижных точек ветвления можно вычислить явно. В рамках численных экспериментов обнаружено, что условия применимости метода требуют дополнительных оговорок, связанных с исключением точек суперсходимости.

Ключевые слова: CROS, метод конечных разностей, Sage, система Калоджеро, свойство Пенлеве

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Simulation of non-stationary event flow with a nested stationary component

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A method for constructing an ensemble of time series trajectories with a non-stationary flow of events and a non-stationary empirical distribution of the values of the observed random variable is described. We consider a special model that is similar in properties to some real processes, such as changes in the price of a financial instrument on the exchange. It is assumed that a random process is represented as an attachment of two processes – stationary and non-stationary. That is, the length of a series of elements in the sequence of the most likely event (the most likely price change in the sequence of transactions) forms a non-stationary time series, and the length of a series of other events is a stationary random process. It is considered that the flow of events is non-stationary Poisson process. A software package that solves the problem of modeling an ensemble of trajectories of an observed random variable is described. Both the values of a random variable and the time of occurrence of the event are modeled. An example of practical application of the model is given.

Key words and phrases: non-stationary time series, non-stationary flow of events, modeling of an ensemble trajectories

Introduction

In [1]–[4], a model is presented for predicting the sample distribution function of a non-stationary time series over a certain horizon determined by the level of non-stationary series. The non-stationarity level is a special statistic that is collected from end-to-end samples of a given length, in the form of a distribution of distances between sample distributions in the C norm. The result of these works was the creation of a software package that generates an ensemble of time series trajectories, the distribution of which evolves in accordance with a kinetic equation that preserves the normalization and meets the observed properties of the series: preserving the trend or changing it to the opposite. The time in these works was considered to be the sequence number of the event, i.e. the observation of a random process was carried out at constant intervals.

In practice, there are often situations where time intervals themselves are a random process. This is the specifics of Queuing systems, a special case of

which is the dynamics of exchange transactions [5]–[7]. The purchase price of a financial instrument and the time interval between two consecutive acts of sale are two dimensions that characterize this time series. Similar properties are found in the series of durations of telephone or Internet connections, sequences of earthquake magnitudes, polluting emissions in megacities, and other events, the moments of occurrence of which, as well as their values, are random.

Stock market forecasting using time series analysis has been considered by a vast number of research papers. Among the works most related to the topic we should note [8], which analyses point process models that account for the market noise, and various applications of theoretical models [9], [10] towards describing the price movements of financial instruments. Notably, neither of the existing models considers a possibility of nonparametric simulation for the ensemble trajectory analysis of two-dimensional time series (a moment of the transaction – a result of the transaction). The given article proposes an approach to modelling price fluctuation trajectories changing their statistical properties with time.

Traditional time series analysis uses assumptions about the stationarity of the corresponding distribution function (hereinafter referred to as FD). The corresponding methods are described in textbooks on mathematical statistics [11] and books on market analysis methods [12]. These methods include: regression trend selection in the sense of MNC; time series co-integration, which forms a stationary time series (Box-Jenkins, 1972); autoregressive models (Dickey-fuller, 1979).

Adaptive time series models are also considered: multiparametric models of short-term forecasting, in which part of the parameters at each next step in time changes depending on the mismatch of the forecast and the fact (brown, Holt, winters, 1990-2000), as well as models of weighted moving averages.

In the case of non-equidistant time series, we consider QMS models with stationary event flows of various types [13] or, alternatively, systems with double stochasticity [14]. Other stochastic models are also used (A. N. Shiryaev [15] et al., see, for example., [16]), in which the properties of stationary random processes are investigated.

As a result, the results of the analysis of stationary models in practice depend on the sample length and on the current time point. This imposes restrictions on the reliability of the results obtained when testing certain management strategies.

Generating an ensemble of trajectories of a non-equidistant non-stationary time series is thus a practically important task, the solution of which will allow modeling various control functions of the observed random process and optimizing them. This paper presents a software package that implements a time series model with embedding processes of different levels of stationarity.

1. Method for generating a non-equidistant time series

Generation of a non-equidistant non-stationary time series is based on the following assumptions about the structure of the event flow [5], [6]:

— there is a certain period of time, called the period, within which the normalized per unit function of the flow intensity is set;

— there is a relatively small part (the first 10–15 % of the period) of the time interval that allows us to estimate the predicted number of events for the period, so that in fact the time series model is built on the remaining part of the period after making the appropriate observations for the start of the process;
— we consider a sequence of events with the same values that are most likely (for example, a sequence of absolute price increments of consecutive transactions excluding zero increments), called the "first series";
— the duration of the first series by the number of events is a non-stationary random process;
— the sequence of values of other events is considered ("second series");
— the duration of the second series by the number of events is a stationary random process;
— the distribution of trend movements over time intervals is a stationary random process, the actual trend is realized by skewing the probability of price increments to take positive or negative values.

The assumptions made allow us to build a model of a time series that has properties close to those observed in practice. In particular, in modeling the price movement of individual transactions on the exchange the most likely increase is one point by absolute value. Let's describe the input data for this particular problem.

At the first stage of preparing data for modeling the trajectory of a time series, statistics are collected:

— distribution function $F(\theta)$ the expert selected the trend of price movements for the duration of time θ general movement of the price trajectory up or down;
— probability $P^{\pm}$ positive and negative price growth on expert selected fragments of trend movements, $P^{+} + P^{-} = 1$;
— parameter of non-stationary Poisson event flow $\Lambda(t, \tau)$ at a time interval $\Delta_t(\tau) = [t - \tau;\ t]$ inside the period T (in relation to stock exchanges this is a single trading session);
— distribution function $G_{x>1}(n)$ series of increments, the absolute value of which x more than one conditional item, depending on the number n events;
— joint distribution density $f_{x=1}(k, k'; K, t)$ lengths k and their increments k' for a series of absolute price increments per conditional item on the sample length K events in a moment of time t.

The statistics collected determine the probability $p_k(t - \tau, t)$ number k events over time $\Delta_t(\tau)$ formula

$$p_k(t - \tau, t) = \frac{(\Lambda(t, \tau))^k}{k!} \exp(-\Lambda(t, \tau)), \quad \Lambda(t, \tau) = \tau\mu(t - \tau, t). \qquad (1)$$

The value entered here $\mu(t - \tau, t)$ called the intensity of the flow in the interval $\Delta_t(\tau)$. This is the average number of events over the specified period. It is defined by the formula

$$\mu(t - \tau, t) = \frac{1}{\tau} \sum_{k=1}^{\infty} k p_k(t - \tau, t). \qquad (2)$$

We believe that events are independent, and the flow is ordinary. We assume the time of aggregation of events to be equal $\tau = 1$ minutes.

Then set the expected number N events on the time horizon T time series simulation. It is necessary in order to perform the normalization of the intensity profile $\Lambda(t, \tau)$ that's the number of events.

At the next stage from the distribution $F(\theta)$ the random series of numbers θ_k is generated in units of measurement of the time taken in the flow parameter, and

$$\sum_k \theta_k = T. \tag{3}$$

Condition (3) determines the total number M macro-movements up and down and their duration, at each interval θ_k the probability P_k^+ of price movements in a single event up is set and thus the probability of $P_k^- = 1 - P_k^+$ price movements down is determined.

Random whole numbers n_j are then generated out of distribution (1), that give a number of events during the minute 1 at intervals $\Delta_1(j)$, where j is the current minute number. There is a number of events for this generation (i.e. deals)

$$N = \sum_{j=1}^{T} n_j. \tag{4}$$

Next, a sample of numbers ± 1 is generated with total length N from the piecewise-stationary distribution of probabilities $P_k^{\pm}$ according to the random number of macro movements out (3). This sample identifies a price increment sign in a single event. From the density of the distribution function $f_{x=1}(k, k'; K, t)$ by method [3] there are features

$$\begin{aligned} \varphi_{x=1}(k; K, t) &= \sum_{k'} f_{x=1}(k, k'; K, t), \\ u(k; K, t)\, \varphi_{x=1}(k; K, t) &= \sum_{k'} k' f_{x=1}(k, k'; K, t), \end{aligned} \tag{5}$$

which are involved in the construction of the Liouville equation to simulate the evolution of distribution $\varphi_{x=1}(k; K, j)$ from a time interval $\Delta_1(j)$ in the interval $\Delta_1(j+1)$:

$$\begin{aligned} \varphi_{x=1}(k; K, j+1) &= \varphi_{x=1}(k; K, j) + \\ &+ \varphi_{x=1}(k-1; K, j)\, u(k-1; K, j) - \varphi_{x=1}(k; K, j)\, u(k; K, j). \end{aligned} \tag{6}$$

Thus, from formula (6), non-stationary distributions of the lengths of series of increments per conditional unit are known. Functions $f_{x=1}(k, k'; K, t)$ are calculated in the sliding window, so that their appearance is also affected by the flow parameters selected in the previous stages of the simulation and the lengths of up and down trend intervals.

After the functions $\varphi_{x=1}(k; K, j)$ are calculated, samples of lengths are constructed from them as from analogs of general aggregates $k_{1,j}, k_{2,j}, \ldots$ in such an amount that their sum is equal to the predicted number of transactions

from (4):

$$\sum_i k_{i,j} = n_j. \tag{7}$$

The length of a series of increments per conditional unit is interrupted by a series of increments of large values. Series of the second type, as already mentioned, have a stationary distribution $G_{x>1}(n)$ in length. A random set of integers is generated from this distribution $n_{1,j}, n_{2,j}, \dots$, equal to the series lengths of the specified second type. Further, the lengths of the series $k_{i,j}$ and $n_{i,j}$ alternate until their total length is equal n_j. Then a similar construction begins in the next time interval $\Delta_1(j+1)$.

The increment signs in all these transactions are determined by a sequence of random signs ± 1, which was generated in the previous stages of the simulation.

The generation of a time series with a stationary distribution function is based on the usual algorithm, which is based on the following statement (see, for example, [3]). Let ξ be a random variable with continuous FD $F(x)$. Then a random variable $\eta = F(\xi)$ has a uniform distribution on [0;1]. Then at the first step we generate an arbitrary sequence of numbers $\{y_k\}$, evenly distributed on $[0; 1]$, then according to the formula

$$y_k = F(x_k), \quad x_k = F^{-1}(y_k) \tag{8}$$

the elements of the series $\{x_k\}$ can be calculated. Appeal FD into (8) is possible because of its strict monotony.

If FD the series is not stationary, we used a model of the evolution of sample density distribution function (next SDDF), so on a given forecast horizon τ by length selection T forecast data SDDF $f_T(x, t+k),\ k = 1, 2, \dots, \tau$ are constructed. After that a stationary evenly distributed on $[0; 1]$ series of numbers $\{y_k\}$ length τ, equal to the forecast horizon. Selective ones FD $F_T(x, t+k),\ k = 1, 2, \dots, \tau$ are also being built according to the model SDDF $f_T(x, t+k)$. Let t_0 be an initial point in time at which the forecast begins to be built. Then, in subsequent moments of time one of the possible trajectories of a random process for which SDDF changes from $f_T(x, t_0)$ to $f_T(x, t_0+\tau)$, is modeled using the formula for the reversal of the corresponding time-local distribution function moving in a sliding window T:

$$y_k = F_T(x_k, t_0 + k). \tag{9}$$

Thus, a model of the event trajectory for a single trading session is built.

2. Algorithm for modeling unsteady flow of events

In practice, the flow parameter $\Lambda(t, \tau)$ is built directly on observations.

Let n_j be a number of events in j minute of the day. Then the number of events in a day is

$$N = \sum_{j=1}^{1440} n_j, \tag{10}$$

and the daily rationed intensity profile is determined by the formula

$$y_j = \frac{n_j}{N}. \tag{11}$$

After that, the average number of events (i.e., the actual flow parameter) in the interval $\Delta_t(\tau)$ can be entered by building a weighted average daily activity profile for a certain period of time T days'. To do this, enter the average intensity $w(j)$ in j minute and average number of ticks N for day. Then the weighted average normalized activity profile is determined:

$$z(m) = \frac{w(m)}{N}. \tag{12}$$

Thus, let the average number of events per day be defined and equal to N. Than average number of ticks for interval τ (minutes) until the time t (minutes) is

$$\Lambda^{(w)}(t,\tau) = \sum_{m=1}^{\tau} w(t-m+1) = N \sum_{m=1}^{\tau} z(t-m+1). \tag{13}$$

Because the profile $z(m)$ is rationed per unit, it can be considered as a probability of intensity by minutes in a day. Its distribution function is there

$$Z(m) = \sum_{k=1}^{m} z(k), \tag{14}$$

that's why

$$\Lambda^{(w)}(t,\tau) = W \cdot (Z(t) - Z(t-\tau)) = \sum_{k=0}^{\tau-1} \Lambda^{(w)}(t-k,1). \tag{15}$$

It defines the event flow model for any moments in time t and intervals τ.

3. The structure of the software complex

This section contains information about the software package for modeling and calculating statistics for non-stationary non-equidistant time series [16].

1. *General information.*
   (a) Program Name – "Module for modeling and calculating statistics for non-stationary non-equidistant time series" NSTS.
   (b) The complex requires.NET 4.0.
   (c) The system is designed using programming languages – C#/C++.
2. *Functional purpose of the complex.*
   (a) The complex is designed to build a set of non-stationary time series that have properties characteristic of a given series or set of series. The possibility of calculating standard sample statistics for non-stationary time series with a random distribution of time intervals between consecutive events is implemented.

(b) Module NSTS implements following main functions:
- — generating a specified number of non-stationary time series with a non-stationary event flow;
- — calculation of sample statistics for a set of time series;
- — calculation of sample statistics for functional values defined along the trajectory of such two-dimensional time series.

(c) There are no functional restrictions on the described operations.

3. *Description of the logical structure.*

(a) The complex consists of the following main components:
- — check the validity of input data;
- — calculation of the tick density mask depending on time;
- — the calculation of the density distribution of tick increments;
- — calculation of the density of the distribution of consecutive tick durations with the value of the most likely increase;
- — statistical functions block;
- — block of possible functions of financial mathematics for building statistics;
- — building a stationary series for the most likely absolute increment;
- — construction of non-stationary series based on tick increment distribution masks;
- — combining stationary and non-stationary components.

(b) The operation of the non-equidistant time series generation module is based on a method based on the decomposition of the considered time series into stationary and non-stationary components. First, the distribution function of the studied random variable is constructed by its value, and the highest probability is found. Next, we consider sequences of events of two types: those consisting only of the values that have the highest probability, and all the others. For each of the two types of sequences, distributions of these sequences by length are constructed, and then the resulting distributions are tested for stationarity. If one of the components is stationary with the accepted accuracy, then we believe that the filtering has been performed and the model is adequate.
The program diagram is shown in Figure 1.

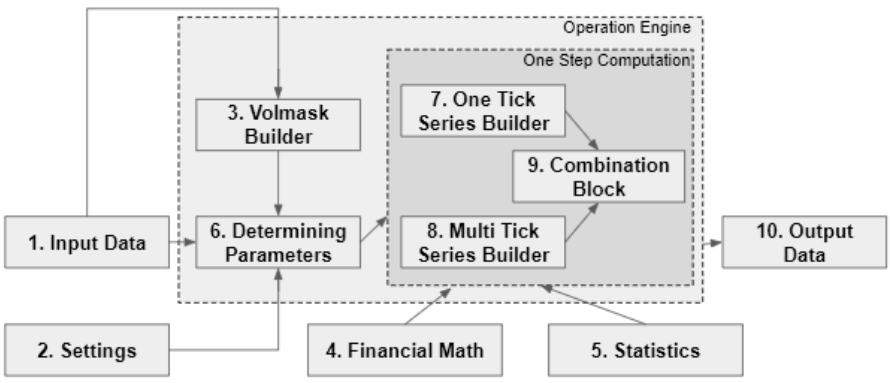

Figure 1. Diagram of the time series generation module

(c) Calculation of statistics for a non-stationary marked time series is carried out in 2 stages. On the first one, a matrix of statistics values is built, where the tick number is located on one line, and the number of points for calculation is located on the column. In the next step, assuming that the event flow is ordinary and a number of event moments are described by the Poisson distribution, the resulting tick density mask is used to move from this matrix to the results in terms of moments and time intervals.

4. *Input data.*
   (a) The input data is:
      — a time series presented in the format of a set of records with values of a random variable;
      — the time points at which these values were recorded.
   (b) The settings block for generating a series bundle includes the following options:
      — target number of rows;
      — time interval for generation;
      — the density distribution of tick increments;
      — the level of trend slope for the target series.
   (c) *Output data.*
      (i) Output in the generation module is a set of time series.
      (ii) The output in the statistical calculation unit is a series of calculated values and moments of time in which these values were obtained.
      (iii) Also in the output are distributions of statistics calculated by the full file as functions from the length of the sample, such as:
         — volatility;
         — autocorrelation;
         — the Hurst exponent.

4. Example of a computational experiment

The time series of tick increments of the RTS index is considered. A fragment of the original series is shown in Figure 2.

Distribution of absolute price increases during the trading session is given on Figure 3. One point on the chart corresponds to an increase of 10 points in the RTS index.

The non-stationary index (see [3]) for this series is shown in Figure 4. A series is considered stationary if the index is less than or equal to one. Otherwise, the series is non-stationary at the election of the corresponding lengths.

From Figure 4 it follows that the distribution function of absolute increments of a number of distinct ticks of the RTS becomes stationary at the length of 7 thousand events and then remains stationary. The most noticeable unsteadiness is manifested at the length of 2 thousand ticks. On the one hand, it would be convenient to work with a stationary distribution. However, the sample distribution becomes stationary around the end of the daily trading session, whereas decisions must be made based on data for shorter periods of time, when the distribution is significantly non-stationary.

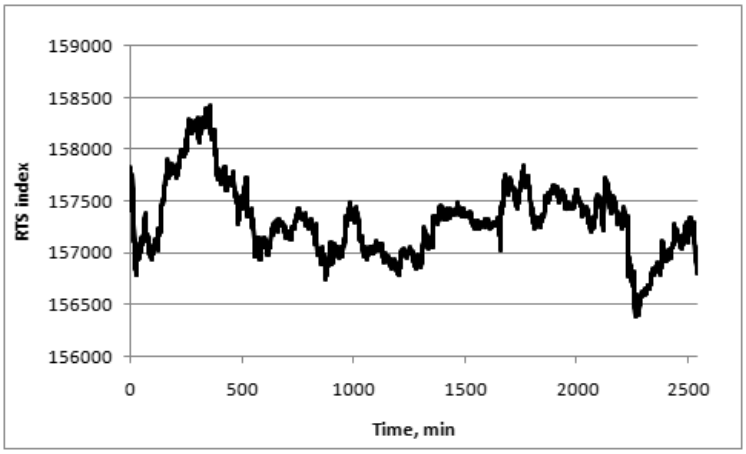

Figure 2. Fragment of the RTS index series

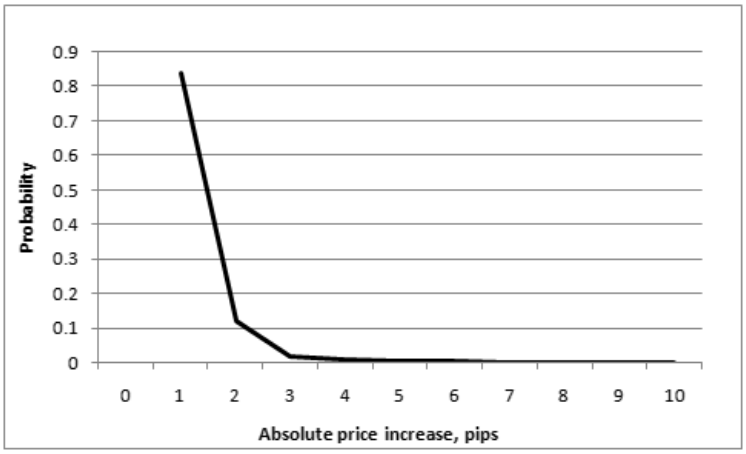

Figure 3. SDDF number of absolute increments in conditional points

The most likely is an increase in the price of one conditional point in absolute value, the probability of this event is 0.84. As a result of filtering, the initial tick series of absolute increments is represented as an alternation of two rows – increments by 1 point and other increments. Elements of each of the ranks are integers in the duration of episodes of each type. For series from the duration of the series, the non-stationary indices are considered (Figure 5).

From the graphs on Figure 5 it can be seen that the non-stationary index of the first row is greater than one in samples up to 10 thousand, while the second row is approximately stationary in almost all samples. This means that the nonstationarity is inherent in the sequence of increments by 1 absolute point, because this series is nonstationally interrupted by the second series, the duration of which is a stationary random process.

Note that a trading day contains an average of 250 thousand ticks, of which about a quarter (i.e. only 60 thousand) are ticks with non-zero increments. Then an average of 4 thousand events of the second type occur per day.

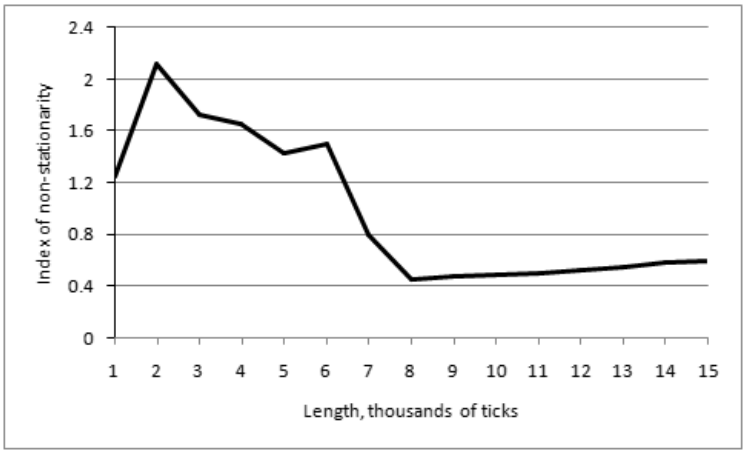

Figure 4. The index of nonstationarity of the absolute increases a number of distinctive tics of RTS

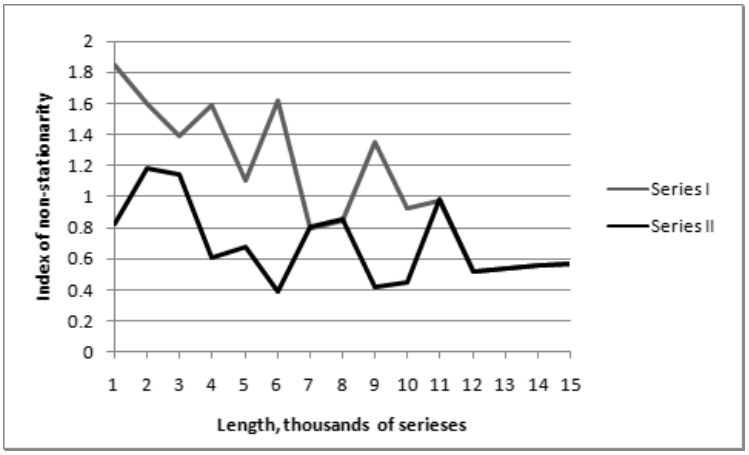

Figure 5. Index of non-stationary series series I and II types

As can be seen from Figure 4, at such lengths, the first row does not yet become stationary, but for the analysis of intra-day changes in the distribution function of the first type, this is no longer relevant, because the day has ended. Therefore, it is interesting to model the time series of durations of the first type of series on samples of smaller lengths, for example, on samples of lengths of 1–2 thousand ticks.

The quasi-stationary distribution over the duration of series of the second type is shown in Figure 6.

The one unit length of the series of the second type is most likely, the remaining lengths fit into an exponential relationship with determination 0,995:

$$P_2(n) = 0,76e^{-0,85n}, \quad n \geqslant 2.$$

The average length of the series of the second type, as well as the standard deviation, is 1.

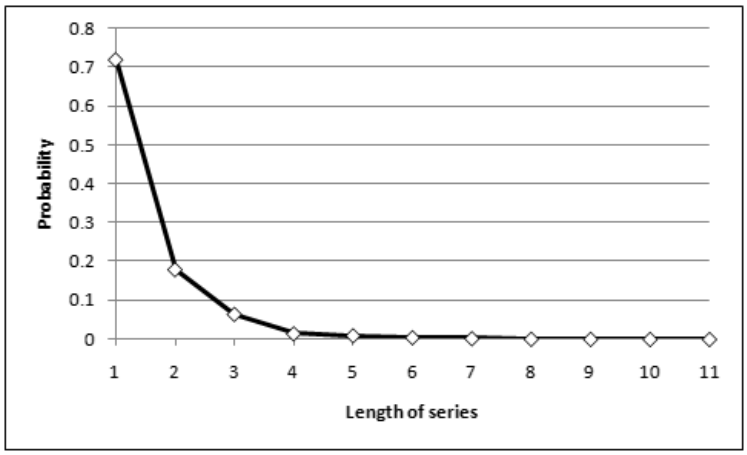

Figure 6. Distribution of series of the second type by duration

The Figure 7 is an example of how the time series generation module works. The bold line indicates the source row, and the remaining curves are the results of the program.

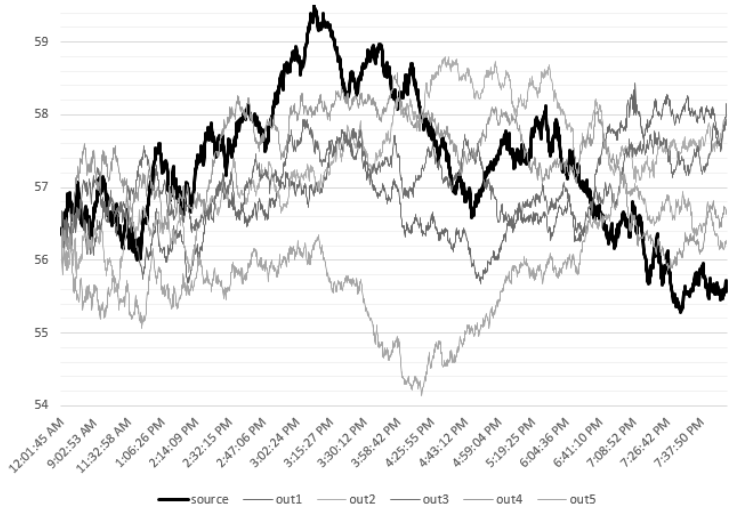

Figure 7. Example of working process of the time series generation module

The module allows you to test a trading algorithm on an ensemble of non-stationary trajectories and more accurately optimize the parameters of this algorithm compared to testing on a stationary trajectory of a large sample.

Conclusion

The described software package allows you to model a non-stationary Queuing system when the event flow and the value of a random variable itself are non-stationary processes. In addition to the exchange series, the

objects of modeling can be the actual CMO, when the flow of phone calls or requests to visit the site and download certain information has non-stationary characteristics. For such systems the built complex allows to optimize the functional of the control. This functional can be the algorithm of the trading system on the exchange, blocking certain requests to the site, etc. In addition, the complex allows you to collect complex nonlinear statistics on an ensemble of trajectories that can not be obtained in practice for a single implementation of a non-stationary time series.

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Моделирование нестационарного потока событий с вложенным стационарным компонентом

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В статье описан метод построения ансамбля траекторий временных рядов с нестационарным потоком событий и нестационарным эмпирическим распределением значений наблюдаемой случайной величины. Мы рассматриваем специальную модель, которая похожа по свойствам на некоторые реальные процессы, такие как изменения цены финансового инструмента на бирже. Предполагается, что случайный процесс представлен как совокупность двух процессов — стационарного и нестационарного. То есть длина ряда элементов в последовательности наиболее вероятного события (например, наиболее вероятное изменение цены в последовательности транзакций) образует нестационарный временной ряд, а длина ряда других событий является стационарным случайным процессом. Считается, что поток событий является нестационарным пуассоновским процессом. В работе описан программный комплекс, решающий задачу моделирования ансамбля траекторий наблюдаемой случайной величины. Моделируются как значения случайной величины, так и время возникновения события. Приведён пример практического применения модели.

Ключевые слова: нестационарные временные ряды, нестационарный поток событий, моделирование ансамблевых траекторий

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Two-way communication retrial queue with unreliable server and multiple types of outgoing calls

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Retrial queue under consideration is the model of call center operator switching between input and outgoing calls. Incoming calls form a Poisson point process. Upon arrival, an incoming call occupies the server for an exponentially distributed service time if the server is idle. If the server if busy, an incoming call joins the orbit to make a delay before the next attempt to take the server. The probability distribution of the length of delay is an exponential distribution. Otherwise, the server makes outgoing calls in its idle time. There are multiple types of outgoing calls in the system. Outgoing call rates are different for each type of outgoing call. Durations of different types of outgoing calls follow distinct exponential distributions. Unsteadiness is that the server crashes after an exponentially distributed time and needs recovery. The rates of breakdowns and restorations are different and depend on server state. Our contribution is to obtain the probability distribution of the number of calls in the orbit under high rate of making outgoing calls limit condition. Based on the obtained asymptotics, we have built the approximations of the probability distribution of the number of calls in the orbit.

Key words and phrases: retrial queue, Poisson process, unreliable server, two-way communication, outgoing calls, incoming calls, asymptotic analysis, Gaussian approximation

1. Introduction

Blended call centers are an efficient and productive implement in modern companies. In addition to the usual call center features they can provide both incoming and outgoing calls. In its idle time the server initiates an outgoing call. Such behaviour allows to increase the effectiveness of the system. To study the functioning of the system with likewise particularity retrial queues are commonly used [1]–[3].

In retrial queues the customers that find the server busy repeat their request for service after a random delay. The pool of waiting customers is

called an orbit. The most extensive research of this type of queueing systems is given in monographs [4], [5]. Retrial queues are usually the models of telecommunication and random access systems and their modifications allow to describe various real systems.

In this paper, we consider retrial queue with two-way communication and unreliable server. Two-way communication arose as a modification of retrial queues for modeling call centers with mixed calls. The Markovian models with outgoing calls are researched by Artalejo and Phung-Duc [6], [7]. Two-way communication queues have a branch of modifications to concrete the performance such as: finite input source [8], server-orbit interaction [9], [10], call-back option [11], etc. In our case there are multiple types of outgoing calls in the system.

The unreliability is explained by the need to take into account possible interruptions in the operation of the server [12], [13]. In our definition the rates of breakdowns are different and depend on the server current state. To study the model we use asymptotic analysis method under high rates of making outgoing calls condition [14].

2. Model description

We consider single server retrial queue with multiple types of outgoing calls. Input process is a stationary Poisson process with rate λ. Incoming calls occupy the server for an exponentially distributed time with rate μ_1. Calls that find the server busy join the orbit and repeat their attempt to take the server after an exponentially distributed delay with rate σ. In its idle time the server makes outgoing calls of type n with rate α_n and provides the service for an exponentially distributed time with parameter μ_n. For convenience, we number the types of outgoing calls from 2 to N. Unsteadiness of the system is a behaviour of the server that it crashes after an exponentially distributed time of functioning. The rate of breakdowns and restorations are distinguished and depend on the server state. Figure 1 depicts the structure of the system.

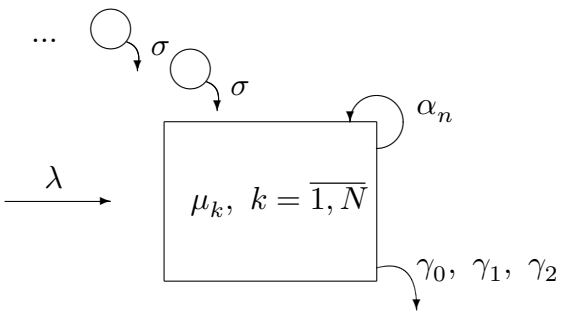

Figure 1. Single server retrial queue with unreliable server and multiple types of outgoing calls

Let $k(t)$ denotes the state of the server at the time $t \geqslant 0$,

$$k(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if an incoming call is in service,} \\ n, & \text{if an outgoing call of type } n \text{ is in service, } n = \overline{2, N}, \\ N+1, & \text{if the server is in recovery state.} \end{cases}$$

As the server breakdowns depend on its state we denote γ_0 is the rate of breakdowns in state 0, γ_1 is the rate of breakdowns in state 1 and γ_2 is the rate of restorations. We assume that when the outgoing call is in service there is no breakdowns as the server calls the customer itself. If the customer is in service the breakdown interrupts the service and the customer joins the orbit.

Let $i(t)$ denotes the number of calls in the orbit at the moment $t \geqslant 0$. It is easy to see that two-dimensional process $\{i(t), k(t)\}$ is a continuous time Markov chain.

Let $P\{i(t) = i, k(t) = k\} = P_k(i, t)$ denotes the probability distribution of the process $\{i(t), k(t)\}$, then it is the solution of Kolmogorov's system of equations. We present the system in stationary regime

$$\begin{cases} -\left(\lambda + i\sigma + \sum\limits_{n=2}^{N} \alpha_n + \gamma_0\right) P_0(i) + \sum\limits_{k=1}^{N} \mu_k P_k(i) + \gamma_2 P_{N+1}(i) = 0, \\ -(\lambda + \mu_1 + \gamma_1) P_1(i) + \lambda P_1(i-1) + \lambda P_0(i) + (i+1)\sigma P_0(i+1) = 0, \\ -(\lambda + \mu_n) P_n(i) + \lambda P_n(i-1) + \alpha_n P_0(i) = 0, \quad n = \overline{2, N}, \\ -(\lambda + \gamma_2) P_{N+1}(i) + \lambda P_{N+1}(i-1) + \gamma_0 P_0(i) + \gamma_1 P_1(i-1) = 0. \end{cases} \quad (1)$$

Let $H_k(u)$ denotes the partial characteristic functions $H_k(u) = \sum\limits_{i=0}^{\infty} e^{jui} P_k(i)$, $k = \overline{0, N+1}$, where $j = \sqrt{-1}$. Multiplying equations of system (1) by e^{jui} and taking the sum over i yields

$$\begin{cases} -\left(\lambda + \gamma_0 + \sum\limits_{n=2}^{N} \alpha_n\right) H_0(u) + j\sigma H_0'(u) + \\ \qquad + \sum\limits_{k=1}^{N} \mu_k H_k(u) + \gamma_2 H_{N+1}(u) = 0, \\ (\lambda(e^{ju} - 1) - \mu_1 - \gamma_1) H_1(u) + \lambda H_0(u) - j\sigma e^{-ju} H_0'(u) = 0, \\ (\lambda(e^{ju} - 1) - \mu_n) H_n(u) + \alpha_n H_0(u) = 0, \quad n = \overline{2, N}, \\ (\lambda(e^{ju} - 1) - \gamma_2) H_{N+1}(u) + \gamma_0 H_0(u) + \gamma_1 e^{ju} H_1(u) = 0, \\ j\sigma e^{-ju} H_0'(u) + (\lambda + \gamma_1) H_1(u) + \lambda \sum\limits_{n=2}^{N+1} H_n(u) = 0, \end{cases} \quad (2)$$

where the last equation is an additional equation that we obtained by summing up the equations of the system.

The characteristic function $H(u)$ of the studied process can be expressed through the partial characteristic functions as follows

$$H(u) = \sum_{k=0}^{N+1} H_k(u).$$

The main contribution of this research is the solution of the system (2) by using an asymptotic analysis method under high rate of outgoing calls limit condition.

3. Asymptotic Analysis of the Model under the High Rate of Making Outgoing Calls

In this section, we will investigate system (2) by asymptotic analysis method under the high rate of making outgoing calls limit condition. In particular, we prove that asymptotic characteristic function of the number of incoming calls in the system corresponds to Gaussian distribution.

To match the asymptotic condition we denote $\alpha_n = \alpha\nu_n$ in system (2)

$$\begin{cases} -\left(\lambda + \gamma_0 + \alpha\sum\limits_{n=2}^{N}\nu_n\right)H_0(u) + j\sigma H_0'(u) + \\ \qquad + \sum\limits_{k=1}^{N}\mu_k H_k(u) + \gamma_2 H_{N+1}(u) = 0, \\ (\lambda(e^{ju}-1) - \mu_1 - \gamma_1)H_1(u) + \lambda H_0(u) - j\sigma e^{-ju}H_0'(u) = 0, \\ (\lambda(e^{ju}-1) - \mu_n)H_n(u) + \alpha\nu_n H_0(u) = 0, \quad n = \overline{2,N}, \\ (\lambda(e^{ju}-1) - \gamma_2)H_{N+1}(u) + \gamma_0 H_0(u) + \gamma_1 e^{ju}H_1(u) = 0, \\ j\sigma e^{-ju}H_0'(u) + (\lambda+\gamma_1)H_1(u) + \lambda\sum\limits_{n=2}^{N+1}H_n(u) = 0. \end{cases} \tag{3}$$

then the limit condition takes the form $\alpha \to \infty$.

3.1. First Order Asymptotic

We denote $\alpha = 1/\varepsilon$ in the system (3), and introduce the following notations

$$u = \varepsilon w, \quad H_0(u) = \varepsilon F_0(w,\varepsilon), \quad H_k(u) = F_k(w,\varepsilon), \quad k = \overline{1,N+1},$$

in order to get the following system

$$\begin{cases} -\left(\lambda\varepsilon+\gamma_0\varepsilon+\sum\limits_{n=2}^{N}\nu_n\right)F_0(w,\varepsilon)+j\sigma\dfrac{\partial F_0(w,\varepsilon)}{\partial w}+ \\ \qquad\qquad +\sum\limits_{k=1}^{N}\mu_k F_k(w,\varepsilon)+\gamma_2 F_{N+1}(w,\varepsilon)=0, \\ (\lambda(e^{jw\varepsilon}-1)-\mu_1-\gamma_1)F_1(w,\varepsilon)+\lambda\varepsilon F_0(w,\varepsilon)- \\ \qquad\qquad -j\sigma e^{-jw\varepsilon}\dfrac{\partial F_0(w,\varepsilon)}{\partial w}=0, \\ (\lambda(e^{jw\varepsilon}-1)-\mu_n)F_n(w,\varepsilon)+\nu_n F_0(w,\varepsilon)=0,\ n=\overline{2,N}, \\ (\lambda(e^{jw\varepsilon}-1)-\gamma_2)F_{N+1}(w,\varepsilon)+\gamma_0\varepsilon F_0(w,\varepsilon)+\gamma_1 e^{jw\varepsilon}F_1(w,\varepsilon)=0, \\ j\sigma e^{-jw\varepsilon}\dfrac{\partial F_0(w,\varepsilon)}{\partial w}+(\lambda+\gamma_1)F_1(w,\varepsilon)+\lambda\sum\limits_{n=2}^{N+1}F_n(w,\varepsilon)=0. \end{cases} \tag{4}$$

Theorem 1. *Suppose $i(t)$ is the number of incoming calls in the aforementioned system of the stationary M/M/1 retrial queue with several types of outgoing calls, then the* (5) *holds*

$$\lim_{\alpha\to\infty} Ee^{jw\frac{i(t)}{\alpha}}=e^{jw\kappa_1}, \tag{5}$$

where

$$\kappa_1=\frac{\lambda}{\sigma}\frac{\gamma_2(\mu_1+\gamma_1)}{\mu_1\gamma_2-\lambda(\gamma_1+\gamma_2)}\sum_{n=2}^{N}\frac{\nu_n}{\mu_n}, \tag{6}$$

Proof. Considering the limit as $\varepsilon\to 0$ in the system (4), then we will get

$$\begin{cases} -\sum\limits_{n=2}^{N}\nu_n F_0(w)+j\sigma F_0'(w)+\sum\limits_{k=1}^{N}\mu_k F_k(w)+\gamma_2 F_{N+1}(w)=0, \\ -(\mu_1+\gamma_1)F_1(w)-j\sigma F_0'(w)=0, \\ -\mu_n F_n(w)+\nu_n F_0(w)=0,\ n=\overline{2,N}, \\ -\gamma_2 F_{N+1}(w)+\gamma_1 F_1(w)=0, \\ j\sigma F_0'(w)+(\lambda+\gamma_1)F_1(w)+\lambda\sum\limits_{n=2}^{N+1}F_n(w)=0. \end{cases} \tag{7}$$

We propose to seek the solution of the system (7) in the following form

$$F_k(w)=r_k\Phi(w),\quad k=\overline{0,N+1}, \tag{8}$$

where r_k is the stationary probability distribution of the system states, then, dividing the equations by $\Phi(w)$, we obtain the system in the following form

$$\begin{cases} -r_0\sum\limits_{n=2}^{N}\nu_n + j\sigma r_0\dfrac{\Phi'(w)}{\Phi(w)} + \sum\limits_{k=1}^{N}\mu_k r_k + \gamma_2 r_{N+1} = 0, \\ -(\mu_1+\gamma_1)r_1 - j\sigma r_0\dfrac{\Phi'(w)}{\Phi(w)} = 0, \\ -\mu_n r_n + \nu_n r_0 = 0, \quad n = \overline{2,N}, \\ -\gamma_2 r_{N+1} + \gamma_1 r_1 = 0, \\ j\sigma r_0\dfrac{\Phi'(w)}{\Phi(w)} + (\lambda+\gamma_1)r_1 + \lambda\sum\limits_{n=2}^{N+1} r_n = 0. \end{cases} \tag{9}$$

As the relation $\dfrac{\Phi'(w)}{\Phi(w)}$ doesn't depend on w, the characteristic function $\Phi(w)$ can be expressed as $\Phi(w) = \exp\{jw\kappa_1\}$, which coincides with (5). Thus, we rewrite the system

$$\begin{cases} -\left(\sum\limits_{n=2}^{N}\nu_n + \sigma\kappa_1\right) r_0 + \sum\limits_{k=1}^{N}\mu_k r_k + \gamma_2 r_{N+1} = 0, \\ -(\mu_1+\gamma_1)r_1 + \sigma\kappa_1 r_0 = 0, \\ -\mu_n r_n + \nu_n r_0 = 0, \\ -\gamma_2 r_{N+1} + \gamma_1 r_1 = 0, \ n = \overline{2,N}, \\ -\sigma\kappa_1 r_0 + (\lambda+\gamma_1)r_1 + \lambda\sum\limits_{n=2}^{N+1} r_n = 0. \end{cases} \tag{10}$$

To obtain the values of r_k we use second, third and fourth equations of the system with normalization condition

$$\begin{cases} r_1 = \dfrac{\sigma\kappa_1}{\mu_1+\gamma_1} r_0, \\ r_n = \dfrac{\nu_n}{\mu_n} r_0, \ n = \overline{2,N}, \\ r_{N+1} = \dfrac{\gamma_1}{\gamma_2} r_1 = \dfrac{\sigma\kappa_1\gamma_1}{\gamma_2(\mu_1+\gamma_1)} r_0, \\ \sum\limits_{k=1}^{N+1} r_k = \dfrac{\sigma\kappa_1}{\mu_1+\gamma_1} r_0 + \sum\limits_{n=2}^{N}\dfrac{\nu_n}{\mu_n} r_0 + \dfrac{\sigma\kappa_1\gamma_1}{\gamma_2(\mu_1+\gamma_1)} r_0 = 1. \end{cases} \tag{11}$$

From the last equality we obtain the value of r_0

$$r_0 = \left(\frac{\sigma\kappa_1(\gamma_1+\gamma_2)}{\gamma_2(\mu_1+\gamma_1)} + \sum_{n=2}^{N}\frac{\nu_n}{\mu_n}\right)^{-1}, \tag{12}$$

using which we can write the values r_k as follows

$$\begin{cases} r_1 = \dfrac{\sigma\kappa_1}{\mu_1+\gamma_1}\left(\dfrac{\sigma\kappa_1(\gamma_1+\gamma_2)}{\gamma_2(\mu_1+\gamma_1)}+\sum\limits_{n=2}^{N}\dfrac{\nu_n}{\mu_n}\right)^{-1}, \\ r_n = \dfrac{\nu_n}{\mu_n}\left(\dfrac{\sigma\kappa_1(\gamma_1+\gamma_2)}{\gamma_2(\mu_1+\gamma_1)}+\sum\limits_{k=2}^{N}\dfrac{\nu_k}{\mu_k}\right)^{-1}, \quad n=\overline{2,N}, \\ r_{N+1} = \dfrac{\sigma\kappa_1\gamma_1}{\gamma_2(\mu_1+\gamma_1)}\left(\dfrac{\sigma\kappa_1(\gamma_1+\gamma_2)}{\gamma_2(\mu_1+\gamma_1)}+\sum\limits_{n=2}^{N}\dfrac{\nu_n}{\mu_n}\right)^{-1}, \end{cases} \tag{13}$$

The explicit value of κ_1 we obtain from the last equation of the system (10) using the equations (11)

$$\kappa_1 = \frac{(\lambda+\gamma_1)r_1+\lambda\sum\limits_{n=2}^{N+1}r_n}{\sigma r_0} = \frac{\lambda}{\sigma}\frac{\gamma_2(\mu_1+\gamma_1)}{\mu_1\gamma_2-\lambda(\gamma_1+\gamma_2)}\sum_{n=2}^{N}\frac{\nu_n}{\mu_n}, \tag{14}$$

which coincides with (6). □

3.2. Second Order Asymptotic

We introduce the following notations in the system (3)

$$H_k(u) = e^{ju\alpha\kappa_1}H_k^{(2)}(u), \tag{15}$$

to obtain the system of equations (16)

$$\begin{cases} -\left(\lambda+\gamma_0+\sigma\alpha\kappa_1+\alpha\sum\limits_{n=2}^{N}\nu_n\right)H_0^{(2)}(u)+j\sigma\dfrac{dH_0^{(2)}(u)}{du}+ \\ \qquad +\sum\limits_{k=1}^{N}\mu_kH_k^{(2)}(u)+\gamma_2H_{N+1}^{(2)}(u)=0, \\ (\lambda(e^{ju}-1)-\mu_1-\gamma_1)H_1^{(2)}(u)+(\lambda+\sigma\alpha\kappa_1)H_0^{(2)}(u)- \\ \qquad -j\sigma e^{-ju}\dfrac{dH_0^{(2)}(u)}{du}=0, \\ (\lambda(e^{ju}-1)-\mu_n)H_n^{(2)}(u)+\alpha\nu_nH_0^{(2)}(u)=0, \quad n=\overline{2,N}, \\ (\lambda(e^{ju}-1)-\gamma_2)H_{N+1}^{(2)}(u)+\gamma_0H_0^{(2)}(u)+\gamma_1e^{ju}H_1^{(2)}(u)=0, \\ j\sigma e^{-ju}\dfrac{dH_0^{(2)}(u)}{du}-\sigma\alpha\kappa_1e^{-ju}H_0^{(2)}(u)+(\lambda+\gamma_1)H_1^{(2)}(u)+ \\ \qquad +\lambda\sum\limits_{n=2}^{N+1}H_n^{(2)}(u)=0. \end{cases} \tag{16}$$

Denoting $\alpha = 1/\varepsilon^2$, and introducing the following notations in the system (16)

$$\begin{gathered} u = w\varepsilon, \\ H_0^{(2)}(u) = \varepsilon^2 F_0^{(2)}(w,\varepsilon), \quad H_k^{(2)}(u) = F_k^{(2)}(w,\varepsilon), \quad k = \overline{1, N+1}, \end{gathered} \tag{17}$$

we obtain

$$\begin{cases} -\left(\lambda\varepsilon^2 + \gamma_0\varepsilon^2 + \sigma\kappa_1 + \sum\limits_{n=2}^{N}\nu_n\right) F_0^{(2)}(w,\varepsilon) + j\sigma\varepsilon\dfrac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} + \\ \qquad + \sum\limits_{k=1}^{N}\mu_k F_k^{(2)}(w,\varepsilon) + \gamma_2 F_{N+1}^{(2)}(w,\varepsilon) = 0, \\ (\lambda(e^{jw\varepsilon} - 1) - \mu_1 - \gamma_1)F_1^{(2)}(w,\varepsilon) + (\lambda\varepsilon^2 + \sigma\kappa_1)F_0^{(2)}(w,\varepsilon) - \\ \qquad - j\sigma e^{-jw\varepsilon}\varepsilon\dfrac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} = 0, \\ (\lambda(e^{jw\varepsilon} - 1) - \mu_n)F_n^{(2)}(w,\varepsilon) + \nu_n F_0^{(2)}(w,\varepsilon) = 0, \quad n = \overline{2, N}, \\ (\lambda(e^{jw\varepsilon} - 1) - \gamma_2)F_{N+1}^{(2)}(w,\varepsilon) + \gamma_0\varepsilon^2 F_0^{(2)}(w,\varepsilon) + \\ \qquad + \gamma_1 e^{jw\varepsilon}F_1^{(2)}(w,\varepsilon) = 0, \\ j\sigma e^{-jw\varepsilon}\varepsilon\dfrac{\partial F_0^{(2)}(w,\varepsilon)}{\partial w} - \sigma\kappa_1 e^{-jw\varepsilon}F_0^{(2)}(w,\varepsilon) + (\lambda + \gamma_1)F_1^{(2)}(w,\varepsilon) + \\ \qquad + \lambda\sum\limits_{n=2}^{N+1}F_n^{(2)}(w,\varepsilon) = 0. \end{cases} \tag{18}$$

Theorem 2. *In the context of Theorem 1 the following equation is true*

$$\lim_{\alpha\to\infty} E\exp\left\{jw\frac{\frac{i(t)}{\alpha} - \kappa_1}{\sqrt{\alpha}}\right\} = \exp\left\{\frac{(jw)^2}{2}\kappa_2\right\}, \tag{19}$$

where

$$\begin{aligned} \kappa_2 = {} & \frac{\gamma_2(\mu_1 + \gamma_1)}{\mu_1\gamma_2 - \lambda\gamma_2 - \lambda\gamma_1}\times \\ & \times\left(\kappa_1 + \lambda\kappa_1\frac{\gamma_1(\gamma_1 + \gamma_2 + \mu_1)(\lambda + \gamma_2) + \lambda\gamma_2^2}{\gamma_2^2(\mu_1 + \gamma_1)^2} + \frac{\lambda^2}{\sigma}\sum_{n=2}^{N}\frac{\nu_n}{\mu_n^2}\right). \end{aligned} \tag{20}$$

Proof. To solve the system of equations (18) we present the functions $F_k^{(2)}(w,\varepsilon)$ in the form

$$F_k^{(2)}(w,\varepsilon) = \Phi_2(w)\{r_k + jw\varepsilon f_k\} + o(\varepsilon^2), \ k = \overline{0, N+1}, \tag{21}$$

here r_k is the probability distribution of the server state obtained in Theorem 1.

Substituting (21) into (18), making some simple conversions and taking (10) into account in the limit by $\varepsilon \to 0$, we obtain the system (22)

$$\begin{cases} \sigma\dfrac{\Phi_2'(w)}{w\Phi(w)}r_0 - \left(\sigma\kappa_1 + \sum\limits_{n=2}^{N}\nu_n\right) f_0 + \sum\limits_{k=1}^{N}\mu_k f_k + \gamma_2 f_{N+1} = 0, \\ -\sigma\dfrac{\Phi_2'(w)}{w\Phi_2(w)}r_0 + \lambda r_1 - (\mu_1+\gamma_1)f_1 + \sigma\kappa_1 f_0 = 0, \\ \lambda r_n - \mu_n f_n + \nu_n f_0 = 0, \quad n = \overline{2,N}, \\ \gamma_1 r_1 + \lambda r_{N+1} - \gamma_2 f_{N+1} + \gamma_1 f_1 = 0, \\ \sigma\dfrac{\Phi_2'(w)}{w\Phi_2(w)}r_0 + \sigma\kappa_1 r_0 - \sigma\kappa_1 f_0 + (\lambda+\gamma_1)f_1 + \lambda\sum\limits_{n=2}^{N+1} f_n = 0. \end{cases} \tag{22}$$

These equations imply that the relation $\dfrac{\Phi_2'(w)}{w\Phi_2(w)}$ doesn't depend on w, thus the function $\Phi_2(w)$ is given in the following form

$$\Phi_2(w) = \exp\left\{\frac{(jw)^2}{2}\kappa_2\right\},$$

which coincides with (19).

We have $\dfrac{\Phi_2'(w)}{w\Phi_2(w)} = -\kappa_2$ and then we obtain the system

$$\begin{cases} -\left(\sigma\kappa_1 + \sum\limits_{n=2}^{N}\nu_n\right) f_0 + \sum\limits_{k=1}^{N}\mu_k f_k + \gamma_2 f_{N+1} = \sigma\kappa_2 r_0, \\ -(\mu_1+\gamma_1)f_1 + \sigma\kappa_1 f_0 = -\sigma\kappa_2 r_0 - \lambda r_1, \\ -\mu_n f_n + \nu_n f_0 = -\lambda r_n, \ n = \overline{2,N}, \\ -\gamma_2 f_{N+1} + \gamma_1 f_1 = -\gamma_1 r_1 - \lambda r_{N+1}, \\ -\sigma\kappa_1 f_0 + (\lambda+\gamma_1)f_1 + \lambda\sum\limits_{n=2}^{N+1} f_n = \sigma\kappa_2 r_0 - \sigma\kappa_1 r_0. \end{cases} \tag{23}$$

The system of equations (23) is heterogeneous form of the system (10), thus we represent the values f_k in the form of

$$f_k(\kappa_2) = Cr_k + g_k(\kappa_2), \tag{24}$$

and substitute (24) into (23). According to the (10) obtained in Theorem 1, the terms containing the constant C are destroyed. We get system of equations

$$\begin{cases} -\left(\sigma\kappa_1 + \sum\limits_{n=2}^{N}\nu_n\right) g_0 + \sum\limits_{k=1}^{N}\mu_k g_k + \gamma_2 g_{N+1} = \sigma\kappa_2 r_0,\ g_k = g_k(\kappa_2), \\ -(\mu_1+\gamma_1)g_1 + \sigma\kappa_1 g_0 = -\sigma\kappa_2 r_0 - \lambda r_1, \\ -\mu_n g_n + \nu_n g_0 = -\lambda r_n, \quad n = \overline{2,N}, \\ -\gamma_2 g_{N+1} + \gamma_1 g_1 = -\gamma_1 r_1 - \lambda r_{N+1}, \\ -\sigma\kappa_1 g_0 + (\lambda+\gamma_1)g_1 + \lambda\sum\limits_{n=2}^{N+1} g_n = \sigma\kappa_2 r_0 - \sigma\kappa_1 r_0, \end{cases} \tag{25}$$

From the second, third and fourth equations of the system (25) we have the following equalities

$$g_1 = \frac{\sigma\kappa_1}{\mu_1+\gamma_1}g_0 + \frac{\sigma\kappa_2}{\mu_1+\gamma_1}r_0 + \frac{\lambda}{\mu_1+\gamma_1}r_1, \quad g_n = \frac{\nu_n}{\mu_n}g_0 + \frac{\lambda}{\mu_n}r_n, \quad n = \overline{2,N},$$

$$\begin{aligned} g_{N+1} &= \frac{\gamma_1}{\gamma_2}r_1 + \frac{\gamma_1}{\gamma_2}g_1 + \frac{\lambda}{\gamma_2}r_{N+1} = \\ &= \frac{\sigma\kappa_1\gamma_1}{\gamma_2(\mu_1+\gamma_1)}g_0 + \frac{\sigma\kappa_2\gamma_1}{\gamma_2(\mu_1+\gamma_1)}r_0 + \frac{\gamma_1}{\gamma_2}\left(1 + \frac{\lambda}{\mu_1+\gamma_1}\right)r_1 + \frac{\lambda}{\gamma_2}r_{N+1}. \end{aligned}$$

Substituting these equalities into the last equation of the system (25) we get

$$\begin{aligned} &\left(-\sigma\kappa_1 + \frac{\sigma\kappa_1(\lambda+\gamma_1)}{\mu_1+\gamma_1} + \lambda\sum_{n=2}^{N}\frac{\nu_n}{\mu_n} + \frac{\lambda\sigma\kappa_1\gamma_1}{\gamma_2(\mu_1+\gamma_1)}\right)g_0 + \\ &\quad + \left(\frac{\sigma\kappa_2(\lambda+\gamma_1)}{\mu_1+\gamma_1} + \frac{\lambda\sigma\kappa_2\gamma_1}{\gamma_2(\mu_1+\gamma_1)} - \sigma\kappa_2 + \sigma\kappa_1\right)r_0 + \\ &+ \left(\frac{\lambda(\lambda+\gamma_1)}{\mu_1+\gamma_1} + \frac{\lambda\gamma_1}{\gamma_2}\left(1 + \frac{\lambda}{\mu_1+\gamma_1}\right)\right)r_1 + \lambda\sum_{n=2}^{N}\frac{\lambda}{\mu_n}r_n + \frac{\lambda^2}{\gamma_2}r_{N+1} = 0. \end{aligned}$$

As the coefficient at g_0 is zero we can rewrite this equation to obtain the explicit expression of κ_2 which coincides with (20):

$$\begin{aligned} \kappa_2 = &\frac{\gamma_2(\mu_1+\gamma_1)}{\mu_1\gamma_2 - \lambda\gamma_2 - \lambda\gamma_1}\times \\ &\times\left(\kappa_1 + \lambda\kappa_1\frac{\gamma_1(\gamma_1+\gamma_2+\mu_1)(\lambda+\gamma_2) + \lambda\gamma_2^2}{\gamma_2^2(\mu_1+\gamma_1)^2} + \frac{\lambda^2}{\sigma}\sum_{n=2}^{N}\frac{\nu_n}{\mu_n^2}\right), \end{aligned}$$

To express the characteristic function $H(u)$ of the number of customers in the orbit we use the notations (15) and (17)

$$H(u) = \exp\left\{ju\alpha\kappa_1 + \left((ju)^2/2\right)\alpha\kappa_2\right\}.$$

Thus, in Theorem 2 we have obtained the variance $\kappa_2\alpha$ of a number of calls in the orbit in prelimit situation of $\alpha \to \infty$. The asymptotic probability distribution of the process $i(t)$ is Gaussian.

4. Conclusions

In this paper, we have considered the Markovian retrial queue with multiple types of outgoing calls and unreliable server. Using the asymptotic analysis method we have shown that the asymptotic behaviour of the number of customers in the orbit coincides with Gaussian distribution and in the limit condition $\alpha \to \infty$, which means high rates of making outgoing calls, we have the mean $\kappa_1\alpha$ and variance $\kappa_2\alpha$.

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RQ-система с ненадёжным прибором и разнотипными вызываемыми заявками

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В статье RQ-система с разнотипными вызываемыми заявками рассматривается как модель оператора call-центра. Входящие звонки образуют простейший поток. В момент поступления заявка из потока занимает прибор для обслуживания, если он свободен. Распределение вероятностей длительностей обслуживания является экспоненциальным. Если прибор занят, поступившая заявка отправляется на орбиту, где осуществляет задержку случайной длительности, распределённой по экспоненциальному закону, после чего снова пытается занять прибор для обслуживания. С другой стороны, когда прибор свободен, он вызывает заявки извне. В системе есть несколько типов вызываемых заявок. Интенсивности вызывания различны для разных типов вызываемых заявок. Длительности обслуживания вызываемых заявок разных типов являются экспоненциальными случайными величинами с различными параметрами. Ненадёжность прибора характеризуется выходом из строя на период времени, длительность которого распределена экспоненциально. Интенсивности выхода из строя и восстановления прибора различны и зависят от состояния прибора. Целью исследования является получение стационарного распределения вероятностей числа заявок на орбите методом асимптотического анализа в предельном условии высокой интенсивности вызывания заявок. На основе полученного асимптотического распределения построена аппроксимация допредельного распределения вероятностей числа заявок на орбите в рассматриваемой RQ-системе.

Ключевые слова: RQ-система, простейший поток, ненадёжный прибор, вызываемые заявки, метод асимптотического анализа, гауссовская аппроксимация

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Calculation of the normal modes of closed waveguides

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The aim of the work is the development of numerical methods for solving waveguiding problems of the theory of waveguides, as well as their implementation in the form of software packages focused on a wide range of practical problems from the classical issues of microwave transmission to the design of optical waveguides and sensors. At the same time, we strive for ease of implementation of the developed methods in computer algebra systems (Maple, Sage) or in software oriented to the finite element method (FreeFem++). The work uses the representation of electromagnetic fields in a waveguide using four potentials. These potentials do not reduce the number of sought functions, but even in the case when the dielectric permittivity and magnetic permeability are described by discontinuous functions, they turn out to be quite smooth functions. A simple check of the operability of programs by calculating the normal modes of a hollow waveguide is made. It is shown that the relative error in the calculation of the first 10 normal modes does not exceed 4%. These results indicate the efficiency of the method proposed in this article.

Key words and phrases: integrated optics, closed waveguide, computer simulation, finite element method, four potential method

1. Introduction

The simplest way to model the phenomena of classical electrodynamics is to use the Maxwell equations describing the electromagnetic field and their subsequent discretization by the finite difference method. The development of computer technology currently allows using the finite difference method directly to discretize the Maxwell equations and conduct numerical studies of applied electrodynamics problems considered in limited domains of space, for example, in a resonator, prism, diffraction grating etc. One of the most widely used methods of this kind is the finite-difference time-domain (FDTD)

method, described in detail in a number of textbooks on modern computational electrodynamics [1]. A specific feature of waveguide problems, e.g., the problem of waveguide diffraction, consists in the fact that the electromagnetic fields is to be calculated at a considerable distance from the studied object that scatters the electromagnetic field, which leads to the need for huge amounts of computation in the framework of the FDTD method and its modifications. It should also be added that this method introduces "numerical dispersion", which leads to errors in determining the phase velocity, and "numerical anisotropy", in which the wave numbers of waves propagating in different directions in an isotropic medium differ [2], [3].

The study of waveguide problems in the full electromagnetic formulation was initiated by the works of A. N. Tikhonov, A. A. Samarsky, P. E. Krasnushkin, and A. G. Sveshnikov, carried out in the second half of the last century. A. N. Tikhonov and A. A. Samarsky investigated the propagation of electromagnetic waves along a cylinder with a constant simply connected cross section, having perfectly conducting walls and filled with a homogeneous substance. In this work, several fundamental theorems were proved that characterize an arbitrary electromagnetic field in such a waveguide, e.g., the theorem on the field decomposition into transverse electric and transverse magnetic (TE and TM) fields and the theorem on the decomposition of a field into normal modes. These results allowed P.E. Krasnushkin to introduce the concept of a normal waveguide wave or mode, and A. G. Sveshnikov [4], [5] to introduce partial conditions of radiation and strictly mathematically pose the problem of diffraction and normal waves in a waveguide.

It should be noted that setting perfect conduction boundary conditions does not limit the scope of the developed methods to the research and design of microwave transmission channels only, since the closed waveguide model is also used to simulate open waveguide systems [6], [7]. When modeling the propagation of guided modes along an open waveguide in the optical range, it is natural to assume that the field at a distance of several wavelengths from the boundary of such a waveguide is zero. Therefore, by placing an open waveguide in a box with perfectly conducting walls, we obtain an approximate model of an open waveguide, as A.G. Sveshnikov first pointed out. The model "open optical waveguide in a box" is a correct mathematical model describing the propagation of waveguide modes, and at the moment it is a correct model describing waveguide diffraction in open optical systems [7]. The limits of applicability of this model can be described quantitatively by comparing the results obtained with different distances of the box walls from the boundary of the waveguide. An obvious drawback of the "waveguide in a box" model is the overestimation of energy flow channeled in the direction of the waveguide axis. This is not essential for modeling the propagation of guided modes, but it is important, e.g., for problems of energy flowing out of such a waveguide through an open end.

The main difficulty in the development of the theory of waveguides was the spectral problem for waveguides filled with optically inhomogeneous matter. As far back as the middle of the last century, waveguides with cores became actively used in practice, i.e., cylinders, the filling of which varies across the section and remains constant along the axis. Below we will call this structure a regular waveguide filled with optically inhomogeneous material. Modern technologies in the field of creating new materials and metamaterials are able to give waveguides with almost any distribution of dielectric constant into

the hands of practitioners. Moreover, they are increasingly trying to use fractal inserts [8], which means that the scalar model, with its assumption that the dielectric constant changes are small and slow, is becoming less and less adequate. The same conclusion can be made with respect to the study of multicore waveguides, the study of which is focused on providing 5G networks [9]–[11].

The method, which goes back to the works of A. N. Tikhonov and A. A. Samarsky, was based on the possibility of introducing two potentials for a hollow waveguide, which are now interpreted as the electric and magnetic Borgnis functions. This circumstance fundamentally distinguishes the computational complexity of the spectral problems for hollow waveguides and for waveguides filled with optically inhomogeneous matter. In the first case, the problems are scalar and well-developed methods are applicable to them, which are equally suitable for problems of acoustics and quantum mechanics. In the case of a waveguide filled with an inhomogeneous optical medium, one has to solve the problem numerically in full vector formulation. A generalization of the classical problems of the mathematical theory of waveguides, particularly, the spectral and diffraction problems, to the case of waveguides with variable permittivity is much more complicated and still not fully explored.

Turning to a discussion of the results obtained in studying the spectral characteristics of waveguides filled with inhomogeneous matter, we agree to work in a Cartesian coordinate system whose axis Oz coincides with the axis of the waveguide. The problem of finding the normal modes of a regular waveguide filled with optically inhomogeneous matter is as follows. Given are:

— the waveguide cross section S,
— the distribution of ε and μ over the cross section S; here and below we assume these functions to take only positive values,
— the frequency ω, and therefore, the wave number $k = \omega/c$.

It is required to find all values of the parameter $\beta \in \mathbb{C}$, for which the Maxwell equations have a nontrivial solution of the form

$$\begin{aligned} &\vec{E}(x,y)e^{ik\beta z - i\omega t}, \\ &\vec{H}(x,y)e^{ik\beta z - i\omega t}, \end{aligned} \tag{1}$$

satisfying the conditions of ideal conductivity of the waveguide walls and the joining conditions at the discontinuities of the permittivity and permeability. The parameter β is called the phase constant. Traditionally, this problem is formulated as an eigenvalue problem with respect to three field components. The choice of these three components from the set of six components of the vectors $\vec{E}$ and $\vec{H}$ can be different, which leads to different formulations of the problem. A. N. Bogolyubov and T. V. Edakina [6] and Frank Schmidt [12], [13] used the components of the vector $\vec{H}$, E. Lezar and D. Davidson [14] working in the framework of the FEniCS Project used the components of the vector $\vec{E}$, A. L. Delitsyn [15]–[17] used the components H_x, H_y, E_z. Normal modes of an axially symmetric waveguide with a dielectric core were considered by N.A. Novoselova, S. B. Raevsky and A. A. Titarenko [18], as well as by A. L. Delitsyn and S. I. Kruglov.

First, for any of the above approaches, we obtain spectral problems for non-self-adjoint operators in function spaces, which are constructed by analogy

with Sobolev spaces, but are much less studied. This greatly complicates the proof of the theorem on the expansion of a monochromatic wave in a waveguide in normal modes, without which it is impossible to proceed to setting the partial radiation conditions in the waveguide diffraction problem [15], [19], [20]. For a hollow waveguide, this theorem was proved by A. N. Tikhonov and A. A. Samarsky as a consequence of Steklov's theorem, while in the case of a waveguide filled with inhomogeneous medium, one has to use the general Keldysh theorem on the completeness of the system of eigenvectors and adjoined vectors [16], [17], [21]–[23]. At present, the completeness of the system of principal vectors of a waveguide has been proved, however, the question about the possibility to use this system as a basis has already been answered in the affirmative sense only for the case of a circular waveguide, the filling of which depends only on the radius [24], [25].

Numerous questions about the distribution of eigenvalues, the conditions for the existence of multiple eigenvalues have remained unexplored due to the difficulty of the spectral theory for non-self-adjoint operators. Are there cases in which the phase constant β of normal modes has both the real and imaginary parts? Are there cases in which adjoined normal modes arise? What is their physical meaning? Numerical experiments do not give an unambiguous answer to these questions. For example, in our experiments performed using the software package presented at the annual Saratov Fall Meeting International Conference in 2017, higher modes appeared to possess complex phase constants. However, numerical methods always calculate higher modes worse than the lower ones, so in those experiments it was not clear whether we discovered a new physical effect, or encountered a computational artifact.

Second, the spectral problem has a zero eigenvalue of infinite multiplicity, because of which, when solving the eigenvalue problem numerically, fictitious ("ghost") modes arise [6], [26]. At the end of their review of the current advances in solving the spectral problem of the waveguide theory, A. N. Bogolyubov and T. V. Edakina [6] wrote:

> "... The appearance of false modes is perhaps the most difficult issue in solving waveguide problems using finite elements or finite differences in a variational formulation, and, in our opinion, researchers will turn to it more than once in search of the simplest and most economical ways to identify the waves actually propagating in the waveguide."

Currently, there are two ways to deal with ghosts: the use of penalties [6] or the use of mixed finite elements [14], [27]. The main drawback of the penalty method is that although an increase in the parameter characterizing the size of the penalty leads to a decrease in the number of fictitious solutions, the accuracy of calculating the characteristics of true modes decreases.

Therefore, since the mid-1990s, the mixed finite element method has been regarded as the only reliable means of combating ghosts [26]. Test examples showed the reliability of this technique, however, in our opinion, the issues of its substantiation as applied to waveguide problems were not given due attention. It should be noted that this issue closely relates to the use of FEM, and not, e.g., the incomplete Galerkin method.

Third, the conditions on the walls of the waveguide $\vec{E} \times \vec{n} = \vec{0}$, $\vec{H} \cdot \vec{n} = 0$ are not classical, and FEA Softwares have no built-in elements for such conditions.

It is not clear how precisely these conditions are approximated in published papers. In papers on optical waveguides [6], [12], [13] the authors assume that the field is zero on the walls. From a physical point of view, this assumption is very reasonable, but unfortunately it leads to a conflict with Müller's theorem on a field equal to zero on an element of an analytic surface [28]. This conflict is removed further, at the stage of introducing penalties, and therefore, the result is a correct mathematical problem.

Fourth, in the examples most interesting for applications, the dielectric constant has discontinuities at the interface of different media filling the waveguide. At these interfaces, the electromagnetic fields suffer discontinuities, because of which a necessity arises to approximate discontinuous functions using the FEM. The results of numerical experiments convincingly support the legitimacy of this operation; however, for this situation, theoretically, the effect of discontinuities on convergence has not been studied. The study of this issue is substantially complicated by the fact that the approximation is carried out in non-standard, poorly studied functional spaces. For example, in the works of A.L. Delitsyn mentioned above, embedding theorems were proved, which for Sobolev spaces were established as early as at the beginning of the last century.

Finally, the achieved accuracy of the calculations is not high. E. Lezar and D. Davidson compared their results with the results obtained earlier by Jin [29] in the same way for the same waveguide, namely, the rectangular half-filled waveguide. Only the first branch of the dispersion curve coincided with graphic precision, while the Jin's next three branches merged into two.

In our works [30], [31], a previously unknown representation of electromagnetic fields in a waveguide using four potentials was proposed.

These potentials do not reduce the number of functions to be determined, i.e., they do not "integrate" Maxwell equations. But even in the case when the permittivity and permeability are described by discontinuous functions, they turn out to be quite smooth functions. The Maple system has developed a symbolic-numerical method for finding normal modes based on a combination of this representation of the field and the incomplete Galerkin method [30], [32], [33]. Comparison of the calculation results with the results obtained using the mixed finite element method was significantly complicated by the lack of a public version of the finite element implementation. The program, written by Lazar and Davidson [14] as part of the FEniCS Project, was only partially published by the authors and has not been updated since 2012, in particular, important changes in the syntax of the project were not taken into account. Y. Yu. Kuziv resumed this program and performed the necessary calculations for comparison, which will be presented in this article.

The ultimate goal of our research is to create numerical methods for solving the major problems of the theory of waveguides and to implement them as software packages focused on a wide range of practical problems, from classical issues of microwave transmission to the design of optical waveguides and sensors. At the same time, we strive for ease of implementation of the developed methods in computer algebra systems (Maple, Sage) or in software oriented at the finite element method (FreeFem ++).

We will not describe finite element methods for solving the spectral problem, since they are described in detail in [14]. As to our method, we intend to dwell on it in more detail.

2. Method of four potentials

Let S be a singly connected waveguide cross section, Z be a segment of the waveguide axis, T be the considered time interval, and ϵ, μ — piecewise continuous functions on S, taking only positive values. The electromagnetic field in the closed waveguide $S \times Z \times T$ with the filling ϵ, μ will be understood as vector fields $\vec{E}, \vec{H}$, whose components are defined on

$$(S - \Gamma) \times Z \times T, \tag{2}$$

under the condition that the sections of $\vec{E}, \vec{H}$ and their partial derivatives in z и t on S for any z and t are piecewise smooth functions that satisfy

1) the Maxwell equations

$$\begin{cases} \operatorname{rot} \vec{E} = -\partial_t \mu \vec{H}, \\ \operatorname{rot} \vec{H} = +\partial_t \epsilon \vec{E}, \\ \operatorname{div} \epsilon \vec{E} = 0, \\ \operatorname{div} \mu \vec{H} = 0, \end{cases} \tag{3}$$

inside the waveguide $S \times Z \times T$,

2) the conditions of perfect conductivity of the waveguide walls

$$\begin{aligned} \vec{E} \times \vec{n} = 0, \\ \vec{H} \cdot \vec{n} = 0 \end{aligned} \tag{4}$$

at regular points of the boundary $\partial S \times Z \times T$,

3) the joining conditions

$$\begin{cases} [\vec{E} \times \vec{n}] = \vec{0}, \quad [\epsilon \vec{E} \cdot \vec{n}] = 0, \\ [\vec{H} \times \vec{n}] = \vec{0}, \quad [\mu \vec{H} \cdot \vec{n}] = 0 \end{cases} \tag{5}$$

at regular points of the boundary where the filling has a discontinuity $\Gamma \times Z \times T$.

In practice, it is important to find at least approximately such a field that satisfies some additional conditions. For example, the spectral problem of the theory of waveguides formulated above consists in finding fields that have all 3 of these properties, and in addition to them having the form (1). In this paper, we focus on the calculation of such fields, leaving aside the theoretical questions of their existence.

Assume for brevity that

$$\vec{A}_\perp = (A_x, A_y, 0)^T,$$

$$\nabla = (\partial_x, \partial_y, 0)^T,$$

and

$$\nabla' = (-\partial_y, \partial_x, 0)^T.$$

According to theorem 2 in [31], the transverse components of such field can be always presented in terms of four scalar potentials in the following form

$$\begin{aligned} \vec{E}_\perp &= \nabla u_e + \frac{1}{\epsilon}\nabla' v_e, \\ \vec{H}_\perp &= \nabla v_h + \frac{1}{\mu}\nabla' u_h. \end{aligned} \tag{6}$$

In this case the potentials u_e и u_h are solutions of the Dirichlet problems

$$\begin{cases} \Delta_\epsilon u_e = -\epsilon \partial_z E_z, \\ u_e|_{\partial S} = 0, \end{cases} \tag{7}$$

and

$$\begin{cases} \Delta_{\frac{1}{\mu}} u_h = +\epsilon \partial_t E_z, \\ u_h|_{\partial S} = 0, \end{cases} \tag{8}$$

while the potentials v_e и v_h are solutions of the Neumann problems

$$\begin{cases} \Delta_{\frac{1}{\epsilon}} v_e = -\mu \partial_t H_z, \\ \left.\dfrac{\partial v_e}{\partial n}\right|_{\partial S} = 0, \end{cases} \tag{9}$$

and

$$\begin{cases} \Delta_\mu v_h = -\mu \partial_z H_z, \\ \left.\dfrac{\partial v_h}{\partial n}\right|_{\partial S} = 0. \end{cases} \tag{10}$$

Here and below for the Laplace operator $\mathrm{div}(k\nabla u)$ we use the notation $\Delta_k u$. The existence of at least generalised solutions of the Dirichlet problems is apparent, for the Neumann problems we have checked the fulfilment of the conditions for the existence of a solution [31]. For simplicity, let us assume that these solutions are classical, which can be proved under certain additional assumptions of the smoothness of boundary S and the considered fields based on the Weyl lemma [34].

Remark 1. In a hollow waveguide the potentials u_e, u_h, as well as v_e, v_h are linearly expressed via the Borgnis functions, so that this representation can be considered as the development of Borgnis' ideas.

It should be emphasized that, due to condition 2, the components of electromagnetic fields suffer discontinuities at the interfaces, and the factors in the formulas (6) are specially selected in [31] so that the potentials are smooth functions. In fact, Eq. (6) allows passing from discontinuous variables to smooth ones. This is very convenient from the point of view of the further application of numerical methods, since not all approximation methods are

applicable to discontinuous functions, and if applicable, the convergence in the discontinuous case is noticeably worse.

Remark 2. Note that any field of the form (6) satisfies theorem 1 of [31] at the interface between media, therefore, the appropriate conditions are automatically fulfilled below.

The method of four potentials consists of a transition from field components to four potentials. Let us apply it to the spectral problem.

3. Numerical example: normal modes of a hollow waveguide

The simplest test of the operability of programs for calculating normal modes is to calculate the normal modes of a hollow waveguide, studied analytically.

Consider a hollow waveguide with the side of a square cross section $l = 3.5 \cdot 10^{-6}$ [m] without filling, i.e., $\epsilon = \mu = 1$. The normal modes, i.e., the solutions of the Maxwell equations, have the form $e^{i\gamma z - i\omega t}$. For brevity let us denote $k = 2\pi\nu/c$ and $\beta = \gamma/k$, we will carry out a numerical calculation for $\nu = 10^{14}$ [Hz]. In a hollow waveguide $\gamma^2 = k^2 - \lambda$, where λ is an eigenvalue of Dirichlet or Neumann problem on the section. For the Dirichlet problem

$$\lambda = \frac{\pi^2}{l^2}(n^2 + m^2), \quad n, m = 1, 2, \ldots; \tag{11}$$

for the Neumann problem

$$\lambda = \frac{\pi^2}{l^2}(n^2 + m^2), \quad n, m = 0, 1, 2, \ldots; \tag{12}$$

the zero eigenvalue corresponding to constant eigenfunction and zero field; it should be removed from the list of normal modes.

For comparison, the relative error is presented

$$\delta_j = \left| \frac{\hat{\beta}_j - \beta_j}{\beta_j} \right|, \tag{13}$$

for the calculation of the first 10 modes. Here β_j is the phase retardation coefficient calculated using an analytical formula, and $\hat{\beta}_j$ is the same quantity found numerically.

Figure 1 presents the errors of mode calculation using finite elements – FEM – with 8 points along each side of the square. The error, as expected, grows with the mode number j. For the first two phase retardation coefficients relative error is less than 10^{-6}, for the next four coefficients the relative error is less than 10^{-4} and for the last four calculated phase retardation coefficients the relative error is less than 4×10^{-3}. The error growth is non-monotonic as follows from the Figure 1.

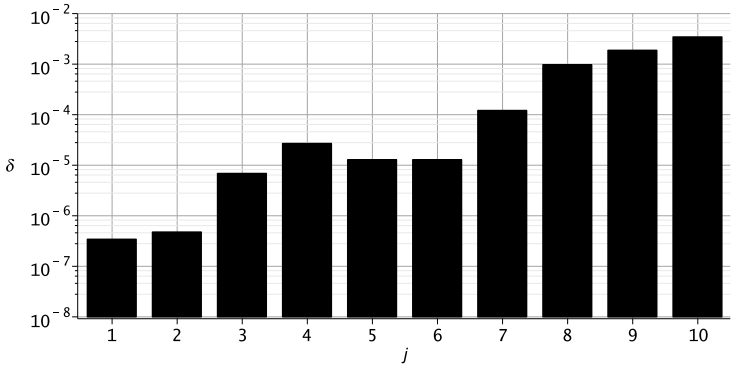

Figure 1. Relative error in the calculation of the first 10 normal modes: a comparison of finite element method (FEM) and analytical solution

Figure 2 presents the errors of mode calculation using four potential method and incomplete Galerkin method (IGM). The sum, which represent the approximate solution in IGM, consists of 1022 terms. The error growth is non-monotonic with the mode number j, the relative error is less than 1.1×10^{-5} for the first ten calculated phase retardation coefficients.

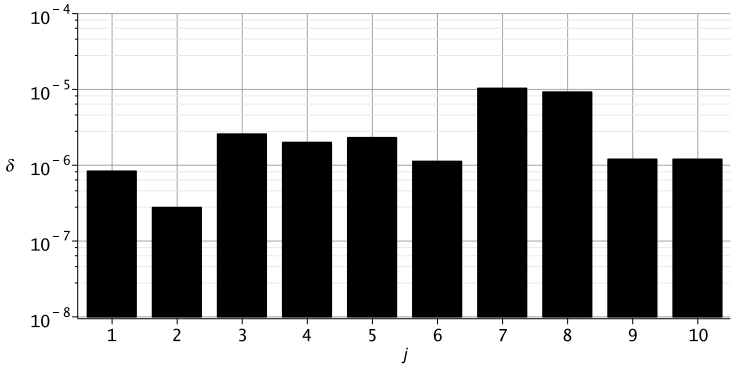

Figure 2. Relative error in the calculation of the first 10 normal modes: a comparison of incomplete Galerkin method (IGM) and analytical solution

The performed numerical experiments show that both methods of calculating normal modes with such a choice of parameters yield reasonable results.

4. Filled waveguide

Now let us consider a square waveguide ($\epsilon_1 = 1$, $\mu_1 = 1$) of width $l = 3.5 \cdot 10^{-6}$ [m] with four square cores of width $d = 10^{-6}$[m] with $\epsilon_2 = 2$, $\mu_2 = 1$ inside, as shown in Figure 3.

The phase retardation coefficients for such a structure cannot be calculated analytically, so we can only compare the phase retardation coefficients calculated numerically using FEM and IGM. For comparison, a relative error Δ_j

is presented in Figure 4, where Δ_j is defined as presented below

$$\Delta_j = \left| \frac{\beta_j^{FEM} - \beta_j^{IGM}}{\beta_j^{IGM}} \right| \tag{14}$$

in the calculation of the first 10 modes using these methods. Here β_j^{FEM} is the phase retardation coefficient calculated by FEM, and β_j^{IGM} is the same quantity calculated by IGM. The relative error grows with increasing mode number j and does not exceed the value 4×10^{-2} (see Figure 4).

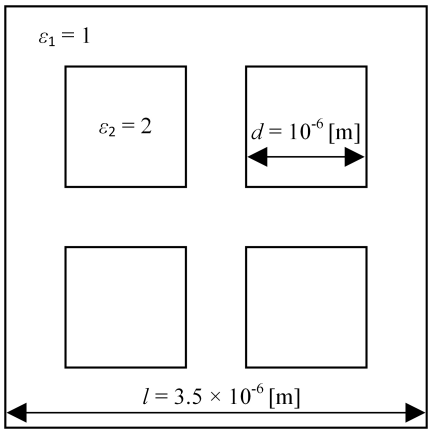

Figure 3. Cross section of a waveguide with 4 cores

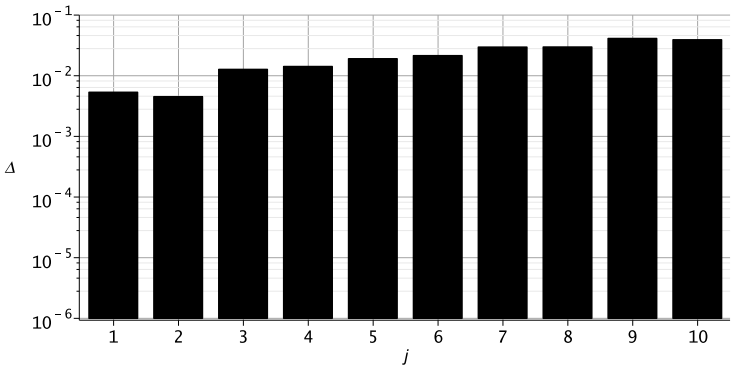

Figure 4. Relative error of calculating the first 10 normal modes: a comparison of the results obtained by FEM and IGM

5. Conclusion

The results obtained demonstrate the coincidence of the numerical results obtained by two different approaches. The first approach is based on the use

of the FEniCS Project libraries and on the use of FEM for an approximate solution of the problem of finding normal modes [14]. The second approach is based on the application of the recently proposed four potential method [30]. Numerical calculation is performed using the incomplete Galerkin method [33].

Both methods are tested on a waveguide structure corresponding to a hollow waveguide, for which phase retardation coefficients are analytically known. The accuracy obtained by both methods is not high, but sufficient for technical calculations. Using an example of a four-core waveguide, for which the phase retardation coefficients are not analytically known, both methods also give quite similar results, differing by less than 4%.

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Вычисление нормальных мод закрытых волноводов

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Целью работы является разработка и создание численных методов решения некоторых задач теории волноводов, а также их реализация в виде комплексов программ, ориентированных на широкий круг практических проблем от классических вопросов передачи СВЧ излучения до проектирования оптических волноводов и датчиков. При этом мы стремимся к простоте реализации разрабатываемых методов в системах компьютерной алгебры (Maple, Sage) или в программном обеспечении, ориентированном на метод конечных элементов (FreeFem++). В работе использовано представление электромагнитных полей в волноводе при помощи четырёх потенциалов. Эти потенциалы не уменьшают число искомых функций, но даже в том случае, когда диэлектрическая и магнитная проницаемости описываются разрывными функциями, они оказываются достаточно гладкими функциями. Сделана простейшая проверка работоспособности программ путём вычисления нормальных мод полого волновода. Показано, что относительная ошибка в вычислении первых 10 нормальных мод не превышает 4%. Эти результаты свидетельствуют о работоспособности предложенного в настоящей статье метода.

Ключевые слова: интегральная оптика, закрытый волновод, компьютерное моделирование, метод конечных элементов, метод четырёх потенциалов