DISCRETE AND CONTINUOUS MODELS AND APPLIED COMPUTATIONAL SCIENCE

Volume 27 Number 2 (2019)

Founded in 1993
Founder: PEOPLES' FRIENDSHIP UNIVERSITY OF RUSSIA

DOI: 10.22363/2658-4670-2019-27-2

Edition registered by the Federal Service for Supervision of Communications, Information Technology and Mass Media
Registration Certificate: ПИ № ФС 77-76317, 19.07.2019

ISSN 2658-7149 (online); 2658-4670 (print)
4 issues per year.
Language: English.

Publisher: Peoples' Friendship University of Russia (RUDN University).
Indexed in Ulrich's Periodicals Directory (http://www.ulrichsweb.com), in Russian Science Citation Index (https://elibrary.ru), EBSCOhost (https://www.ebsco.com), CyberLeninka (https://cyberleninka.ru).

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Computer Design: *A. V. Korolkova*

Address of editorial board:
Ordzhonikidze str., 3, Moscow, Russia, 115419
Tel. +7 (495) 955-07-16, e-mail: publishing@rudn.ru
Editorial office:
Tel. +7 (495) 952-02-50, miphj@rudn.ru, dcm@sci.pfu.edu.ru
site: http://journals.rudn.ru/miph

Paper size 70×100/16. Offset paper. Offset printing. Typeface "Computer Modern".
Conventional printed sheet 5,16. Printing run 500 copies. Open price. The order 620.
PEOPLES' FRIENDSHIP UNIVERSITY OF RUSSIA
6 Miklukho-Maklaya str., 117198 Moscow, Russia
Printed at RUDN Publishing House:
3 Ordzhonikidze str., 115419 Moscow, Russia,
Ph. +7 (495) 952-04-41; e-mail: publishing@rudn.ru

Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 104
http://journals.rudn.ru/miph

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UDC 512, 519.6
DOI: 10.22363/2658-4670-2019-27-2-105-123

On algebraic integrals of a differential equation

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(received: October 24, 2019; accepted: November 14, 2019)

We consider the problem of integrating a given differential equation in algebraic functions, which arose together with the integral calculus, but still is not completely resolved in finite form. The difficulties that modern systems of computer algebra face in solving it are examined using Maple as an example. Its solution according to the method of Lagutinski's determinants and its implementation in the form of a Sagemath package are presented.

Necessary conditions for the existence of an integral of contracting derivation are given. A derivation D of the ring A will be called contracting, if such basis $B = \{m_1, m_2, \dots\}$ exists in which $Dm_i = c_i m_i + o(m_i)$. We prove that a contracting derivation of a polynomial ring R admits a general integral only if among the indices $c_1, c_2, \dots$ there are equal ones. This theorem is convenient for applying to the problem of finding an algebraic integral of Briot–Bouquet equation and differential equations with symbolic parameters. A number of necessary criteria for the existence of an integral are obtained, including those for differential equations of the Briot and Bouquet. New necessary conditions for the existence of a rational integral concerning a fixed singular point are given and realized in Sage.

Key words and phrases: Darboux polynomials; algebraic integrals of differential equations; finite solution; Sage; Sagemath; Maple

1. De Beaune problem

In the theory of differential equations, it is common from the very beginning to choose a class of functions in which solutions of differential equations are sought so wide that the initial problem has solutions for almost all initial data. In the case of symbolic integration, or finding the solution in finite form, on the contrary, this class is constricted to make it possible in a finite number of operations, first, to find out whether the given differential equation has a general solution in this class, and second, to write out this solution explicitly. The simplest class, which could be expected to possess the above two properties, is the set of algebraic functions.

The problem of integrating differential equations in algebraic functions arose as early as the 1630s, when Forimond de Beaune proposed to Descartes several "inverse tangent problems" [1, Pp. 510–518]. We formulate this purely algebraic problem as follows.

Problem 1 (de Beaune). *Clarify whether a given differential equation*

$$p(x,y)dx + q(x,y)dy = 0, \quad p, q \in k[x,y], \tag{1}$$

has an integral r in the field $k(x,y)$; in the case of a positive answer, write out this integral.

Here k is the field of constants, commonly represented by $\mathbb{Q}$, $\mathbb{C}$ or $\mathbb{Q}[a,b,\dots]$, where $a, b, \dots$ are the parameters that enter the differential equation. There is no reason to consider these cases separately, so we assume that k is an infinite field of characteristic zero.

The interest to the De Beaune problem sometimes faded away, sometimes arose again. At the turn of the XIX–XX centuries, it was due to successes in proving the nonexistence of algebraic integrals of dynamical systems; among the papers of this period worth particular attention are the Poincaré memoir [2, Pp. 35–95] and a series of articles by M.N. Lagutinski [3, 4]; the biographical data were published by J.-M. Strelcyn [5, 6].

Recently, the classical problem of finding an algebraic integral has again become relevant in connection with the development of algorithms for the symbolic solution of differential equations suitable for implementation in modern computer algebra systems [7, 8]. First of all, it should be noted that popular computer algebra systems cannot efficiently recognize differential equations having algebraic integrals.

Example 1. To confirm this statement the following test was used. Let u, v — be arbitrary polynomials, then $w = u/v$ is an integral of the differential equation

$$\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dx + \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right)dy = 0.$$

Taking randomly u and v, we get the differential equation

$$p(x,y)dx + q(x,y)dy = 0, \quad p, q \in \mathbb{Q}[x,y].$$

An attempt to apply standard methods of solving differential equations to this differential equation in the Maple computer algebra system reduces the differential equation to a quadrature of the form

$$\int r dx + s dy = C, \quad r, s \in \mathbb{Q}(x,y),$$

occupying many screens, moreover, Maple cannot take the written integrals.

It is worth noting that for the symbolic solution of differential equations in Maple the package DETools [9] is used. Within the second algorithm of DETools the search for integrating factors in the ring $\mathbb{C}[x, y]$ is executed. The equation generated in the test has several such integrating factors, namely, u and v, so that Maple would have to cope with the test. However, the following occurs:

— `symgen` returns two integrating factors, whose ratio yields the rational desired integral,
— `dsolve` ignores the second factor and write out a quadrature which it cannot calculate in elementary functions, although the full implementation of Ostrogradski algorithm would cope with this difficulty.

Thus, usually Maple cannot recognize an algebraic integral, however, the user can do it himself, looking at the result of applying the function `symgen`.

Insurmountable difficulties arise when p and q have common factors. The methods implemented by Maple, first of all, relieve the ordinary differential equation to be solved from common factors. The reduced equation may not have integrating factors in the ring $\mathbb{C}[x, y]$, and finding factors from $\mathbb{C}(x, y)$ leads to nonlinear equations for the coefficients and requires completely different computational costs for which the developers of `symgen` did not go. As a result, e.g., when

$$u = (x^2 + y)^5(x - y^6 + 1) + 1, \quad v = (13xy^8 + y^5 + 3xy + 2)(x^2 + y)^4,$$

`symgen` finds one factor from $\mathbb{Q}[x, y]$ and nothing else.

Despite the antiquity of the de Beaune problem, we do not have an algorithm to solve it in a finite number of operations. The de Beaune problem is equivalent to the problem of integrating a partial differential equation

$$p\frac{\partial r}{\partial y} - q\frac{\partial r}{\partial x} = 0$$

in the field $k(x, y)$; we will further briefly write it as $Dr = 0$. By the method of uncertain coefficients, we can substitute into the equation $Dr = 0$ the expression

$$r = \frac{a + \dots + by^n}{1 + \dots + cy^n}$$

and obtain a system of nonlinear algebraic equations for finding the coefficient $a, b, c, \dots$. The solvability of this system can be determined in a finite number of steps and in a purely algebraic way. Therefore, in a finite number of operations one can find out whether a given differential equation has rational integrals whose degree does not exceed a given number n.

The problem of finding the upper bound for the degree of the sought integral was noted by Descartes, and in some cases was resolved by Poincaré [2], pp. 35-95. The idea of the Poincaré method is as follows. If a differential equation admits a rational integral, then its integral curves form a linear sheaf of algebraic curves of some order n, this immediately follows from a comparison of the Cauchy theorem from the analytic theory of differential equations [10] and Bertini's theorem from the theory of algebraic curves [11]. Two arbitrary curves of the sheaf intersect at n^2 fixed points. On the other hand, according to the Cauchy theorem, these curves can intersect only at those points at which the polynomials p and q vanish simultaneously; in the analytical theory of differential equations, such singular points are called fixed points. If the orders of the curves $p(x, y) = 0$ and $q(x, y) = 0$ do not exceed m, then $n \leqslant m$. However, it is impossible to bring this idea to a rigorous statement: among the intersection points of the integral curves there may be multiple and infinitely distant ones, as well as at fixed singular points of the differential equation, the solutions may have various kinds of "degeneracies". That is why M.N. Lagutinski carefully notes that the "French scientist in the work just referred deduces a number of equalities and inequalities that in some cases achieve the goal of indicating the upper bound of the order n" [3, P. 181]. Taking into account that "the difficulties of this way for solving this problem have stopped even H. Poincaré" [3], it is not hard to understand why in all modern implementations of algorithms for finding integrals, the order of the integral is assumed to be given [12].

The de Beaune problem, in which a bound for the orders of considered integrals is given, will be referred to as a bounded problem.

Problem 2 (The bounded de Beaune problem). *Clarify whether a given differential equation*

$$p(x, y)dx + q(x, y)dy = 0, \quad p, q \in k[x, y], \tag{2}$$

admits an integral r in the field $k(x, y)$ whose order does not exceed a given number N, and in case of positive answer, write out this integral.

Practically the described solution of a system of nonlinear algebraic equations requires considerable computation resources even at $N = 3$. Therefore, the authors of algorithms for solving this problem try to avoid the solution of nonlinear systems. Among the implemented algorithms, worth special attention are the Lagutinski's method of determinants and the method proposed by Jacques–Arthur Weil in 1985 based on power series expansion [12].

2. The bounded de Beaune problem and Lagutinski's method of determinants

Lagutinski's method allows searching for particular and general integrals of ring derivations of sufficiently general form. An up-to-date presentation of this method for the case of the $\mathbb{C}[x, y]$ ring is given in [13, 14], and the general case is considered in [15]. For convenience of reference we present here a brief description of the method.

Let R be a ring with derivation D and field of constants k. Consider k to be an arbitrary field of characteristic zero and $\mathbb{Q} \in k$. Let us call a general integral of this derivation a pair of elements ψ_1, ψ_2 linearly independent over the field k, satisfying the equality

$$\psi_1 D\psi_2 = \psi_2 D\psi_1. \tag{3}$$

If the ring R is integral, then the derivation is naturally continued on its field of quotients, and the fraction ψ_1/ψ_2 satisfies the equation

$$D(\psi_1/\psi_2) = 0.$$

We will deal with rings where a basis can be introduced in the following sense.

Definition 1. A countable ordered set B of elements m_j of a ring R will be called a basis of the ring if

1) any element of the ring R can be presented as a linear combination of a finite number of elements of the set B with constant coefficients;
2) a product of any two elements of the set B belongs to B, and follows strictly after both efficients, i.e., $m_i m_j = m_n$ and n is strictly greater than i and j.

Let us introduce the ordering relationship in the basis, i.e., the inequality $m_i < m_j$ means that $i < j$ and assume that the notation $u = o(m_i)$ means that the representation of the element u of the ring R in the form of a linear combination of basis elements contains the basis elements whose numbers are strictly larger than i. If $u = am_i + o(m_i)$, $a \neq 0$, then the addend am_i will be called the lowest term in u.

In contrast to the common agreement, we call the number of the greatest basis term entering the decomposition of an element u in the basis an order of this element.

Example 2. In the ring $R = \mathbb{Q}[x, y]$ a system of various monomials may be taken to be a basis by accepting the glex-ordering:

$$1, y, x, y^2, xy, x^2, y^3, y^2x, yx^2, x^3, ...$$

Below this basis will be referred to as glex-basis. In this case, for example,

$$y^2 + xy + 3x^3 = y^2 + o(y^2),$$

and the order of this element equals 10.

The calculations of integrals is closely related to Lagutinski's determinants.

Definition 2. Compose an infinite matrix with the first row

$$m_1, m_2, ... ,$$

the second row being the first derivative of the first one,

$$Dm_1, Dm_2, ... ,$$

the third row being the second derivative of the first one,

$$D^2m_1, D^2m_2, ... ,$$

and so on to infinity. A determinant of the corner minor of the n-th order of this matrix, i.e.,

$$\det \begin{pmatrix} m_1 & m_2 & ... & m_n \\ Dm_1 & Dm_2 & ... & Dm_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}m_1 & D^{n-1}m_2 & ..., & D^{n-1}m_n \end{pmatrix} \qquad (4)$$

will be denoted by Δ_n and called Lagutinski's determinant of the n-th order.

The following theorem provides a complete solution of the bounded de Beaune problem.

Theorem 1 (by M. N. Lagutinski). *Let R be a ring of polynomials.*

1. *A general integral exists then and only then, when all Lagutinski's determinants of sufficiently high order are equal to zero.*
2. *A general integral of the order N exists then and only then, when $\Delta_N = 0$; in this case the integral can be calculated as a ratio of the corresponding minors of this determinant.*

The proof of Lagutinski's theorem and the rule of choosing minors to construct integrals is given in [15].

Remark 1. *From this theorem, in particular, it follows that finding a rational integral does not require the field extension. If p and q belong to $\mathbb{Q}[x, y]$ and there is an integral in $\mathbb{C}(x, y)$, then applying this theorem at $k = \mathbb{C}$, we see that for a certain N $\Delta_N = 0$. The calculation of Lagutinski's determinants does not lead beyond the field $\mathbb{Q}$. Therefor, applying this theorem at $k = \mathbb{Q}$, we arrive at the existence of an integral in the field $\mathbb{Q}(x, y)$. For this reason, below we mean the integral of an equation with integer coefficients to be an element of $\mathbb{Q}(x, y)$.*

Lagutinski's method agrees well with the concept of operating with rings, accepted in Sage [16]. We have written a package `Lagutinski` [17] in Sage, which allows calculation of Lagutinski's determinants and integrals in this environment. The package was presented in 2016 at a number of conferences on computer algebra [18–20]. Here we restrict ourselves to one example illustrating the application of this package. In more detail the technique of its application is described in [21].

Example 3. Let the Bernoulli differential equation be given,

$$y(x + 1)dy - (y^2 + x + 2)dx = 0,$$

which for certain possesses an algebraic integral. Let us find it using Lagutinski's method. For this purpose we specify in a usual manner the corresponding differential ring and its basis:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: D=lambda phi: y*(x+1)*diff(phi,x)+(y^2+x+2)*diff(phi,y)
sage: B= sorted(((1+x+y)^5).monomials(),reverse=0)
```

and load our package

```
sage: load(’’lagutinski.sage’’)
None
```

Now we can calculate Lagutinski's determinants, e.g.,

```
sage: lagutinski_det(2,B)
y^2 + x + 2
sage: lagutinski_det(3,B)
x^3 + x*y^2 + 5*x^2 + y^2 + 8*x + 4
```

Let us find that of the determinants, which equals zero:

```
sage: lagutinski_det(5,B)==0
False
sage: lagutinski_det(6,B)==0
True
```

Since $\Delta_5 \neq 0$, and $\Delta_6 = 0$, the integral will be:

```
sage: lagutinski_integral(6,B)
(-54*x^2 + 18*y^2 - 72*x)/(-18*y^2 - 36*x - 54)
```

Since the calculations are cumbersome, the first argument of this function should coincide with the smallest number of zero determinant.

The theory and its implementation are illustrated by Yu Ying by the examples taken from the book of problems by A. F. Filippov, the report is published in [22]. The numerical experiments carried out show that Lagutinski's method practically allows fast and resource-saving detection of the presence of a rational integral. However, the method requires considerable computational costs for the calculation of this integral. Note that the problem of determining the boundary for the integral order, always discussed in theory, appeared insignificant in practice, since there were no differential equations in the book of problems, whose integral curves had the order of 10 or higher.

3. Necessary conditions for the existence of an integral of contracting derivation

In application to a non-bounded de Beaune problem the Lagutinski method yields a sequence of determinants $\Delta_1, \Delta_2, \dots$.

According to the theorem 1 this sequence is finite then and only then when an integral in $k(x,y)$ exists. However, its condition cannot be checked constructively, moreover, the calculation of determinants of the order of $20 \div 30$ already requires considerable computational costs. Therefore, it is important to transform this statement into a necessary condition of the integral existence, at least for some classes of derivations.

Definition 3. A derivation D of the ring A will be called contracting, if such basis $B = \{m_1, m_2, \dots\}$ exists in which

$$Dm_i = c_i m_i + o(m_i). \tag{5}$$

Any basis, in which the differentiation operation satisfies the conditions 5, will be called a contracted derivation of D, c_i will be called indices of contraction in the basis B.

Generally, there can be several contracting bases, and the indices of contraction n then can be different. The possibility of applying the integral existence criteria presented below essentially depend on the possibility to choose a basis that contracts a given derivation.

Remark 2. *The proposed name refers to the theory of contracting operators in Banach spaces. In the present case, of course, there is no norm, but the basis specifies a certain "topology", and the condition contained in the definition* *indicates the fact that the derivation D transforms the basis element m_i into the element Dm_i, which is a linear combination of basis elements whose numbers are greater than i.*

Example 4. In the ring $R = \mathbb{Q}[x, y]$ the derivation

$$D = (ay + cx + \dots)\frac{\partial}{\partial y} - (bx + \dots)\frac{\partial}{\partial x}, \qquad (6)$$

is contracting with respect to glex-basis $B = \{1, y, x, y^2, yx, x^2, \dots\}$.
Indeed,

$$D(y^n x^m) = n(ay + cx + \dots)y^{n-1}x^m - m(bx + \dots)y^n x^{m-1} = \\ = (an - mb)y^n x^m + o(y^n x^m).$$

The numbers $an - mb$ that appeared here are indices of contraction.

Theorem 2 (necessary criterion for existence of general integrals). *A contracting derivation of a polynomial ring R admits a general integral only if among the indices of contraction there are equal ones.*

This simple criterion follows from theorem 1 using the following lemma.

Lemma 1. *Let the derivation D be contracting, then in a suitable basis*

$$\Delta_n = W(c_1, c_2, \dots, c_n) \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right),$$

where W is a Vandermonde determinant.

Proof. In a suitable basis

$$Dm_i = c_i m_i + o(m_i),$$

from where

$$Do(m_i) = o(m_i)$$

and further

$$D^m m_i = c_i^m m_i + o(m_i).$$

The Lagutinski determinant Δ_n is formed by linear combinations of the products

$$D^{i_1} m_1 D^{i_2} m_2 \cdots = (c_1^{i_1} c_2^{i_2} \dots) \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right),$$

and, therefore, is a sum of the expression

$$\sum_{i_1, i_2, \dots} (-1)^{\sigma(i_1, i_2, \dots)} c_1^{i_1} c_2^{i_2} \dots \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right)$$

and higher-order terms. In the expression written out it is easy to recognize a Vandermonde determinant

$$W(c_1, \dots, c_n) = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{n-1} & c_2^{n-1} & \dots & c_n^{n-1} \end{pmatrix}.$$

Example 5. The derivation

$$D = (x + x^4 y)\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}$$

of the ring $\mathbb{Q}[x, y]$ is contracting, since in the glex-basis

$$B = \{1, y, x, y^2, xy, x^2, \dots\}$$

is true

$$D(x^n y^m) = n x^n y^m + (x+y) m x^n y^{m-1} + o(x^n y^m) = (n+m) x^n y^m + o(x^n y^m).$$

The indices of contraction form a sequence $0, 1, 1, 2, \dots$, in which equal elements are present. Therefore

$$\Delta_2 = W(0, 1) y + o(y) = y + o(y),$$

and then we obtain only

$$\Delta_n = W(0, 1, 1, \dots) \prod_{i=1}^{n} m_i + o\left(\prod_{i=1}^{n} m_i\right) = o\left(\prod_{i=1}^{n} m_i\right).$$

For small orders n the validity of this formula is easily checked by direct calculation:

```
sage: D=lambda phi: (x+x^4*y)*diff(phi,x)+(x+y)*diff(phi,y)
sage: prod(B[:2])
y
sage: sorted(lagutinski_det(2,B).monomials(),reverse=0)
[y, x]
sage: prod(B[:3])
x*y
sage: sorted(lagutinski_det(3,B).monomials(),reverse=0)
[x^2, x^4*y^2, x^5*y, x^6, x^7*y^3, x^8*y^2]
sage: prod(B[:4])
x*y^3
sage: sorted(lagutinski_det(4,B).monomials(),reverse=0)
[x^2*y^2, x^3*y, x^4, x^4*y^4, x^5*y^3, x^6*y^2, x^7*y, x^8,
x^7*y^5, x^8*y^4, x^9*y^3, x^10*y^2, x^11*y, x^10*y^6,
x^11*y^5, x^12*y^4, x^13*y^3]
```

4. Necessary conditions for the existence of a rational integral of the Briot–Bouquet equation

The theorem 2 is convenient for applying to the problem of finding an algebraic integral of the differential equation

$$(ay + cx + \dots)dx + (bx + \dots)dy = 0, \tag{7}$$

which we, following E. Ains [23, n. 12.6], will refer to as the Briot–Bouquet equation.

Remark 3. *Equation* (7) *possesses a number of unexpected analytical properties and for a long time attracts the attention of researchers. The initial problem*

$$\begin{cases} (ay + cx + \dots)dx + (bx + \dots)dy = 0, \\ y|_{x=0} = 0 \end{cases}$$

does not satisfy the conditions of the Cauchy theorem. Nevertheless, in 1856 Briot and Bouquet [23, n. 12.6], has proved that at $a/b \notin \mathbb{Z}$ this problem admits a unique solutions holomorphic in the vicinity of zero. The question of whether the initial problem admits other solutions having a singularity at zero, was the subject of research by Briot and Bouquet, Picard and Poincaré [24, n. 426].

An integral of the equation (7) is also an integral of the derivation

$$D = (ay + cx + \dots)\frac{\partial}{\partial y} - (bx + \dots)\frac{\partial}{\partial x}, \tag{8}$$

which as it has been shown in the example 4, contracts the glex-basis

$$B = \{1, y, x, y^2, xy, x^2, \dots\}.$$

From here, as a consequence of theorem 2, immediately follows:

Theorem 3 (about the Briot–Bouquet equation). *The differential Briot–Bouquet equation* (7) *can have a rational integral in $k(x, y)$ only if a and b are linearly dependent over the field $\mathbb{Q}$.*

Proof. Applying the derivation (8) to a monomial, we get

$$Dx^n y^m = (ay + cx + \dots)\frac{\partial x^n y^m}{\partial y} - (bx + \dots)\frac{\partial x^n y^m}{\partial x} = \\ = (ma - nb)x^n y^m + o(x^n y^m).$$

If there are no integer relations between a and b, then among the indices of contraction $ma - nb$ there are no equal ones, so that according to theorem 2 this derivation does not admit general integrals. □

Example 6. The general solution of the linear equation

$$(ay + cx)dx + bxdy = 0$$

is easy to write out

$$x^{a/b}\left(y+\frac{cy}{b+a}\right)=C,$$

where C is the integration constant. Whether the written integral is algebraic or not, depends on whether the ratio a/b is a rational number or not, which completely agrees with the proved lemma.

Example 7. According to the proved theorem the equation

$$(ay+cx)dx+(bx+xy)dy=0.$$

has no algebraic integral at arbitrary a and b.

5. Necessary conditions for the existence of a rational integral concerning a fixed singular point

It is easily seen that the point $(0,0)$ is a fixed singular point of the differential equation (7). Recall that the Cauchy theorem is applicable to all points of the xy-plane except those in which the polynomials p and q from $\mathbb{C}[x,y]$ simultaneously turn into zero. These points are called fixed singular points of the differential equation [10]. If we put the origin of the coordinate system into a fixed singular point, then

$$pdx+qdy=(a_{11}x+a_{12}y+\dots)dx+(a_{21}x+a_{22}y+\dots)dy,$$

where ... denote the terms of the order higher than the first one. The coefficient a_{22} prevents the application of theorem 3, however, it is easy to get rid of it by a linear change of variables.

Theorem 4. *Let neither p, nor q be reducible to a constant and the field of constants k is algebraically closed. Then to any fixed singular point (x_0,y_0) of the differential equation we can relate a new system of coordinates*

$$\begin{cases} x=x_0+\xi+\alpha\eta, & \alpha\in k\\ y=y_0+\eta, \end{cases} \tag{9}$$

in which this differential equation takes the form of Briot–Bouquet equation, i.e.,

$$(a\eta+c\xi+\dots)d\xi+(b\xi+\dots)d\eta,$$

where ... denotes the higher-order terms.

Proof. Since the field k is algebraically closed, the curves $p(x,y)=0$ and $q(x,y)=0$ intersect at some points of the xy-plane. Let us denote one on these points as (x_0,y_0) and put the origin of coordinates into this point. Then

$$pdx+qdy=(a_{11}x+a_{12}y+\dots)dx+(a_{21}x+a_{22}y+\dots)dy,$$

where ... denotes higher-order terms. The differential equation $pdx + qdy = 0$ corresponds to the derivation

$$D = (a_{11}x + a_{12}y + \dots)\frac{\partial}{\partial y} - (a_{21}x + a_{22}y + \dots)\frac{\partial}{\partial x}.$$

If $a_{22} \neq 0$, then it can be eliminated by a linear transformation

$$\begin{cases} x = \xi + \alpha\eta, & \alpha \in k, \\ y = \eta. \end{cases}$$

Under this transformation the form changes as follows:

$$pdx + qdy = (\dots)d\xi + [(a_{11}(\xi + \alpha\eta) + a_{12}\eta)\alpha + a_{21}(\xi + \alpha\eta) + a_{22}\eta + \dots]d\eta.$$

Equating the coefficient at $\eta d\eta$ to zero, we arrive at the quadratic equation

$$a_{11}\alpha^2 + (a_{12} + a_{21})\alpha + a_{22} = 0$$

for finding the parameter α. Since the field k is algebraically closed, this quadratic equation has roots in k, and for such a choice of the parameter the expression will get the desired form

$$pdx + qdy = (a\eta + c\xi + \dots)d\xi + (b\xi + \dots)d\eta.$$

Collecting the results of theorems 3 and 4 together, we get the following algorithm that allows clarifying whether the given differential equation (1) has a rational integral in the field $k(x, y)$:

1) find the fixed singular point (x_0, y_0);
2) execute a linear transformation, containing the parameter α, in the form

$$pdx + qdy = (a_{11}\xi + a_{12}\eta + \dots)d\xi + (a_{21}\xi + a_{22}\eta + \dots)d\eta;$$

3) determine the value of the parameter α from the quadratic equation $a_{22} = 0$;
4) check whether for such value of α the coefficients a_{12} and a_{21} are linearly dependent over $\mathbb{Q}$.

If yes, they are linearly dependent, then the differential equation can admit a rational integral, otherwise it does not exist. It is worth noting that the formulated criterion is necessary, but not sufficient.

Our Lagutinski package includes the function `lagutinski_ab`, which for specified p and q returns `true`, if at the first fixed singular point the above quantities are linearly dependent.

Example 8. For checking, let us begin with the linear equation

$$(x + y)dx + xdy = 0,$$

the general solution of which is expressed as

$$y(x) = -\frac{x}{2} + \frac{C}{x}.$$

We have:

```
sage: x,y=var('x,y')
sage: lagutinski_ab(x+y,x)
True
```

Example 9. Maple cannot make any definite conclusion about the equation

$$(2 - x^2 - y^2)dx + (x - y)dy = 0.$$

The application of our criterion yields

```
sage: x,y=var('x,y')
sage: lagutinski_ab(2-x^2-y^2,x-y)
False
```

Therefore, this equation does not admit a rational integral in the field $\mathbb{C}(x, y)$.

It is well known that an arbitrary differential equation (1) cannot be integrated in elementary functions. The proposed algorithm specifies the "degeneracies" that should occur with the coefficients p and q of the differential equation considered to make it integrable in such functions. If the polynomials p, q belong to $\mathbb{Q}[x, y]$, then the application of the described algorithm introduces algebraic numbers twice: first, in finding the fixed singular points and, second, in searching for the parameter α. Therefore, generally the ratio of the coefficients a_{12} and a_{21} appears to be an algebraic number, so that the equation does not admit an algebraic integral even in $\mathbb{C}(x, y)$.

Remark 4. *It is natural to draw an analogy here with the integration of rational functions: in the general case, the denominator of a rational function has simple zeros, and the integral of such a function consists of logarithmic terms; the integral will be rational only in the exotic case when all the singularities are multiple.*

6. Differential equations with symbolic parameters

The theorem 3 is seen useless in the case, when the coefficients of Briot–Bouquet equation belong to the field $\mathbb{Q}$. Actually, theorem 2 provides a convenient criterion of unsolvability when the considered equation contains indefinite parameters $a, b, \ldots$, in other words, when as the field k we consider the field $\mathbb{Q}(a, b, \ldots)$, generated by the variables $a, b, \ldots$ algebraically independent over $\mathbb{Q}$. With their appearance the problem of finding an algebraic integral is separated into two problems:

— to clarify whether the differential equation admits a rational integral in the field $k(x, y)$, i.e., "in the general case";
— to find particular values of the parameters $a, b, \ldots$ in $\mathbb{C}$, for which the differential equation admits a rational integral in the field $\mathbb{C}(x, y)$.

The first problem for the equation (7) was completely solved by theorem 3: this equation has no rational integral in the general case.

Now let us proceed to the second problem. Without loss of generality, we can assume that p and q are polynomials with respect to x, y and all parameters $a, b, \dots$; for clarity let us consider the set of complex values of the parameters $a, b, \dots$ as a point in a finite-dimensional affine space A over the field $\mathbb{C}$. Accepting this agreement and using common notations of algebraic geometry [25], the theorem 1 for the field $R = Q[x, y, a, b, \dots]$ can be reformulated in the following way.

Theorem 5. *Let the coefficients Δ_n of monomials $x^n y^m$ generate an ideal J_n of the ring $\mathbb{Q}[a, b, \dots]$. The set of points $(a, b, \dots)$ of the affine space A, for which the differential equation*

$$pdx + qdy = 0, \quad p, q \in \mathbb{Q}[x, y, a, b, \dots],$$

admits a rational integral from $\mathbb{Q}(x, y)$, whose order does not exceed, is an algebraic affine set $Z(J_n)$ in A.

Proof. If the point $(a, b, \dots)$ belongs to $Z(J_n)$, then $\Delta_n(x, y, a, b, \dots)$ at such values of parameters $a, b, \dots$ indentically turns into zero, and due to the Lagutinski theorem 1 the differential equation admits a rational integral. Conversely, if for some values of the parameters $a, b, \dots$ the differential equation admits a rational integral of the order n, then the Lagutinski determinant of the same order turns into zero identically and, therefore, $(a, b, \dots)$ belongs to $Z(J_n)$. □

Generally, the set $Z(J_n)$ can be empty or reducible.

Example 10. Consider again the linear equation

$$(ay + cx)dx + bxdy = 0.$$

Let us specify the appropriate ring, derivation, and basis:

```
sage: R.<x,y,a,b,c> = PolynomialRing(QQ, 5)
sage: D=lambda phi: (a*y+c*x)*diff(phi,y) -b*x*diff(phi,x)
sage: B= sorted(((1+x+y)^30).monomials(),reverse=0)
```

Calculate the Lagutinski determinants:

```
sage: lagutinski_det(2,B).factor()
y*a + x*c
sage: lagutinski_det(3,B).factor()
b * a * x * (y*a + y*b + x*c)
sage: lagutinski_det(4,B).factor()
(-2) * b * a * x * (y*a + x*c) * (y*a + y*b + x*c) * (-
2*y*a^2 - y*a*b - 2*x*a*c + 2*x*b*c)
```

In the three-dimensional affine space A the set $Z(J_2)$ represents a straight line $\{a = 0, c = 0\}$, the sets $Z(J_3), Z(J_4)$ represent a union of two planes $\{a = 0\} \cup \{b = 0\}$, and so on.

According to the theorem 5 the values of the parameters $a, b, \dots$, for which the differential equation

$$pdx + qdy = 0, \quad p, q \in \mathbb{Q}[x, y, a, b, \dots]$$

admits a rational integral of any order in $\mathbb{C}(x, y)$ for the set $\cup Z(J_n)$. It could be expected that this set is also algebraic, as it usually happens in algebraic problems. However, this is not true.

Example 11. The differential equation from the example 6 has a rational integral then and only then, when the ratio a/b is a rational number or when $b = 0$. Therefore $\cup Z(J_n)$ represents a union of various planes

$$na + mb = 0, \quad n, m \in \mathbb{Z}$$

in the three-dimensional affine space A.

Now let us reformulate the theorem 3 in these terms.

Theorem 6. *The projection onto the plane ab of a set of all values of the parameters $a, b, \dots$, at which the differential equation* (7) *admits a rational integral in $\mathbb{C}(x, y)$, is a union of a certain number of straight lines of the form*

$$na + mb = 0, \quad n, m \in \mathbb{Z}$$

and points.

Proof. According to the theorem 5 the set of all points of affine space A, at which the differential equation admits an integral from $\mathbb{C}(x, y)$, is a sum of algebraic affine sets and, therefore, represents a union of irreducible affine manifolds. And according to the theorem 3 a projection of this set onto the plane ab is formed by points that are linearly dependent over $\mathbb{Q}$.

This projection cannot coincide with the entire plane, therefore, it can be decomposed into irreducible lines and points. Assume, in contradiction to the theorem, that among these lines there is an irreducible line C of the order r, different from straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

According to the theorem 3 for any point $(a, b) \in C$ of this curve it is possible to specify one and only one such pair of mutually simple integer numbers (n, m) that

$$na + mb = 0, \quad m \geqslant 0.$$

From a geometric point of view this means that any point $(a, b) \in C$ corresponds to the point (n, m) of a projective straight line $P^1_{\mathbb{Q}}$, i.e., we get a mapping

$$f : C \to P^1_{\mathbb{Q}}.$$

The prototype of the point (n, m) is the set of points (a, b) of the line C at which the equality

$$na + mb = 0,$$

i.e., the points of intersection of the straight line $na + mb = 0$ and the line C in the plane $A^2_{\mathbb{C}}$. By Bézout's theorem, there are exactly r such points, therefore, there is a $(1, r)$-correspondence between the affine line C over $\mathbb{C}$ and the projective straight line P^1 over $\mathbb{Q}$. As soon as the set $\mathbb{Q}$ is countable and the set of points of the algebraic line over $\mathbb{C}$ is uncountable, the above is impossible. Hence, the projection is a union of straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

and points. □

As shown by example 11, the projection of a set of parameter sets $a, b, \dots$, at which the differential equation admits an algebraic integral can be a union of countable sets of affine manifolds, projected into a family of straight lines

$$na + mb = 0, \quad n, m \in \mathbb{Z}.$$

If, as Lagutinski hoped for, it would be possible to replace an infinite sequence of determinants $\Delta_2(x, y, a, b, \dots)$, $\Delta_3(x, y, a, b, \dots), \dots$ with a finite set of conditions, then this set would be an affine set. Thus, the appearance of the infinite sequence is not a defect of the Lagutinski method, it indicates the non-algebraic component of the theory of integration of differential equations in algebraic functions. Thus, the bounded de Beaune problem is completely solved by the Lagutinski method, and the unbounded de Beaune problem with parameters inevitably introduces non-algebraic sets and therefore, it does not admit a purely algebraic method of solution.

Conclusion

To summarize, let us list the main results of our consideration:

— Lagutinski's method allows solving the bounded de Beaune problem 2 using a finite number of operations, its implementation in Sage faces but one difficulty: with the growth of the boundary N the calculation of determinants requires more and more computer resources. The calculations can be made faster by choosing a suitable basis; in contracted bases the calculations are considerably more rapid (see lemma 1).

— For the unbounded problem 1 it appears possible to derive from Lagutinski theorem the necessary and easily checked conditions of existence of a rational integral. These criteria are applicable also in the cases, when the standard approaches implemented in Maple yield no definite information, see example 9.

— The above criterion appears to be rather useful for that problems with parameters, when for a given differential equation, containing indefinite parameters, one has to choose their values in a way providing the particular differential equation to admit an algebraic integral. This case clearly demonstrates the reasons why the full solution of an unbounded de Beaune problem is impossible: the desired set of the parameter values is not always an algebraic set.

Acknowledgments

The Lagutinski package was developed by M.D. Malykh and Yu Ying within the frameworks of a project, supported by the Russian Foundation for basic Research (grant No. 18-51-18005). The studies of necessary conditions for the existence of a contracting derivation integral were carried out by Yu Ying under the guidance of L. A. Sevastianov and supported by the Program RUDN 5-100. The calculations presented in the paper were performed in the computer algebra system Sage [16].

References

1. R. Descartes, Œuvres, Vol. 2, Léopold Cerf, Paris, 1898.
2. H. Poincaré, Œuvres, Vol. 3, Gautier, Paris, 1934.
3. M. N. Lagutinski, Applying polar operations to the integration of ordinary differential equations in finite form [Prilozhenie poljarnyh operacij k integrirovaniju obyknovennyh differencial'nyh uravnenij v konechnom vide], Soobshh. Har'kov. matem. obshh. Vtoraja serija 12 (1911) 111–243, in Russian.
   URL `http://mi.mathnet.ru/khmo117`
4. M. N. Lagutinski, On some polynomials and their relationship to algebraic integration of ordinary differential algebraic equations [O nekotoryh polinomah i svjazi ih s algebraicheskim integrirovaniem obyknovennyh differencial'nyh algebraicheskih uravnenij], Soobshh. Har'kov. matem. obshh. Vtoraja serija 13 (1912) 200–224, in Russian.
   URL `http://mi.mathnet.ru/khmo104`
5. V. A. Dobrovol'skij, N. V. Lokot', S. J.-M., Mikhail Nikolaevich Lagutinskii (1871–1915): un mathématicien méconnu, Historia Mathematica 25 (1998) 245–64.
6. A. J. Maciejewski, J.-M. Strelcyn, On the algebraic non-integrability of the Halphen system, Physics Letters A 201 (1995). doi:10.1016/0375-9601(95)00285-B.
7. Ngoc Thieu Vo, F. Winkler, Algebraic general solutions of first order algebraic ODEs, Vol. 9301, Springer, Cham, 2015, pp. 479–492. doi:10.1007/978-3-319-24021-3_35.
8. M. D. Malykh, On integration of the first order differential equations in finite terms, IOP Conf. Series: Journal of Physics: Conf. Series 788, article number 012026 (2017). doi:10.1088/1742-6596/788/1/012026.
9. E. S. Cheb-Terrab, Computer algebra solving of first order ODEs, Computer physics communications 101 (1997) 254–268. doi:10.1016/S0010-4655(97)00018-0.
10. W. W. Golubew, Vorlesungen über Differentialgleichungen im Komplexen, Deutscher Verlag der Wissenschaften, Berlin, 1958.
11. Fr. Severi, Lezioni di geometria algebrica, Angelo Graghi, Padova, 1908.
12. A. Bostan, G. Chéze, T. Cluzeau, J.-A. Weil, Efficient Algorithms for Computing Rational First Integrals and Darboux Polynomials of Planar Polynomial Vector Fields, Mathematics of Computation 85 (2016) 1393–1425. doi:10.1090/mcom/3007.
13. C. Christopher, J. Llibre, J. Vitório Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Pacific Journal of Mathematics 229 (1) (2007) 63–117. doi:10.2140/pjm.2007.229.63.
14. G. Chèze, Computation of Darboux polynomials and rational first integrals with bounded degree in polynomial time, Journal of Complexity 27 (2) (2011) 246–262. doi:10.1016/j.jco.2010.10.004.
15. M. D. Malykh, On the computation of the rational integrals of systems of ordinary differential equations by Lagutinski's method [Ob otyskanii ratsional'nykh integralov sistem obyknovennykh differentsial'nykh uravneniy po metodu M.N. Lagutinskogo], Bulletin of NRNU MEPhI [Vestnik Natsional'nogo issledovatel'skogo yadernogo universiteta "MIFI"] 5 (24) (2016) 327–336, in Russian. doi:10.1134/S2304487X16030068.

16. The Sage Developers, SageMath, the Sage Mathematics Software System (Version 7.4) (2016).
URL `https://www.sagemath.org`
17. M. D. Malykh, Lagutinski.sage, ver. 1.5., RUDN University (2016).
URL `http://malykhmd.neocities.org`
18. M. D. Malykh, On M.N. Lagutinski method for integration of ordinary differential equations, in: International conference “Polynomial Computer Algebra’2016”, 2016, pp. 57–58.
URL `http://pca.pdmi.ras.ru/2016/pca2016book.pdf`
19. M. D. Malykh, On the integration of ordinary differential equations [Ob integrirovanii obyknovennyh differencial’nyh uravnenij], in: Computer algebra. Proceedings of the international conference, June 29 – July 2, 2016, Moscow, Russia, 2016, pp. 25–29, in Russian.
20. M. D. Malykh, On the integration of first-order differential equations in finite form [Ob integrirovanii differencial’nyh uravnenij pervogo porjadka v konechnom vide], in: Fifth International Conference on Problems of Mathematical and Theoretical Physics and Mathematical Modelling. Moscow, April 5–7, 2016. Collection of reports, 2016, pp. 81–82, in Russian.
21. M. D. Malykh, On application of M. N. Lagutinski method to integration of differential equations in symbolic form. Part 1 [O yavnom atribute M.N. Lagutinskogo k integrirovaniyu differentsial’nykh uravneniy 1-go poryadka. Chast’ 1. Otyskaniye algebraicheskikh integralov], RUDN Journal of Mathematics, Information Sciences and Physics 25 (2) (2017) 103–112, in Russian. doi:10.22363/2312-9735-2017-25-2-103-112.
22. M. D. Malykh, Yu Ying, The Method of finding algebraic integral for first-order differential equations [Metodika otyskaniya algebraicheskikh integralov differentsial’nykh uravneniy pervogo poryadka], RUDN Journal of Mathematics, Information Sciences and Physics 26 (3) (2018) 285–291, in Russian. doi:10.22363/2312-9735-2018-26-3-285-291.
23. E. L. Ince, Ordinary differential equations, Courier Corporation, 1956.
24. É. Goursat, Cours d’analyse mathématique, Vol. 2, Gauthier-Villars, Paris, 1925.
25. R. Hartshorne, Algebraic geometry, Springer, 1977.

For citation:
M. D. Malykh, L. A. Sevastianov, Yu. Ying, On algebraic integrals of a differential equation, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 105–123. DOI: 10.22363/2658-4670-2019-27-2-105-123.

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Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 124–132
http://journals.rudn.ru/miph

Mathematical modeling

Research article

UDC 519.6;699.841
DOI: 10.22363/2658-4670-2019-27-2-124-132

Seismic stability of oscillating building on kinematic supports

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(received: February 4, 2019; accepted: November 14, 2019)

The design of kinematic supports is considered, which allows to damp the oscillation energy of seismic waves during earthquakes. The building rests on supports that have the geometry of straight cylinders. When horizontal ground oscillations occur, the supports are deflected at a small angle ψ. At the same time, their centre of gravity rises and tends to return to its original position under the action of two forces on each support: the weight of the building evenly distributed over all the supports, and the weight of the support itself. The first force is applied to the highest point of the support, the second one is applied to the centre of gravity of the support, so that the rotational moments of two forces act on the support.

It should be noted that under very strong vibrations of the ground, the projection of the centre of gravity could move beyond the base of the support. In this case, the supports will begin to tip over. We confine ourselves to considering such deviations that the rotational moments of the forces of gravity still tend to return the supports to their initial state of equilibrium.

Key words and phrases: ensuring seismic stability of buildings during earthquakes, the equation of motion of a physical pendulum, vibration damping.

1. Introduction

The amount of energy transferred to the building depends on the relation between the spectra of seismic effects and natural oscillations of the building. The closer the peaks of the spectra, the greater the energy transferred to the building under similar conditions. This energy is mainly absorbed by the inelastic deformations of the structure. Based on the above facts, two main tasks can be formulated aimed at ensuring the seismic resistance of a building:

— to separate the spectra and thereby reduce the amount of energy transmitted to the building and

— to provide the absorption of the remaining energy using special measures.

In the publications on deterministic analysis of the dynamics of constructions, describing the time-dependent motion of the system under the action of seismic load (strongly oscillating and irregular) [1–8], various types of the building supports are studied, aimed at efficient damping the energy of the spatial movement of the building caused by an earthquake. Among them the most successful solutions have been proposed by A. M. Kurzanov and Yu. P. Cherepinsky [6]. In this paper, we consider in detail the design and operation of Kurzanov's kinematic supports, which have been well-proven in experimental studies [1, 7, 8]. Our theoretical study of the functioning of kinematic supports is aimed at their mathematical modelling and the subsequent selection of parameters of the model of supports in order to solve both of the above problems of seismic stability of buildings. Note that in recent years the interest in the study of kinematic supports that provide the seismic resistance of buildings has noticeably increased [9–11]. See also [12–19].

2. Mathematical model of the functioning of kinematic supports

The design of kinematic supports is considered, which allows damping the oscillation energy of seismic waves during earthquakes. The building having the weight Mg rests on n supports, each having the weight mg. The supports have the geometry of straight cylinders of height h and base diameter a.

Under the action of an incident seismic wave, the whole construction "building + n supports" comes into a complex movement. Of all the seismic waves, let us consider horizontal waves. When horizontal ground oscillations occur, the supports are deflected by an angle ψ. At the same time, their centre of gravity rises and due to this fact the building, moving horizontally, rises and acquires additional potential energy. After that, the building tends to return to its original position and acquires additional kinetic energy, with the result that each of the supports acquires additional kinetic energy under the action of two forces on each support: the weight of the building evenly distributed on each support, and the weight of the support itself. The first force is applied to the highest point of the support (see Figure 1), the second one is applied to the centre of gravity of the support (see Figure 2), so that the rotational moments of two forces act on the support. Below we consider only a part of the system, namely, a support isolated from the seismic impact.

Each support has the shape of a cylinder with the height h and diameter $a = 2r$. It should be noted that under very strong vibrations of the ground, the projection of the centre of gravity can move horizontally beyond the base of the support, i.e., $\varphi > \alpha$, where α is the angle between the diagonal of the support and its height, so that $\alpha = \arctan\left(\frac{a}{h}\right)$. In this case, the supports will begin to tip over. We confine ourselves to considering such deviations, when the rotational moments of the forces of gravity still tend to return the supports to the initial state of equilibrium, that is, we will consider the case $\varphi < \alpha$. Then $\alpha = \varphi + \psi$, where ψ is the angle of deviation of the flat base of the support from the horizontal, while the supports perform non-linear oscillations. To compose the equation of motion of a support, we take into account that it is a physical pendulum.

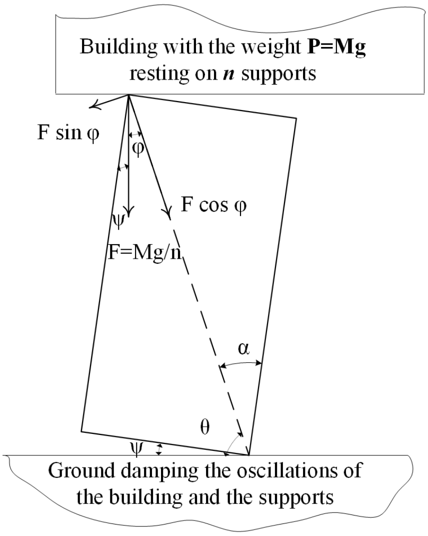

Figure 1. Rotational force generated by the pressure of the building on the supports

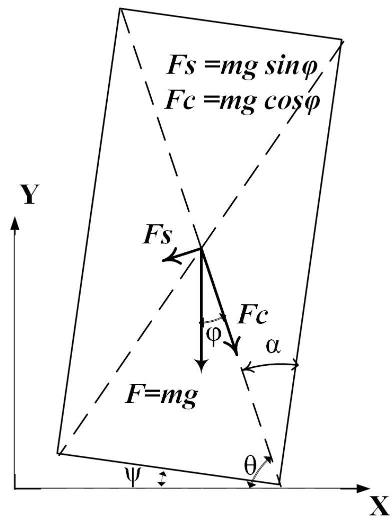

Figure 2. Rotational force generated by the weight of the support

Before the appearance of seismic effects, the "building + n supports" system is in a state of stable equilibrium, i.e., the supports stand on the ground, the building rests on the supports, with the centre of gravity in the lowest position and the lowest potential energy. After the deviation of the "building + n supports" system from the equilibrium position of the supports (along with the building), they rise and fall, rotating around one of their edges during the first half-period of the oscillatory motion. Then the supports touch their bases to the ground with a blow, followed by the loss of a part ε of the energy due to the inelastic impact of the ground. After that, due to the remaining $(1 - \varepsilon)$ kinetic energy of the support (along with the building), they rise and fall, rotating around its other edge during the second half-period of the oscillatory motion. Then the next blow occurs and the oscillatory motion continues with damping.

During each of the half-periods, the movement occurs under the action of gravity forces and their rotational moments.

The returning force of a uniformly distributed building weight acting on a support is $F_M = -\frac{Mg}{n} \sin\varphi$. The distance from the point of application of the force F_M to the axis of rotation is equal to $L_M = 2l = \sqrt{a^2 + h^2}$. The torque of the force is expressed as $F_M \bullet L_M = -\frac{Mg}{n}\sqrt{a^2 + h^2} \sin\varphi$.

The returning force generated by the weight of the support itself is calculated as $F_m = mg \sin\varphi$. The distance from the point of application of the force F_M to the axis of rotation is l. The torque of the force is calculated using the formula $F_m * l$.

The moment of inertia of the support is equal to $J = \frac{m}{2} r^2 + \frac{m}{6} h^2$. The equation of motion of a support, assuming that it is a physical pendulum, has the form

$$F_m + F_M = -J\ddot{\varphi}. \tag{1}$$

Taking into account the explicit form of the restoring forces, we obtain the relation

$$mgl\sin\varphi + \frac{2Mg}{n} l\sin\varphi = -\frac{12l^2+h^2}{24}\ddot{\varphi}, \tag{2}$$

which is reduced to the Lagrange differential equation describing the dynamics of the motion of the supports after a seismic shock, bringing the entire system out of equilibrium:

$$-\ddot{\varphi} = -\left(\frac{nm+2M}{12l^2+h^2}\right)\frac{24gl}{n}\sin\varphi. \tag{3}$$

In the case of limited oscillations, i.e., when $\psi < \alpha$, we get

$$\ddot{\psi} - \left(\frac{nm+2M}{12l^2+h^2}\right)\frac{24gl}{n}\sin(\alpha-\psi) = 0. \tag{4}$$

The obtained Eq. (4) is the equation of free oscillations (not disturbed by the continuing seismic effect).

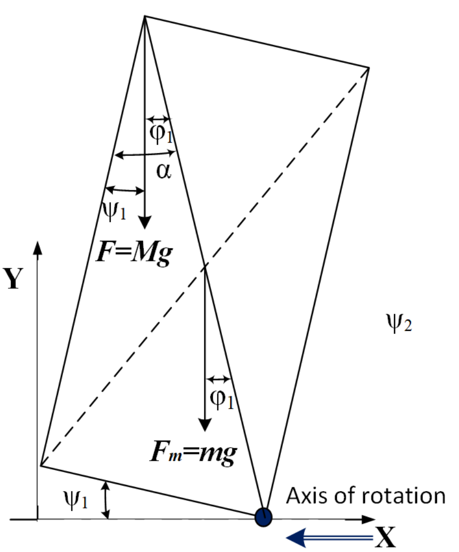

Figure 3. Position of the support at the moment of maximum lifting of the centre of gravity ($t = 0$)

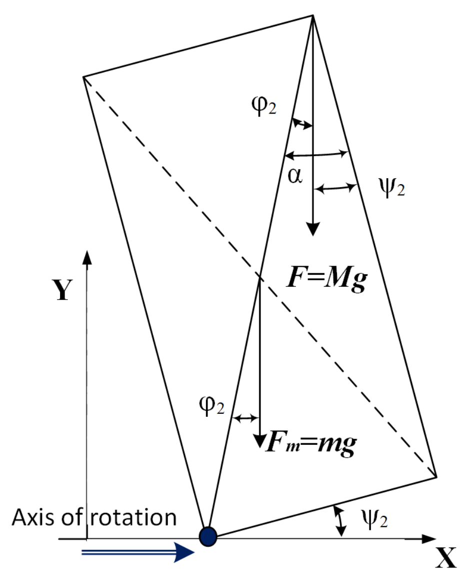

Figure 4. The support of the building. Maximum deviation after changing the axis of rotation

For a complete description of the evolutionary process of support oscillations, we supplement Eq. (4) with the initial conditions, taking the maximum deflection of the support after the seismic shock shown in Figure 3 for the initial position at zero time. Then at the moment of time corresponding to the highest ascent of the centre of gravity of the support (the centre of rotation of the support is located at its lower right point, see Figure 3) the movement of an individual support is described by Eq. (4) with the initial conditions (we assume the deflection angle positive)

$$\psi_1(0) = \psi_{const}, \quad \dot{\psi}_1(0) : \psi_{const} > 0. \tag{5}$$

Under the influence of the gravity of the support and the entire building, the centres of gravity of the supports will tend to return to their original position. In this case, the angle $\psi_1 > 0$, $\psi_1 \to 0$, in accordance with the solution of the Cauchy problem (4), will tend to zero. At that moment, when the magnitude of the angle equals zero, the inertial forces will force the building and the supports to move further in the same horizontal direction, which will lead (after the impact of the support on the base surface) to a change of the rotation axis of the supports, see Figure 4. The motion of the support system and the building itself is still described by exactly the same equations, but with the variable ψ_1 changed for ψ_2:

$$\ddot{\psi}_2 - \left(\frac{nm + 2M}{12l^2 + nm}\right) \frac{24gl}{n} \sin(\alpha - \psi_2) = 0 \tag{6}$$

and with other initial conditions (again we consider the deflection angle ψ_2 to be positive)

$$\begin{aligned} \psi_2(0) &= 0, \\ \dot{\psi}_2(0) &= \dot{\psi}_1(0). \end{aligned} \tag{7}$$

It is important to note that at the time of the collision of the support with the ground surface, the system consisting of the building and the supports loses some fraction of the kinetic energy. Setting the restitution coefficient to be equal to $C_r = (1 - \varepsilon) < 1$ and considering the kinetic energy losses, we arrive at the relation

$$\frac{J}{2}\dot{\varphi}^2_{after} = C_r \frac{J}{2}\dot{\varphi}^2_{befor} \Rightarrow \dot{\varphi}_{after} = \sqrt{C_r}\dot{\varphi}_{befor}, \tag{8}$$

which, in turn, makes it possible to determine the new velocities of the supports after their collisions with the surface and recalculate the initial conditions of the problems (5) and (7) when going through the zero value of the rotation angle using the formulas

$$\begin{aligned} \psi_{k+1}(t + 0) &= \psi_k(t - 0), \\ \dot{\psi}_{k+1}(t + 0) &= -\sqrt{C_r}\dot{\psi}_k(t - 0), \end{aligned} \tag{9}$$

for the generalized equation of motion of the supports and the building:

$$\ddot{\psi} = sign(\psi) \left(\frac{nm + 2M}{12l^2 + nm}\right) \frac{24gl}{n} \sin(\alpha - sign(\psi)\psi). \tag{10}$$

The formulated system can be solved by means of any stable numerical method, for example, the 4^{th}-order Runge–Kutta method with automatic step selection. We get the following plots for the solutions (Figures 5–7).

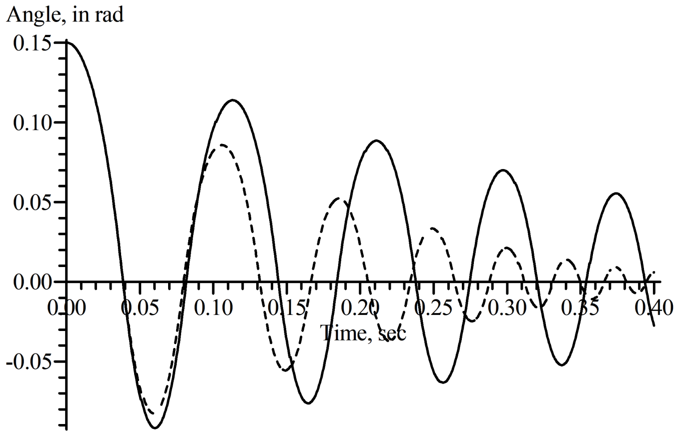

Figure 5. The angle of deviation from the vertical. The solid line corresponds to the recovery coefficient $C_r = 0.9$, the dashed line — to $C_r = 0.8$

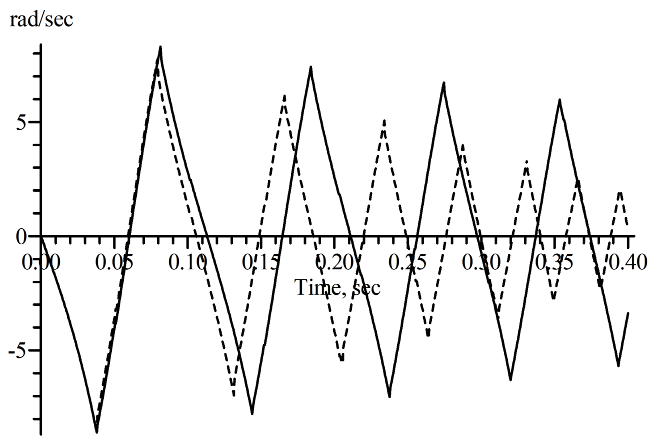

Figure 6. The rate of change of the deviation angle from the vertical. The solid line corresponds to the recovery coefficient $C_r = 0.9$, the dashed line — to $C_r = 0.8$

3. Conclusion

The model (4) is a nonlinear conservative dynamical system. The motion of the supports and the building body during the non-linear oscillations can be considered as oscillations of the coupled physical pendulums. At the same time, the coupling of pendulums is not conservative, but contains a factor proportional to the rolling friction of the supports on the base of the building body. Moreover, the rolling radius depends on the angle: the larger the angle ψ, the smaller the radius, which means greater friction force between the n supports and the base of the building.

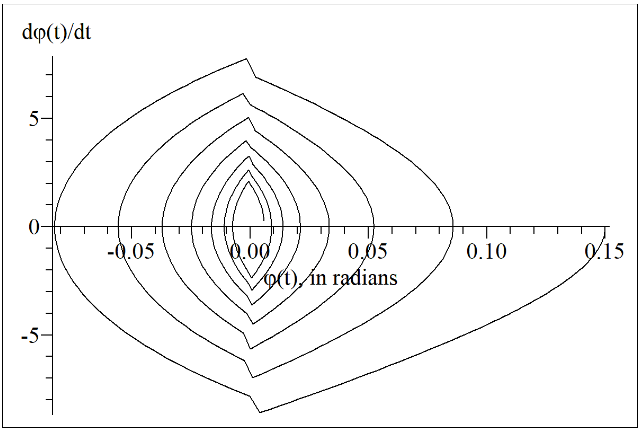

Figure 7. Phase portrait of the "oscillatory system" ($C_r = 0.8$).

In addition, of course, this entire unified system, including the reaction of the soil, moves under the action of a seismic strongly oscillating and irregular disturbance generated by an earthquake. Only as a result of the joint consideration of all these factors and we can count on an adequate description of the deterministic dynamics of the building under the influence of seismic perturbations from an earthquake.

However, even the analysis of equation (4) allows detuning of the natural frequencies of free vibrations of a building system on supports by varying the weight of the building and the number of supports.

The equation obtained by us is valid for angles $\psi < \alpha$, but it can be easily generalized to the case $\psi > \alpha$, when the movement of the supports will cease to be oscillatory, and the whole structure will lose stability. However, this case, interesting while considering the driving forces of horizontal seismic vibrations, will be analysed elsewhere.

Acknowledgments

The publication has been prepared under support from the "RUDN University Program 5-100" and the Russian Foundation for Basic Research, grants No. 18-07-00567 and No. 18-51-18005.

References

1. A. M. Kurzanov, S. Yu. Semenov, Dynamic tests of a multi-storey monolithic house in Sochi [Naturnyye dinamicheskiye ispytaniya stroyashchegosya mnogoetazhnogo seysmo izolirovannogo monolitnogo doma v Sochi], Industrial and civil construction [Promyshlennoye i grazhdanskoye stroitel'stvo] (3) (2005) 42–43, in Russian.
2. A. K. Chopra, Dynamic of structures. Theory and applications to earthquake engineering, Prentice-Hall, New Jersey, 2006.
3. A. Martelli, M. Forny, Seismic isolation: present application and perspectives, Yerevan, Armenia, 2006, pp. 1–26.
4. H. Masahiko, O. Shin, Response Control and Seismic Isolation of Buildings, Taylor & Francis, New York, 2006.

5. V. L. Kharlanov, Deterministic analysis of metal frames for high-intensity dynamic loads: a monograph [Determinirovannyy analiz metallicheskikh karkasov na dinamicheskiye nagruzki vysokoy intensivnosti], VolgGASU, 2006, in Russian.
6. G. E. Avidon, E. A. Karlina, Vibration propertie of buildings with base isolating foundations by A. M. Kurzanov and Yu.D.Cherepinsky [Osobennosti kolebaniy zdaniy s seysmoizoliruyushchimi fundamentami A. M. Kurzanova i Yu. D. Cherepinskogo], Earthquake engineering. Building safety [Seysmostoykoye stroitel'stvo. Bezopasnost' sooruzheniy] (1) (2008) 42–45, in Russian.
7. A. M. Kurzanov, Modern state of standardization of design of structures for seismic load [Sovremennoye sostoyaniye normirovaniya rascheta sooruzheniy na seysmicheskuyu nagruzku], Industrial and civil construction [Promyshlennoye i grazhdanskoye stroitel'stvo] (11) (2009) 52–53, in Russian.
8. E. L. Schesnyak, et al., STO RUDN 02066463-001-2011 "Standards for Designing Structures in Seismic Areas" [STO RUDN 02066463-001-2011. Stroitel'stvo v seysmicheskikh rayonakh. Normy proyektirovaniya sooruzheniy], RUDN, Moscow, 2011, in Russian.
9. S. P. Karnilovich, K. P. Lovetsky, L. A. Sevastyanov, E. L. Schesnyak, Seismic systems based on A.M. Kurzanov's kinematic supports [Seysmoizolyatsiya zdaniy na osnove kinematicheskikh opor Kurzanova A.M.], in: XIX international Conference on Distributed Computer and Communication Networks: Control, Computation, Communications (DCCN-2016). Proceedings of the Nineteenth International Scientific Conference "Mathematical Modelling, Simulation and Control Problems", Vol. 2, 2016, pp. 159–164, in Russian.
10. O. V. Druzhinina, L. A. Sevastianov, S. A. Vasilyev, D. G. Vasilyeva, Lyapunov stability analysis for the generalized Kapitza pendulum, Journal of Physics: Conference Series 937, article number 012011 (2017). doi:10.1088/1742-6596/937/1/012011.
11. O. V. Druzhinina, L. A. Sevastianov, S. A. Vasilyev, D. G. Vasilyeva, Numerical analysis of kurzanov bearing oscillation, in: M. B. Kochanov (Ed.), 7th International conference "Problems of Mathematical Physics and Mathematical Modelling": Books of abstracts (Moscow, NRNU MEPhI, 25–27 June), National Research Nuclear University MEPhI, Moscow, 2018, pp. 164–166.
12. Y. M. Parulekar, G. R. Reddy, Passive response control systems for seismic response reduction: a state-of-the-art review, International Journal of Structural Stability and Dynamics 09 (01) (2009) 151–177. doi:10.1142/S0219455409002965.
13. M. Palermo, S. Silvestri, L. Landi, G. Gasparini, T. Trombetti, A "direct five-step procedure" for the preliminary seismic design of buildings with added viscous dampers, Engineering Structures 173 (2018) 933–950. doi:10.1016/j.engstruct.2018.06.103.
14. S. Wang, S. A. Mahin, High-performance computer-aided optimization of viscous dampers for improving the seismic performance of a tall building, Soil Dynamics and Earthquake Engineering 113 (2018) 454–461. doi:10.1016/j.soildyn.2018.06.008.

15. C.-M. Chang, S. Shia, C.-Y. Yang, Design of buildings with seismic isolation using linear quadratic algorithm, Procedia Engineering 199 (2017) 1610–1615. doi:10.1016/j.proeng.2017.09.069.
16. J. M. Kelly, The role of damping in seismic isolation, Earthquake Engineering and Structural Dynamics 28 (1) (1999) 3–20. doi:10.1002/(SICI)1096-9845(199901)28:1<3::AID-EQE801>3.0.CO;2-D.
17. I. Politopoulos, A review of adverse effects of damping in seismic isolation, Earthquake Engineering and Structural Dynamics 37 (3) (2008) 447–465. doi:10.1002/eqe.763.
18. G. P. Warn, K. L. Ryan, A review of seismic isolation for buildings : historical development and research needs, Buildings (2) (2012) 300–325. doi:10.3390/buildings2030300.
19. S. Moretti, A. Trozzo, V. Terzic, G. P. Cimellaro, S. Mahin, Utilizing base-isolation systems to increase earthquake resiliency of healthcare and school buildings, Procedia Economics and Finance 18 (2014) 969–976. doi:10.1016/S2212-5671(14)01024-7.

For citation:

S. P. Karnilovich, K. P. Lovetskiy, L. A. Sevastianov, E. L. Shchesnyak, Seismic stability of oscillating building on kinematic supports, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 124–132. DOI: 10.22363/2658-4670-2019-27-2-124-132.

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Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 133–142
http://journals.rudn.ru/miph

Research article

UDC 533.9
DOI: 10.22363/2658-4670-2019-27-2-133-142

On the radiation losses during motion of an electron in the field of intense laser radiation

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(received: April 20, 2019; accepted: November 14, 2019)

Motion of the relativistic electron in the field of intense laser pulse of the arbitrary shape is considered. The pulse dimension is supposed to be of the order of the Gaussian laser beam dimension in the focal plane. It is supposed that the pulse is propagating along the external constant magnetic field. In the paraxial approximation the corrections of the first order to the vectors of the field of radiation as well as the force of the radiation friction are taken into account. Averaged relativistic equations of motion of electron are obtained with the help of averaging over the fast oscillations of the laser radiation. It is shown that with taking into account corrections of the first order to the field vectors an averaged force arises. This force is defined by pulsed character of radiation and proportional to the intensity but not to gradient of intensity. It is shown that radiation losses are of little importance in the transverse plane but may considerably act on the longitudinal motion of electron.

Key words and phrases: relativistic electron, intense laser pulse, paraxial approximation, Gaussian beam, radiation friction

1. Introduction

When a relativistic charged particle moves in an electromagnetic field, radiation friction forces can play a significant role [1–3]. Various aspects of the radiation friction forces problem, including the refinement of the classical Lorentz–Abraham–Dirac expression, were discussed in [4–7]. The problem of radiation losses has become particularly relevant in connection with the creation of powerful laser radiation sources [8–10]. The charged particle motion pattern substantially depends on the properties of electromagnetic radiation. In the case of a sufficiently weak radiation intensity, the parameter $g = \dfrac{e|E|}{\omega mc}$, which is the ratio of the particle momentum oscillations amplitude in the wave field to a rest momentum, is small: $g \ll 1$. Here, $|E|$ is the amplitude of the electric wave field, ω is the wave frequency, e is the charge, m — the mass of the particle, c is the light speed in vacuum. Availability of a small parameter allows one to find a solution to the equations of motion for a particle in the form of expansions over this parameter and averaging over fast oscillations. When a particle moves in the high-power laser radiation field which is pulsed by its nature and of high intensity, the parameter g becomes large. In the case of an electron, this parameter is represented as

$$g = 0.85 \cdot 10^{-9} \lambda \sqrt{I}, \tag{1}$$

where $I[W/\mathrm{cm}^2]$ is the laser pulse intensity, and $\lambda\,[\mu\mathrm{m}]$ is the wavelength. This shows that the parameter $g \geqslant 1$ at an intensity of $I \geqslant 10^{18} W/\mathrm{cm}^2$. Intensities of such an order and even higher by their value are well achieved in modern laser devices. In this case, a solution of the equations of motion for a particle in the form of expansions in the parameter (1) is impossible. Therefore, they usually use numerical methods for solving the problem. Analytical solution turns out to be possible if propagation of laser radiation is adequately described within the paraxial approximation in the form of Gaussian beams [11–17]. In this approximation, there is a small parameter

$$\mu = \frac{2}{ka} = \frac{a}{z_r} \ll 1, \tag{2}$$

where a is the laser beam radius at focus (waist of the laser beam), $z_r = ka^2/2$ is the Rayleigh length which determines the diffraction divergence of the laser beam, $k = 2\pi/\lambda$ is the wave number, and λ is the radiation wavelength. The presence of such a small parameter (2) allows finding the solution of the equations of motion for a particle in the form of expansions in this parameter and averaging over fast oscillations of the radiation field. In the case of sufficiently long pulses, the radiation field vectors in the zero approximation of the expansion of Maxwell's equations in parameter (2) are represented in the form of Gaussian beams of different modes as a solution to the parabolic equation [11, 14]. Upon that, the longitudinal components of the field vectors in the form of first-order corrections, and the corrections to the transverse components of the field vectors in the form of second-order quantities arise. In the case of sufficiently short pulses, the length of which is of the order of the beam waist, the corrections to the transverse components of the field vectors are first order quantities [13, 17]. These amendments were not taken into account in papers [8–10]. Note that in the case of highly-focused laser radiation, paraxial approximation is inapplicable [18].

This work is devoted to the study of a relativistic electron motion in the field of high-power pulsed laser radiation in the form of a Gaussian beam with the circularly polarized basic mode, taking into account first-order corrections to field vectors and radiation friction force.

2. Initial equations and problem statement

In case of laser radiation with the circularly polarized basic mode, as a zero approximation of expansions in a small parameter, the field vectors are described by the formulas [17]:

$$E_x^0 = \frac{fE_x(0)}{\sqrt{1+Z^2}} e^{\frac{-\rho^2}{1+Z^2}} \cos(\varphi+\theta) = A\cos(\varphi+\theta),$$
$$E_y^0 = -\frac{fE_x(0)}{\sqrt{1+Z^2}} e^{\frac{-\rho^2}{1+Z^2}} \sin(\varphi+\theta) = -A\sin(\varphi+\theta). \tag{3}$$

In the formulas written out: $\theta = \omega\left(\frac{z}{c} - t\right)$ is the "fast" wave phase, $\varphi(\rho, Z) = \frac{\rho^2 Z}{1+Z^2} - \arctan(Z)$ is the slow phase, $f(\sigma)$ is the function which takes into account the pulsed nature of the radiation, and where the parameter $\sigma = \frac{(t - \frac{z}{c})}{\Delta t}$, and Δt is pulse duration. Dimensionless quantities are used for the longitudinal coordinate $Z = \frac{z}{z_r}$, the distance from the Gaussian beam axis in the transverse plane is $\rho = \frac{r}{a}$, where $r = \sqrt{x^2+y^2}$. In the first approximation for expansions of the field vectors with respect to the small parameter μ, not only the longitudinal field components, but also corrections to the transverse components arise in the case of sufficiently short pulses [17]:

$$E_x^1 = -\frac{\lambda Z}{2\pi c\Delta t}\frac{E_x(0)f'}{(1+Z^2)^{3/2}}\sqrt{(1-\rho^2)^2+Z^2}e^{\frac{-\rho^2}{1+Z^2}}\times$$
$$\times\cos\left(\arctan\frac{Z}{1-\rho^2} + \frac{\rho^2 Z}{1+Z^2} - 3\arctan(Z) + \theta\right) \equiv -A_1\cos(\varphi_1+\theta),$$
$$E_y^1 = \frac{\lambda Z}{2\pi c\Delta t}\frac{E_x(0)f'}{(1+Z^2)^{3/2}}\sqrt{(1-\rho^2)^2+Z^2}e^{\frac{-\rho^2}{1+Z^2}}\times$$
$$\times\sin\left(\arctan\frac{Z}{1-\rho^2} + \frac{\rho^2 Z}{1+Z^2} - 3\arctan(Z) + \theta\right) \equiv A_1\sin(\varphi_1+\theta),$$
$$E_z^1 = \frac{2f\rho E_x(0)}{ka(1+Z^2)}e^{\frac{-\rho^2}{1+Z^2}}\sin\left(\frac{Z\rho^2}{1+Z^2} - \arctan\frac{y}{x} - 3\arctan(Z) + \theta\right). \tag{4}$$

Here $f'(\sigma) \equiv \frac{\partial f}{\partial \sigma}$.

The relationship between the transverse components of the electric and magnetic vectors is determined by the relations:

$$H_x = -E_y, \quad H_y = E_x. \tag{5}$$

There is also a longitudinal component of the magnetic field of laser radiation:

$$H_z^1 = \frac{2f\rho E_x(0)}{ka(1+Z^2)}e^{\frac{-\rho^2}{1+Z^2}}\sin\left(\frac{Z\rho^2}{1+Z^2} - \arctan\frac{y}{x} - 3\arctan(Z) + \theta\right). \tag{6}$$

The relativistic electron motion in an electromagnetic field is described by the equations:

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = -e\left(\vec{E} + \frac{1}{mc\gamma}[\vec{p}\vec{H}]\right) + \vec{f}_r, \tag{7}$$

where p is a particle momentum vector, m is its mass; $\gamma = \sqrt{1 + \frac{p^2}{m^2c^2}}$ is a relativistic factor (dimensionless particle energy), $\vec{f}_r$ is a radiation friction force.

The laser radiation field is given by formulas (3–6). The constant magnetic field is directed along the z axis: $\vec{H}_0 = (0, 0, H_0)$.

The system of equations (7) must be supplemented by an equation for the wave phase θ which is considered as a “fast” variable:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\omega\left(1 - \frac{v_z}{c}\right) \equiv -\frac{\omega}{\gamma}G. \tag{8}$$

Here v_z is the longitudinal velocity of the particle; and the quantity G has the following meaning

$$G \equiv \gamma - \frac{p_z}{mc}. \tag{9}$$

It follows from equation (8) that the phase θ becomes a “slow” variable with $1 - \frac{v_z}{c} \approx 0$. If $1 - \frac{v_z}{c} \gg \mu$, then the phase θ can be considered as a “fast” variable over which it is possible to carry out averaging. This is assumed to perform in the future. According to the equations of motion (7), the value G is described by the equation:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = -\frac{e}{mc}\left(1 - \frac{v_z}{c}\right)E_Z. \tag{10}$$

This shows that in the first approximation, the quantity G is on average preserved and is determined by the initial conditions. It is usually assumed that $G = 1$ [1]. From the equation (10) it follows that the quantity G can be represented as an expansion:

$$G = G_0 + \mu G_1 + \dots . \tag{11}$$

Here $G_0 \approxeq 1$ is the value averaged over the fast phase, and G_i are quickly oscillating components of the quantity G.

It follows directly from the equations of motion for a particle that the relativistic factor satisfies the equation:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{1}{\gamma(mc)^2}\vec{p}(-e\vec{E} + \vec{f}_r). \tag{12}$$

The main task of this work is to obtain averaged equations for electron motion using the Bogolyubov’s method [19], by averaging over the fast oscillations of laser radiation, taking into account the averaged radiation friction force. In their sense, the radiation friction forces are small compared with the effects of laser radiation. Therefore, we first consider the motion of a particle without taking radiation losses into account.

3. Averaged equations of motion for a charged particle

The initial equations of motion for a particle (7) contain members oscillating with the laser frequency, the amplitude of which is determined by parameter (1). In the conditions under consideration, this parameter is large. Therefore, it is impossible to carry out averaging according to the Bogolyubov's method [19] using expansions in the parameter g. However, the equations of motion also contain parameter (2) which is small. In principle, this allows to seek a solution in the form of expansions in the parameter μ. However, it is necessary that the quickly oscillating members have small amplitude in order to apply the averaging method. This can be achieved if we firstly exclude in the equations of motion the quickly oscillating members with large amplitudes [20, 21]. In this regard, we will make the replacement of the transverse components of the particle momentum vector, which is convenient to use in the complex form:

$$p = \pi + Ce^{i(\varphi+\theta)}. \tag{13}$$

Here, $p = p_x + ip_y$ is the complex transverse momentum of a particle, and π is the complex generalized momentum. Substituting (13) into system (7), we obtain the equation:

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} + \frac{\mathrm{d}C}{\mathrm{d}t}e^{i(\varphi+\theta)} + iC\frac{\mathrm{d}\theta}{\mathrm{d}t}e^{i(\varphi+\theta)} + iC\frac{\mathrm{d}\varphi}{\mathrm{d}t}e^{i(\varphi+\theta)} = \\ - e\left(1 - \frac{v_z}{c}\right)Ae^{i(\varphi+\theta)} + \frac{i\omega_{co}}{\gamma}(\pi + Ce^{i(\varphi+\theta)})\left(1 + \frac{H_z}{H_0}\right). \tag{14}$$

In the equation (14) the first member on the right is quickly oscillating with large amplitude. It can be eliminated by appropriate selection of the amplitude C in the transformation (13). In its meaning, the amplitude C must be determined by the amplitude of the radiation field in zero approximation. Then, equating the members of zero approximation in the equation (14) considering equation (8) and expansion (11), we get:

$$C = \frac{-ieA}{\omega + \frac{\omega_{co}}{G_0}}. \tag{15}$$

Here $\omega_{co} = \dfrac{eH_0}{mc}$ is a classic cyclotron frequency. Thus, for the found value of C, equation (14) for the variable π contains only members of the first order in the parameter μ. We can obtain the following expression from the definition of the relativistic factor, taking into account (9)

$$\gamma = \frac{1 + G^2 + \frac{p_\perp^2}{m^2c^2}}{2G}, \tag{16}$$

where $p_\perp^2 = p_x^2 + p_y^2$. Given the replacement (13), we get

$$p_\perp^2 = \pi_\perp^2 + \bar{C}^2 - 2\bar{C}\pi_\perp \sin(\varphi + \theta - \beta). \tag{17}$$

Here: $\pi_\perp^2 = \pi_x^2 + \pi_y^2$, $\pi_x = \pi_\perp \cos\beta$, $\pi_y = \pi_\perp \sin\beta$.

The value of $\bar{C} = \dfrac{-eA}{\omega + \omega_{co}/G_0}$ represents the amplitude of electron momentum oscillations in a field of laser radiation propagating along a constant magnetic field. Amplitude A is determined by formulas (3). After a series of transformations, formula (16) takes the form:

$$\gamma = \frac{1}{2G_0} + \frac{G_0}{2} + \frac{\bar{C}^2}{2G_0 m^2c^2} + \frac{G_1}{2} - \frac{\bar{C}\pi_\perp \sin(\varphi+\theta-\beta)}{G_0 m^2 c^2} - \frac{\bar{C}^2 G_1}{2G_0^2 m^2 c^2}. \tag{18}$$

Here the expansion $\dfrac{1}{G_0 + G_1} \approx \dfrac{1}{G_0}\left(1 - \dfrac{G_1}{G_0}\right)$ is taken into account. It is also assumed that $\pi_\perp \ll \bar{C}$. This is a natural limitation in particle acceleration problems. From the formula (18) it follows that the relativistic factor contains "constant" and fast oscillating parts:

$$\gamma = \Gamma + \gamma_1, \tag{19}$$

where

$$\Gamma = \frac{1 + G_0^2 + \frac{\bar{C}^2}{m^2c^2}}{2G_0}, \quad \gamma_1 = \frac{G_1}{2} - \frac{\bar{C}^2 G_1}{2G_0^2 m^2c^2} - \frac{\bar{C}\pi_\perp \sin(\varphi+\theta-\beta)}{G_0 m^2c^2}. \tag{20}$$

Let us further consider the equation of longitudinal motion from system (7):

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = eE_z^1 + \frac{e}{mc\gamma}\left\{ p_x \left[\frac{fE_x(0)}{\sqrt{1+Z^2}} e^{\frac{-\rho^2}{1+Z^2}} \cos(\varphi+\theta) + H_y^1\right] - \right.$$
$$\left. - \left[\frac{fE_x(0)}{\sqrt{1+Z^2}} e^{\frac{-\rho^2}{1+Z^2}} \sin(\varphi+\theta) + H_x^1\right] p_y \right\}.$$

After a series of transformations we get:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = e\frac{2f\rho E_x(0)}{ka(1+Z^2)} e^{\frac{-\rho^2}{1+Z^2}} \sin\left(\varphi - \arctan(Z) - \arctan\frac{y}{x} + \theta\right) +$$
$$+ \frac{2e}{m^2c^2 + G_0^2 m^2c^2 + \bar{C}^2}\left[G_0 - \frac{G_0^2 G_1 m^2c^2 - \bar{C}^2 G_1 - 2G_0\bar{C}\pi_\perp \sin(\varphi+\theta-\beta)}{m^2c^2 + G_0^2 m^2c^2 + \bar{C}^2}\right] \times$$
$$\times \Big[A\pi_\perp \cos(\varphi+\theta+\beta) - A_1\pi_\perp \cos(\varphi_1+\theta+\beta) -$$
$$- A\bar{C} \sin 2(\varphi+\theta) + A_1\bar{C} \sin\left(\arctan\frac{Z}{1-\rho^2} - 2\arctan(Z)\right) +$$

$$+ A_1\bar{C}\sin\left(\frac{2\rho^2 Z}{1+Z^2} - 4\arctan(Z) + \arctan\frac{Z}{1-\rho^2} + 2\theta\right)\Bigg]. \quad (21)$$

We omit the members containing $\pi_\perp$, and select the members that are independent on the fast phase θ. As a result, we obtain the averaged equation of the longitudinal motion:

$$\frac{\mathrm{d}\bar{p_z}}{\mathrm{d}t} = \frac{e^2\lambda_0 Z}{2m\pi c^2\Delta t\Gamma}\frac{E_x^2(0)ff'}{(1+Z^2)^2}\frac{1}{\omega+\omega_{co}/G_0}\sqrt{(1-\rho^2)^2+Z^2}e^{\frac{-2\rho^2}{1+Z^2}}\times$$
$$\times\sin\left(\arctan\frac{Z}{1-\rho^2} - 2\arctan(Z)\right). \quad (22)$$

Here $\bar{p_z}$ is the average value for the longitudinal momentum of the particle. The right side of this equation, which has the first order of smallness in the parameter μ, describes the averaged force with which high-power pulsed laser radiation acts on an electron in the direction of the magnetic field. A similar expression was obtained in [15] without taking into account the external magnetic field. However, due to the large difference between the cyclotron frequency and the laser radiation frequency, the influence of an external magnetic field is insignificant. As can be seen from (22), the averaged force is due to the pulsed nature of the radiation: $f' \equiv \dfrac{\partial f}{\partial\sigma} \neq 0$.

4. Radiation friction force

The radiation friction force problem is one of the frequently discussed in the literature [3–10]. In [6, 7], some shortcomings were noted and refinements of the classical expression for the radiation friction force were made [1, 2]. In the present work, the classical Lorenz–Abraham–Dirac formula is used for the radiation friction force [1–3]:

$$\vec{f_{fr}} = \frac{2e^4}{3m^2c^4}\left\{[\vec{E}\vec{H}] + \frac{1}{c}\left[\vec{H}[\vec{H}\vec{v}]\right]\right\} - \frac{2e^4\gamma}{3mc^3}\vec{v}\left\{\left(\vec{E}+\frac{1}{c}[\vec{v}\vec{H}]\right)^2 - \frac{1}{c^2}(\vec{E}\vec{v})^2\right\}. \quad (23)$$

For the longitudinal motion, the radiation friction force when electron is moving in the laser field (3), taking into account the transformation (13), is represented as;

$$f_z = \frac{2e^4}{3m^2c^4}\Bigg\{A^2\cos^2(\varphi+\theta) + A^2\sin^2(\varphi+\theta) + \frac{H_0}{cm\gamma}\Big[-A\bar{C}\sin^2(\varphi+\theta)+$$
$$+ A\bar{C}\cos^2(\varphi+\theta)\Big] - \frac{p_z}{mc\gamma}\Big[A^2\cos^2(\varphi+\theta) + A^2\sin^2(\varphi+\theta)\Big]\Bigg\} -$$
$$- \frac{2e^4\gamma}{3m^2c^3}p_z\Bigg\{A^2\cos^2(\varphi+\theta) + A^2\sin^2(\varphi+\theta) - \frac{2}{mc\gamma}\Big[A\bar{C}H_0\sin^2(\varphi+\theta) -$$
$$- A\bar{C}H_0\cos^2(\varphi+\theta) + p_z(A^2\cos^2(\varphi+\theta) + A^2\sin^2(\varphi+\theta))\Big] -$$

$$- \frac{1}{c^2m^2\gamma^2}\Big[A^2\bar{C}^2\sin^4(\varphi+\theta) + A^2\bar{C}^2\cos^4(\varphi+\theta) - 2A\bar{C}H_0p_z\sin^2(\varphi+\theta) + $$
$$+ 2A\bar{C}H_0p_z\cos^2(\varphi+\theta) + 2A^2\bar{C}^2\sin^2(\varphi+\theta)\cos^2(\varphi+\theta)\Big]\Big\}.$$

Averaging over the fast oscillations of the radiation field and holding only the main members, we obtain the expression for the radiation friction force in the direction of the laser pulse propagation:

$$f_{avz} = 2e^4\frac{f^2E_x^2(0)}{3m^2c^4(1+Z^2)}e^{\frac{-2\rho^2}{1+Z^2}}\left\{1-\frac{\bar{p}_Z}{mc\Gamma}\right\} - 2\frac{f^2E_x^2(0)e^4\Gamma}{3m^2c^3(1+Z^2)}e^{\frac{-2\rho^2}{1+Z^2}}\bar{p}_Z\times$$
$$\times\left\{1-\frac{2\bar{p}_Z}{mc\Gamma} - \frac{e^2f^2E_x^2(0)}{m^2c^2\Gamma^2(1+Z^2)(\omega-\frac{\omega_{co}}{G_0})^2}e^{\frac{-2\rho^2}{1+Z^2}}\right\}. \quad (24)$$

Expression (24) is still quite complicated. For a rough estimate, we will assume that the particle velocity is close enough to the velocity of light. Then the main member of this expression is $-\frac{2A^2e^4\Gamma}{3m^2c^3}\bar{p}_Z$. From here, it can be concluded that the radiation reaction force can be sufficiently large in the field of high-intensity laser radiation. The calculations showed that the averaged transverse components of the radiation friction force are absent:

$$f_{avx} = 0, \quad f_{avy} = 0.$$

This means that when a particle is moving in the transverse plane, the radiation losses on average play no role. When a particle is moving in the direction of radiation propagation, they can be significant in the field of super-intense laser radiation.

5. Conclusion

An expression for the averaged force acting on a relativistic electron in the field of an intense short laser pulse propagating along an external constant magnetic field is obtained. This force is due to the pulsed nature of the radiation, the description of which requires the inclusion of first-order members in expansions with respect to a small parameter of the paraxial approximation. It was taken into account that in the case of high-power laser radiation, the ratio of the amplitude of the particle oscillatory velocity in a wave field to the velocity of light can be large. To use the averaging over the fast oscillations of laser radiation, the transverse components of the particle momentum vector have been transformed. Thus, in the equations of motion, quickly oscillating members with large amplitude determined by the radiation field were eliminated. An expression was obtained for the averaged radiation friction force determined by the classical Lorentz–Abraham–Dirac formula. It was shown that this force in the transverse plane is zero and can be significant in the direction of laser pulse propagation.

Acknowledgments

The publication has been prepared with the support of the “RUDN University Program 5-100”.

References

1. L. D. Landau, E. M. Lifshitz, Field theory [Teoriya polya], Nauka, Moscow, 1988, in Russian.
2. J. D. Jackson, Classical electrodynamics, J. Wiley, NY-L., 1962.
3. V. L. Ginzburg, Theoretical physics and astrophysics [Teoreticheskaya fizika i astrofizika], Nauka, Moscow, 1975, in Russian.
4. N. P. Klepikov, Radiation damping forces and radiation from charged particles, Soviet Physics Uspekhi 28 (1985) 506–520. doi:10.1070/PU1985v028n06ABEH005205.
5. V. S. Krivitskii, V. N. Tsytovich, Average radiation-reaction force in quantum electrodynamics, Soviet Physics Uspekhi 234 (1991) 250–258. doi:10.1070/PU1991v034n03ABEH002352.
6. G. F. Efremov, Radiative damping of a relativistic electron in classical electrodynamics, Journal of Experimental and Theoretical Physics 89 (1998) 899–904. doi:10.1134/1.558738.
7. I. V. Sokolov, Renormalization of the Lorentz–Abraham–Dirac equation for radiation reaction force in classical electrodynamics, Journal of Experimental and Theoretical Physics 109 (2009) 207–212. doi:10.1134/S1063776109080044.
8. A. L. Galkin, Dynamics of an electron in a relativistically intense laser field including radiation reaction, Journal of Experimental and Theoretical Physics 115 (2012) 201–207. doi:10.1134/S1063776112070072.
9. A. V. Bashinov, A. A. Gonoskov, A. V. Kim, M. Marklund, G. Mourou, A. M. Sergeev, Electron acceleration and emission in a field of a plane and converging dipole wave of relativistic amplitudes with the radiation reaction force taken into account, Quantum Electronics 43 (2013) 291–299. doi:10.1070/QE2013v043n04ABEH015101.
10. K. Seto, H. Nagamoto, J. Koga, K. Mima, Equations of motion with radiation reaction in ultrarelativistic laser-electron interactions, Physics of Plasmas 18 (2011) 123404. doi:10.1063/1.3663843.
11. L. W. Davis, Theory of electromagnetic beams, Physical Review A 19 (1979) 1177–1179. doi:10.1103/PhysRevA.19.1177.
12. D. Bauer, P. Mulser, W. Steeb, Relativistic ponderomotive force, uphill acceleration and transition to chaos, Physical Review Letters 75 (1995) 4622–4625. doi:10.1103/PhysRevLett.75.4622.
13. B. Quesnel, P. Mora, Theory and simulation of the interaction of ultraintense laser pulses with electrons in vacuum, Physical Review E 58 (1998) 3719–3732. doi:10.1103/PhysRevE.58.3719.
14. A. M. Goncharenko, Gaussian light beams [Gaussovy puchki sveta], Nauka i tekhnika, Minsk, 1997, in Russian.
15. D. R. Bituk, M. V. Fedorov, Relativistic ponderomotive forces, Journal of Experimental and Theoretical Physics 89 (1999) 640–646.

16. N. B. Narozhny, M. S. Fofanov, Scattering of relativistic electrons by a focused laser pulse, Journal of Experimental and Theoretical Physics 90 (2000) 753–768. doi:10.1134/1.559160.
17. V. P. Milant'ev, S. P. Karnilovich, Y. N. Shaar, Description of high-power laser radiation in the paraxial approximation, Quantum Electronics 45 (2015) 1063–1068. doi:10.1070/QE2015v045n11ABEH015800.
18. S. G. Bochkarev, V. Y. Bychenkov, Acceleration of electrons by tightly focused femtosecond laser pulses, Quantum Electronics 37 (2007) 273–284. doi:10.1070/QE2007v037n03ABEH013462.
19. N. N. Bogoljubov, Y. A. Mitropolskij, Asymptotic methods in the theory of nonlinear oscillations [Asimptoticheskiye metody v teorii nelineynykh kolebaniy], Nauka, Moscow, 1974, in Russian.
20. V. P. Milant'ev, A. J. Castillo, On the theory of the relativistic motion of a charged particle in the field of intense electromagnetic radiation, Journal of Experimental and Theoretical Physics 116 (2013) 558–566. doi:10.1134/S1063776113040067.
21. A. J. Castillo, V. P. Milant'ev, Relativistic ponderomotive forces in the field of intense laser radiation, Technical Physics 59 (2014) 1261–1266. doi:10.1134/S1063784214090138.

For citation:

E. V. Dobrova, V. P. Milantiev, On the radiation losses during motion of an electron in the field of intense laser radiation, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 133–142. DOI: 10.22363/2658-4670-2019-27-2-133-142.

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Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 143–153
http://journals.rudn.ru/miph

Research article

UDC 535.421, 535.44, 535015
PACS 42.25.Fx, 42.62 Fi
DOI: 10.22363/2658-4670-2019-27-2-143-153

Characteristics of optical filters built on the basis of periodic relief reflective structures

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(received: March 15, 2019; accepted: November 14, 2019)

A scheme of a new type optical filter, built using a relief reflective periodic diffraction structure, which has a specific rectangular profile, is proposed. The input radiation beam is directed to the relief structure at a certain angle of incidence. The zero diffraction order beam is our output beam, which is separated from the other diffraction order beams with the help of a diaphragm. The incidence-reflection plane is parallel to the relief lines of the diffraction structure. The dependence of the output beam power on the angle of incidence and on the wavelength of the radiation is investigated. It is shown that the power transfer coefficient from the input to the output of the scheme substantially depends on the wavelength of the optical beam. The scheme can be used as an optical signal filter. The spectral characteristic of this type of filter has an oscillating character. The zero (minimum) values of the power transfer coefficient of radiation from the input to the output of the filter alternate with maximum values close to unity. The spectral characteristic of the filter is easy to change by changing the angle of incidence of the input beam to the relief reflecting structure. Filters of this type can be built for the ultraviolet, visible, and infrared range. Calculations of the dependence of the filter parameters on the relief depth and on the angle of incidence of the input optical beam to the relief structure are presented.

Key words and phrases: filtering of the optical spectrum, diffraction structure, rearrangement of the spectral characteristics of the filter

Introduction

The need of filtering optical radiation appears in the conduction of physical experiments and in solving technical problems, in which it is necessary to separate a useful signal with specific wavelength from the background radiation or eliminate a certain part of the spectrum that interferes with measurements [1]. Filters based on colored glasses [2], filters built using dye solutions, as well as filters built using multilayer dielectric coatings [3] are widely used in practice. However, the above types of filters do not have the capability to rearrange their spectral characteristics. To rearrange the spectral characteristics is possible in acousto-optic filters [4], but it is difficult to manufacture filters of this type, and they are expensive devices.

In this paper, we consider the spectral characteristics of optical filters of a new type [5], which are constructed using relief reflective structures (RRS) with a rectangular profile, with "square wave" form, whose protrusion length is equal to depression length. The RRS profile is shown in Figure 1. The depth of the relief structure H_g, usually lies in the range from half the wavelength to several wavelengths of optical radiation in a given range. Relief structures of this type can be fabricated on a glass substrate using the method of photolithography and etching, and then covering the structure with metal thin film having high reflection coefficients [6]. Also, polished metal plates with high reflectance in a given spectral region can be used as substrates. For example, silver or aluminum [7]. The range of wavelengths in which the implementation of filters of this type is possible is quite wide: it includes part of the ultraviolet range, the visible range and part of the infrared range.

A characteristic feature of filters of this type is the possibility to change its spectral characteristics by changing the angle of incidence of the input optical beam to RRS. The dependence of the power transfer coefficient on the radiation wavelength, in filters of this type, has smooth shape with a certain number of minima and maxima. At the minima, the transfer coefficient is equal or close to zero, and at the maxima, the transfer coefficient is equal to the reflection coefficient from the surface of the metal thin film or a metal plate on which the RRS is located. It should be mentioned that relief structures with a rectangular shape have been used as optical filters before [8]. But a scheme was considered, in which the optical beam passed through a transparent substrate, with a transparent relief structure on its surface. For such structures, the dependence of the transmission coefficient on the wavelength is similar to the dependences that we describe in this paper. However, it is quite difficult to change the spectral characteristics of the filter changing the angle of incidence, because when we change the angle of incidence, it is necessary to rearrange the scheme. Thus, this paper focuses on the possibility of changing the spectral characteristics of filters when the angle of incidence of the input optical beam is changed.

1. Filter Scheme, Basic Relations for Calculating Characteristics

The filter scheme is shown in Figure 1.

The radiation source can be a laser (1), or a light source with a collimator forming a parallel optical beam. The radiation beam from the source is directed to the reflecting relief structure (2) at an angle θ, so that the incidence-reflection plane is parallel to the grooves forming the RRS. After reflection from the RRS, the radiation beam breaks down into diffraction orders and is directed to the diaphragm (3). Only zero diffraction order passes the diaphragm, all higher diffraction orders are cut off. Zero order beam is the output beam in this filter. The power transfer coefficient of the filter is defined as

$$k_p = \frac{P_0}{P_{\text{in}}}, \tag{1}$$

where P_{in} is the radiation power at the input of the optical scheme, and P_0 is the zero diffraction order radiation power at the output of the optical scheme.

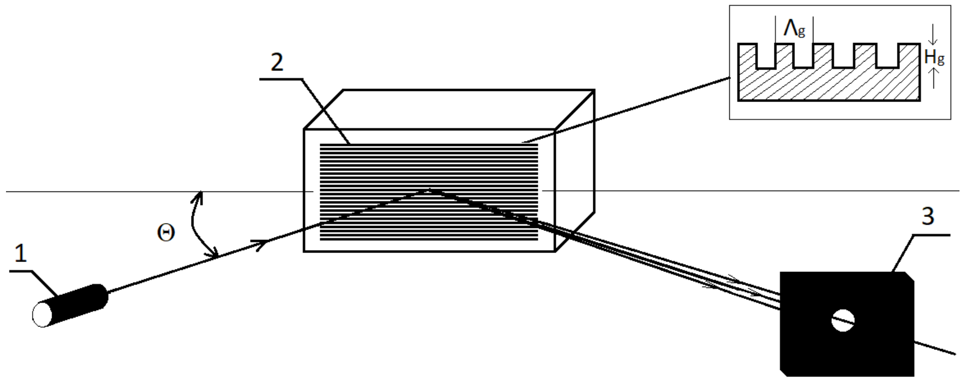

Figure 1. Filter scheme. 1 — radiation source, 2 — relief reflective structure (RRS), 3 — diaphragm

The spatial spectrum of an optical wave after reflection from the RRS consists of a set of diffraction orders. In this scheme we separate only the zero diffraction order. A detailed analysis of the spatial spectrum during the diffraction of an optical wave by phase gratings, which create a rectangular shape modulation of the wave-front phase, is given in [9]. We choose, in this work, periodic reflecting gratings with rectangular profile of the "square wave" form, in which the width of the protrusion is equal to the width of the depression. After wave diffraction on such a structure, there are no even orders in the spatial spectrum. The formulas for calculating the powers of the diffraction orders are given in the appendix. From the analysis of these formulas it follows that when the RRS depth varies in the range from zero to several wavelengths, the power of the zero diffraction order changes from zero to the maximum value.

For different wavelengths, and for different angles of incidence, the power transfer coefficient from input to output of the scheme is different. Dependence of the transfer coefficient on the radiation wavelength λ, angle of incidence θ, and RRS depth H_g for zero diffraction order, can be expressed by the formula:

$$k_p = \frac{P_0}{P_{\text{in}}} = R\left(0.5 + 0.5\cos\left(\frac{4\pi}{\lambda} H_g \cos\theta\right)\right). \tag{2}$$

Here R is the reflection coefficient of the RRS surface.

As follows from formula (2) at a given relief depth H_g, and at a given angle of incidence θ, the dependence of the transfer coefficient on the wavelength λ, is determined by a function of the form: $\cos\left(\lambda^{-1}\right)$. The transfer coefficient can take the minimum values equal to zero and the maximum values equal to reflection coefficient R. If the relief structure is made on a glass substrate, in order to obtain high values of the transfer coefficient at the maxima, we cover the structure with a metal thin film with a high reflection coefficient R.

From formula (2) it can be derived that the zero (minimum) values of the transfer coefficient correspond to the following wavelengths:

$$\lambda_{\min} = \frac{4H_g}{(2n+1)} \cos\theta, \qquad (n = 0, 1, 2, \dots). \tag{3}$$

The maximum values of the transfer coefficient correspond to the following wavelengths:

$$\lambda_{\max} = \frac{2H_g}{k} \cos\theta, \qquad (k = 1, 2, 3, \dots). \tag{4}$$

As can be seen from formulas (3) and (4), the spectral characteristics of the filter can be changed by changing the angle of incidence of the optical beam to the RRS. But, since changing the angle of incidence of the input beam changes the angle of the reflected optical beam, then in order to separate the zero diffraction order in the scheme shown in Figure 1, it will be necessary to change the position of the diaphragm (3).

As shown in Figure 2, another filter scheme is possible, in which the direction of the output beam does not change when the angle of incidence of the input beam changes. The input beam is directed from radiation source (1) to the RRS (2), which is located on one of the faces of the corner reflector. A mirror is located on the other face of the corner reflector. The angle between the RRS plane and the mirror plane is 90°. The beam reflected from the mirror is directed to the diaphragm (4), which permits the passing of only the zero diffraction order and does not permit the passing of higher orders. When the corner reflector is rotated, the angle of incidence of the input beam changes related to RRS. This changes the spectral characteristic of the filter. But the direction of the output optical beam relative to the base coordinate system and relative to the input optical beam does not change.

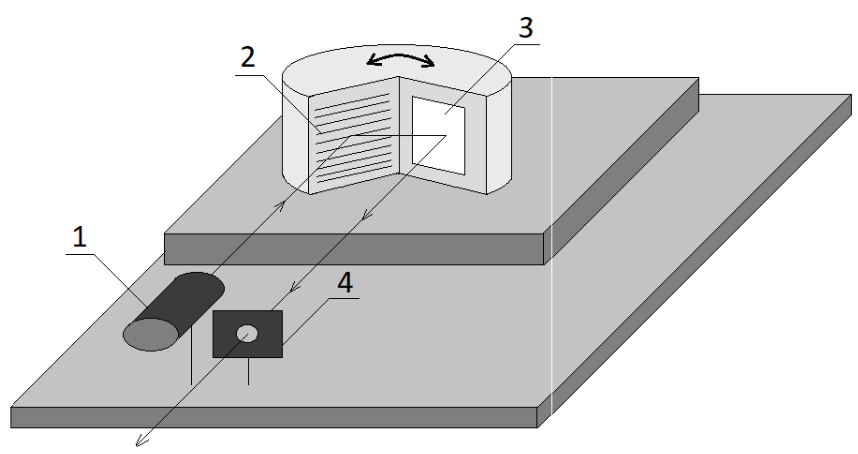

Figure 2. Scheme of the filter using a corner reflector. 1 — radiation source, 2 — RRS, 3 — mirror, 4 — diaphragm

However, in practice, the use of a corner reflector limits the range of angles of incidence of the input beam to RRS. If the angle of incidence of the input beam is 45°, then the point of incidence of the input beam at RRS and the point of reflection of the beam from the mirror (3) are at the same distance from the plane of symmetry of the corner reflector. Let us assume that the axis of rotation of the corner reflector passes through the point of incidence of the input beam on the RRS. Then, when the corner reflector is rotated, the position of the point of incidence on the RRS remains unchanged, and the reflection point at the mirror moves, and at certain angles of rotation it may go out from the mirror limits, and as a consequence the scheme would stop working.Thus, the scheme shown in Figure 2 has smaller range of possibilities for changing the spectral characteristics of the filter compared with the first scheme (Figure 1).

2. Estimated characteristics of filters based on RRS

Figure 3 shows the dependence of the transmission coefficient on the wavelength, estimated for RRS with a relief depth of $H_g = 0.4\ \mu m$ at three values of the angle of incidence (20°, 40°, 60°).

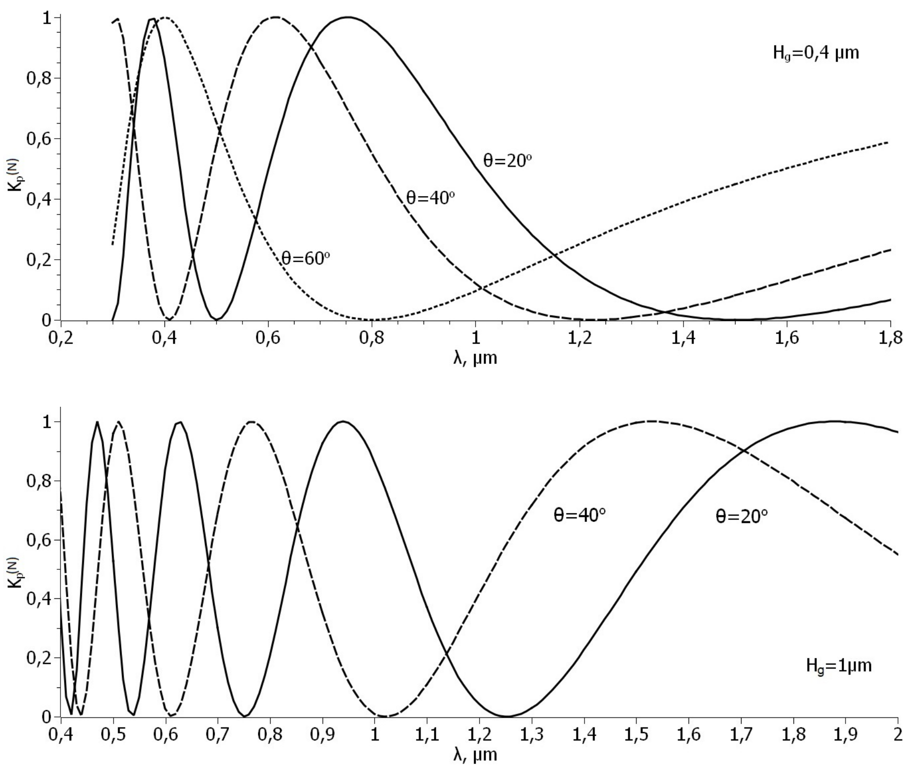

Figure 3. Dependence of the normalized transmission coefficients of the filters on the incident beam wavelength. The upper graph (a) corresponds to the depth $H_g = 0.4\ \mu m$ at different angles of incidence $\theta = 20^\circ$, 40° and 60°. The bottom graph (b) corresponds to the depth $H_g = 1\ \mu m$, at angles of incidence $\theta = 20^\circ$ and 40°

The curves are normalized, i.e. the vertical axis represents the value $k_p^{(\mathrm{N})} = k_p/R$. In other words, the curves correspond to the transfer coefficient, provided that the reflection coefficient of the RRS surface would be equal to unity. As can be seen from the graphs, even with small changes in the angle of incidence, it is possible to obtain significant changes in the spectral characteristics of the transfer coefficient. For example, you can set the task: to completely suppress a certain wavelength. Substituting the value of the wavelength in the formula (3), we calculate the angle at which the transfer coefficient will be zero. Thus we find the solution to this problem. Practically fine tuning of the angle of incidence can be carried out according to the measurement result of the output signal.

Analyzing the dependencies presented in the graphs, you can notice the following features:

— the number of maxima and minima in a certain range of wavelengths increases with increasing of the RRS depth;
— the distance between adjacent maxima and minima decreases with decreasing of wavelength;
— as the angle of incidence increases, the minima and maxima of the curves move toward shorter waves.

At RRS depths more than 1 micron, the curves have several minima and maxima in the visible region of the spectrum.

In order to get a more general idea of the positions of the maxima and minima in a wide wavelength region and for different angles of incidence of the input beam, we constructed the diagrams of the positions of the maxima and minima on the plane of the coordinates $\lambda - \theta$, which are shown in Figure 4(a,b).

Using the diagrams given in Figure 4(a,b), it is easy to estimate how the spectral characteristic of the filter changes as the angle of incidence of the input optical beam changes. If we draw a horizontal line at a given angle of incidence, then the projections onto the horizontal axis of the points of its intersection with the lines of maxima and minima give the corresponding coordinates of minima and maxima in the dependence of the transfer coefficient on the wavelength $\left(k_p^{(\mathrm{N})}(\lambda)\right)$.

The longest wavelength corresponding to a minimum of the dependence $\left(k_p^{(\mathrm{N})}(\lambda)\right)$, at a certain angle of incidence θ, can be determined by setting $n = 0$ in expression (3). This wavelength is: $\lambda_{\min}(n = 0) = 4H_g \cos\theta$. In the region of wavelengths exceeding this $\lambda_{\min}$, the curve $\left(k_p^{(\mathrm{N})}(\lambda)\right)$ have a monotonic increasing character and asymptotically tend to the level $\left(k_p^{(\mathrm{N})}(\lambda)\right) = 1$. At a wavelength equal to $\lambda_{\max}(k = 1) = 2H_g \cos\theta$ is located the maximum of the transfer coefficient closest to the longest wavelength corresponding to minimum $(\lambda_{\min}(n = 0))$. In the region of wavelengths exceeding $\lambda_{\max}(k = 1) = 2H_g \cos\theta$, we observe a smooth curve with one minimum.

In the region where the wavelength satisfies the condition $\lambda \ll 2H_g \cos\theta$, there are frequent oscillations of the transfer coefficient as the radiation wavelength changes. Using this region you can build a filter that suppresses one radiation wavelength and passes another radiation wavelength, and these wavelengths are close to each other. As an example, we give the characteristic of the filter, using the RRS with a depth of 3 microns. Figure 5 shows the dependence of the transmission coefficient on the wavelength of such a filter at two different angles of incidence.

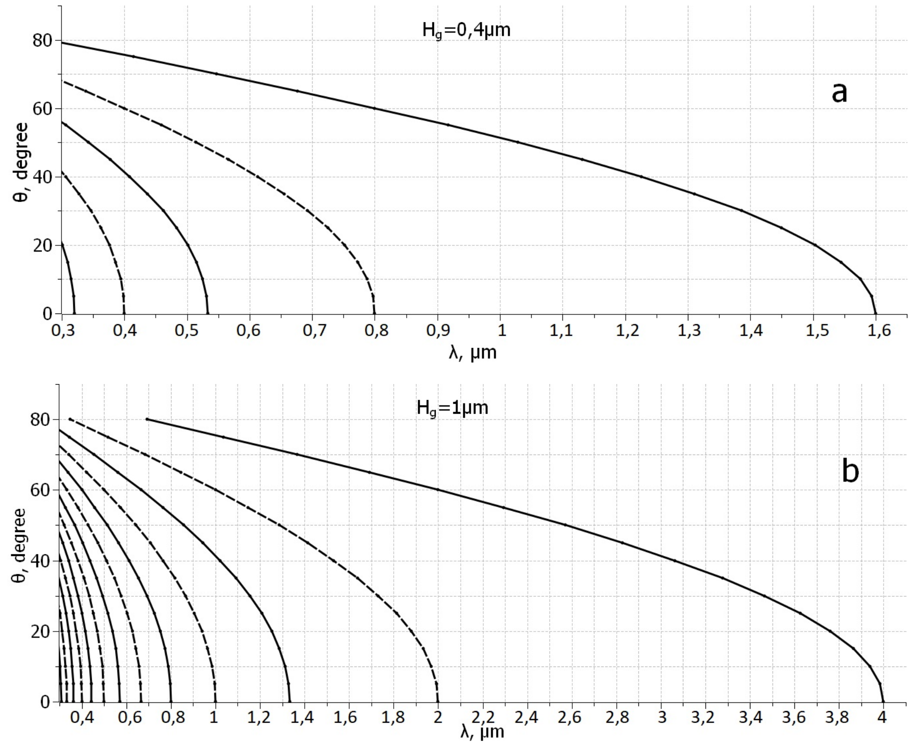

Figure 4. (a, b). Diagram of the positions of the maxima and minima of the output beam power in the zero diffraction order, on a scale of wavelengths, depending on the angle of incidence of the optical beam, for a relief depth equal to: 0.4 μm (a) and 1 μm (b). The solid lines indicate the positions of the minima, and the dotted lines indicate the positions of the maxima

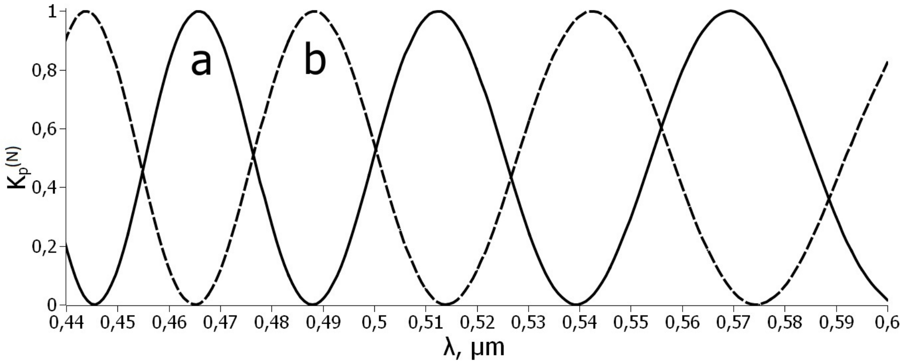

Figure 5. (a, b). Dependence of the transmission coefficient of the filter constructed using RRS on the radiation wavelength. Graph (a): $H_g = 3\ \mu m$, $\theta = 31.35^{\circ}$. Graph (b): $H_g = 3\ \mu m$, $\theta = 35.53^{\circ}$

Let us assume that an argon laser beam is directed to our filter. It is known that the argon laser has two strong emission spectral lines: at 0.488 μm and at 0.514 μm [10]. When the angle of incidence is $\theta = 31.35°$, radiation with 0.488 μm wavelength will not pass to the output of the filter, and for radiation with 0.514 μm wavelength the normalize transfer coefficient will be close to 1.

If you change the angle of incidence and set it equal $\theta = 35.53°$, then the situation will change to the opposite. The transmission coefficient at a wavelength of 0.514 μm will be equal to zero, and the transmission coefficient at a wavelength of 0.488 μm will be close to the maximum.

It should be noted that the part of the radiation power that did not pass to the filter output in the zero diffraction order is distributed between the first and higher diffraction orders and is absorbed by the diaphragm.

Consider another example of constructing a filter for wavelength ranges of part of the visible and ultra violet light. As a light source we will consider a mercury lamp. The spectrum of a mercury lamp contains a series of lines with wavelengths in the range from 280 nm to 630 nm [11]. These wavelengths are indicated by thin vertical lines in Figure 6, which also shows the theoretical dependences of the transmission coefficient of two kinds of filters on the wavelength. The filter of the first type, whose spectral characteristic is drawn with a dotted line, is made as a RRS on the surface of a polished aluminum plate, which reflectance is about 90%. This filter can pass two groups of lines: one in the region 280–320 nm and the other in the region 540–620 nm. At the same time, the intensity of the lines in the region of 360–440 nm is significantly reduced. The filter of the second type, the spectral characteristic of which is drawn by a solid line, is made on the surface of a polished silver plate. The reflection coefficient of silver in the region of wavelengths less than 350 nm has a sharp decline. With this in mind, the transmission coefficient of this type of filter has a much smaller value in the region of 280–320 nm compared with the transmission coefficient of the filter of the first type. In both types of filters, the relief depth was assumed to be 405 nm and the angle of incidence was 41.4°.

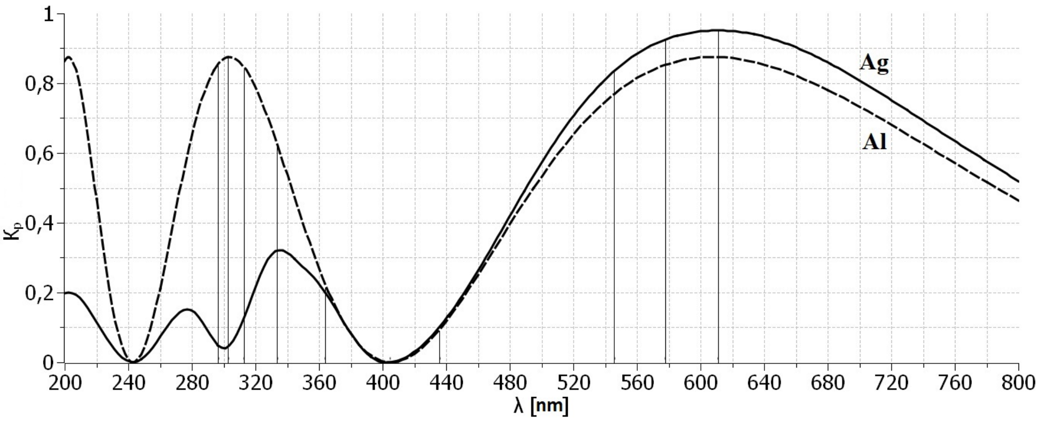

Figure 6. Dependence of the transmission coefficients of the two types of filters built using RRS on the wavelength. The RRS depth is 405 nm. The angle of incidence of the radiation is 41.4°. The solid line corresponds to the RRS on a silver plate, and the dotted line corresponds to the RRS on an aluminum plate

Parameters such as the RRS period and the diameter of the optical beam are not directly included in the above calculation formulas, therefore, we will discuss here suggestions for choosing these parameters.

The period of the diffraction structure must be significantly longer than the maximum wavelength of the wavelength range in which the filter is supposed to be used. The transverse size of the optical beam must be several times greater than the radiation wavelength, in order for the diffraction beams directions of zero and first orders to be successfully separated in space.

For separation of diffraction beams at the plane of the diaphragm, it is necessary that the distance from the RRS to the diaphragm satisfy the condition:

$$L = p\,D\frac{\Lambda}{\lambda}, \tag{5}$$

L is the distance from the RRS to the diaphragm, D is the diameter of the optical beam, Λ is the period of the diffraction structure, the number p is the safety factor (no less than 2), which is necessary to prevent penetration of the radiation beams of the first orders of diffraction into the aperture of the diaphragm.

Conclusion

Filters based on RRS have a number of positive properties. Their spectral characteristics can be easily changed by changing the angle of incidence of the optical beam to the RRS. The design of the filter is quite simple. The relief structure is fabricated using well-developed photolithography and etching technologies. For the manufacture of filters designed to work in the infrared wavelength range, there is no need to use special materials. Regular polished metal plates are quite suitable: aluminum, copper, silver. The transfer coefficient at the maximum is equal to the reflection coefficient of the RRS surface.

Also, we should mention a number of disadvantages of filters of this type. The spectral characteristics of filters of this type are smooth. Their shape is far from rectangular. In this respect they cannot be compared with widely known filters based on multilayer dielectric structures. As can be seen from the principle of operation, filters based on the RRS will work well when they are irradiated by directed laser beams with a small divergence. The question of the interaction of diverging beams with a filter of this type was not considered in this paper.

Appendix

As a result of the reflection of a coherent optical beam from a relief structure, which has a rectangular profile with "square wave" form, a spatial phase modulation of the wave front takes place. The phase modulation function form is rectangular, and the amplitude of this function is equal to:

$$\Phi_M = \frac{2\pi}{\lambda} H_g \cos\theta.$$

The function of the wave front phase modulation can be expressed as:

$$f(x) = e^{i\Phi(x)}, \quad \text{where } \Phi(x) = \begin{cases} \Phi_M, & \text{if} \quad \frac{\Lambda}{2} + k\Lambda > x > 0 + k\Lambda, \\ -\Phi_M, & \text{if} \quad 0 + k\Lambda > x > -\frac{\Lambda}{2} + k\Lambda, \end{cases} \quad k = 0, 1, 2, 3, \ldots$$

We find the spatial spectrum of the coherent optical wave after reflection from the relief structure applying the Fourier transform to this expression. The spatial spectrum consists of a zero order and a set of diffraction orders at spatial frequencies that are multiple of the spatial frequency $\xi = \Lambda^{-1}$.

The expressions for the Fourier coefficients are:

$$C_0 = \cos\Phi_M, \quad C_1 = \frac{2}{\pi}\sin\Phi_M, \quad C_m = \frac{1}{m\pi}(\sin\Phi_M - \sin(\Phi_M + \pi m)).$$

The ratio of the radiation power in one diffraction order to the input radiation power P_{in} is equal to the square of the corresponding Fourier coefficient (assuming that there are no losses).

In particular, the ratio of the power in zero diffraction order to the power at the input of the diffraction structure (provided that the reflection coefficient is 100%) is:

$$\frac{P_0}{P_{\text{in}}} = \cos^2\Phi_M = 0.5 + 0.5\cos 2\Phi_M = \left(0.5 + 0.5\cos\left(\frac{4\pi}{\lambda}H_g\cos\theta\right)\right).$$

Acknowledgments

The publication has been prepared with the support of the "RUDN University Program 5-100".

References

1. V. V. Lebedeva, Experimental optics [Eksperimental'naya optika], 3rd Edition, Publishing House of Moscow University, Moscow, 1994, in Russian.
2. V. I. Puchkov, E. A. Iosep, G. T. Petrovsky, L. S. Iutinskaya, A. P. Ivanova, A. V. Smirnova, S. P. Lunkin, Coloured optical glass. Specifications [Steklo opticheskoye tsvetnoye], State Standard of the USSR, GOST 9411-91, `http://docs.cntd.ru/document/gost-9411-91`, 1991, in Russian.
3. A. M. Prokhorov (Ed.), Handbook of Lasers [Spravochnik po lazeram], Vol. 2, Soviet Radio, Moscow, 1978, in Russian.
4. I. P. Golyamin (Ed.), Ultrasound (small encyclopedia) [Ul'trazvuk (Malen'kaya entsiklopediya)], Soviet Encyclopedia, Moscow, 1979, in Russian.
5. V. A. Komotsky, Y. M. Sokolov, N. V. Suetin, A device for filtering optical signal spectra [Ustroystvo dlya fil'tratsii spektrov opticheskogo signala], Patent for utility model 181381, in Russian (2018).

6. V. P. Lavrintsev (Ed.), Introduction to photolithography [Vvedeniye v fotolitografiyu], Energy, Moscow, 1977, in Russian.
7. G. V. Rosenberg, Optics of thin-layer coatings [Optika tonkosloynykh pokrytiy], State publishing house of physical and mathematical literature, Moscow, 1958, in Russian.
8. Machine construction encyclopedia XXL [Entsiklopediya po mashinostroyeniyu XXL], in Russian.
URL http://mash-xxl.info
9. N. M. Kashchenko, Determination of the depth of the reference diffraction gratings based on the measurement and analysis of the intensities of the diffraction orders [Opredeleniye glubiny opornykh difraktsionnykh reshetok na osnove izmereniya i analiza intensivnostey difraktsionnykh poryadkov], Bulletin of the Peoples' Friendship University of Russia. Series: Physics 7 (1) (1999) 16–27, in Russian.
10. Gas argon laser LEXEL 85/95 (visible) [Gazovyy argonovyy lazer LEXEL 85/95 (vidimyy)], in Russian.
URL http://scientific-technology.ru/laserslink/gaslaser/lexel85ar
11. Spectrum of a natural sunshine fluorescent light.
URL https://commons.wikimedia.org/wiki/File:Spectra-Philips_32T8_natural_sunshine_fluorescent_light.svg

For citation:

V. A. Komotskii, J. A. Huaman, V. D. Evstigneeva, Characteristics of optical filters built on the basis of periodic relief reflective structures, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 143–153. DOI: 10.22363/2658-4670-2019-27-2-143-153.

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Discrete & Continuous Models
& Applied Computational Science

2019, 27 (2) 154–164
http://journals.rudn.ru/miph

Research article

UDC 51-7(630)
DOI: 10.22363/2658-4670-2019-27-2-154-164

Numerical analysis of ecology–economic model for forest fire fighting in Baikal region

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(received: September 2, 2019; accepted: November 14, 2019)

Forest fires lead to the serious damage of ecological state and national economy of the country. This problem is especially relevant for Siberians. According to Greenpeace, Siberian forest fires in 2019 reached record levels in the entire history of observation in terms of burning area and the amount of carbon dioxide emitted into the atmosphere. It leads not only to a deterioration in the health of Siberians, but also to environmental problems of the region. Note that the large-scale fire-prevention measures entails enormous financial costs. Therefore, economical, ecological and mathematical modeling of the situations, arose in forest fires countering, becomes actual.

The paper is devoted to optimal control problem of forest fires fighting. Its prototype is the well-known Parks model. To investigate the model, we apply the modern programming language Julia, which is designed to mathematical calculations and numerical studies. We made an extensive computational experiment in this model and a numerical analysis of corresponding optimal control problems. The obtained results were examined both on the adequacy of the model, and on the possibility of using the Julia language and the included solvers of mathematical problems.

Key words and phrases: forest fires fighting, mathematical modeling, optimal control, numerical analysis, Julia programming language

1. Introduction

The Irkutsk Region is one of the largest constituent entities of the Russian Federation. Its area is about 774.846 square km., it is slightly less than the area of the Republic of Turkey (780.580 sq. km.), and also exceeds the territory of France, Germany and many other European countries. Most of the territory of the Irkutsk region, namely 71.5 million hectares, or 92% of its territory, is occupied by the forest. About 12% of timber reserves of ripe forests of the country are concentrated here, and the share of especially valuable coniferous species is significant even on the scale of the planet.

According to the press service of the Ministry of Emergency Situations of Russia, on July 28, the 143 forest fires with a total area of 597,298 hectares are active in the Irkutsk Region. Totally, in Siberia and the Far East, forest fires are burning in the area comparable to Belgium. Of course, such an essential loss of natural resources leads to negative environmental, economic and social consequences [1–3]. For instance, in the work [2] it was established that during intensive burning of the taiga, the concentration of carbon monoxide increases by almost 30 times in comparison with the background content in the air, methane by 2 times, carbon dioxide in 8%. Such exceedences lead to the health deterioration of the inhabitants of the Irkutsk region [3]. In addition, due to the annual, large-scale forest fires blazing near Lake Baikal, chemical components, such as ammonium, expedite the reproduction of various microorganisms that destroy the aquatic ecosystem of the Baikal region.

The most important problem in forest fires fighting, besides the protecting of people's lives, is a quick and effective fire suppression, planned to minimize the total damage. Controlling of the process of suppression, transportation of forces to the place of fire is made by employees of forest protection organizations. In the most cases they make decisions based on their personal experience. But even with experience, defining an optimal fire fighting plan is, often, a quite difficult task. For many years, scientists have been studying models that allow them to find optimal solutions of fire fighting forces control under an active forest fire. These attempts are being made to take into account the characteristics of the spread of fire, the capabilities of the available fire-fighting forces and equipment, topographic features and other factors. Here we mention works [4–16]. Note that the papers [4, 5] continues the research originated in [7, 8].

The analysis of some foreign works on the subject [9–11] shows that more investigations use modern programming languages, such as Python, R, Java, and so on. These software products are applied to analyze and visualize the data. Actually, it improves the quality of the research. However, despite a sufficient number of software tools, in explorations of Russian scientists such products are not so widely used as abroad. In this paper, we apply rather new universal programming language Julia [17]. Its development was begun by scientists in 2009, and its first version was published in 2012. The research uses the latest version of Julia, presented in 2018. Julia is a modern high-level programming language with dynamic typing for mathematical calculations, which is used to develop research software, approbation and test of new problem-solving methods. The essential advantages of this language are simple syntax and speed of program execution. It is often chosen by astronomers, robotics and financiers. In the computational experiment, the IPOPT (Interior Point OPTimizer) package is used. This package is meant to numerical solving of optimization problems of a large dimension.

2. Problem statement

The paper focuses on the optimization model for the dispatching and withdrawal of fire-fighting forces to the place of a forest fire [6, 8]. After certain transformations, it can be written as the following optimal control problem (P):

$$J = \int_{t_a}^{t_c} (2C_S u_2(t) - \alpha(t)x(t) + \beta(t))dt + 2C_S x(t_c) \to \min,$$

$$\dot{x} = u_1 - u_2,$$
$$x(t_a) = x_a,$$
$$-m \leqslant u_1(t) - u_2(t) \leqslant M, \quad u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0,$$
$$0 \leqslant x(t) \leqslant X,$$
$$g(t_c, x(t_c)) = x(t_c) - \frac{r(t_c)}{E(t_c - t_a)} = 0,$$

where $\alpha(t) = C_B E(t - t_a) - C_X$ and $\beta(t) = C_T + C_B r(t)$.

The state variable $x(t)$ denotes the size of the fire fighting force at the moment t. The pair of control variables $u_1(t)$ and $u_2(t)$ represent dispatching and withdrawal rates of the reinforcements at the time t, respectively. The trajectory $x(\cdot)$ is supposed to be a piecewise smooth function, while the control functions $u_1(\cdot)$, $u_2(\cdot)$ are piecewise continuous.

Let us give an economical interpretation of model's parameters:

— t_a and t_c are the time moment of initial attack and the final time moment, when the fire is brought under control, respectively; t_c, in general, is supposed to be non-fixed;
— $r(t)$ is a function of the fire spread rate in the absence of fire fighting forces;
— C_S is the cost to transportation (i.e., dispatching or withdrawal) of fire fighting forces (currency unit per force unit);
— C_T is the parameter characterizing the loss of the forest during uncontrolled burning per time unit (currency unit per time);
— C_B is the cost per unit area of forest damaged by fire (currency unit per area unit);
— C_X is the cost of the fire-fighting (currency unit per unit of forces – time);
— m is the maximal rate of fire-fighters withdrawal (force unit per time);
— M is the maximal rate of fire-fighters dispatching (force unit per time);
— E is the ratio of the effectiveness of fire fighting in this area (unit of forces per time);
— x_a is the initial attack force (force unit);
— X is the maximal limit of fire fighting forces (force unit).

We point out some features of problem (P). This linear optimal control problem contains

a) the terminal state constraint in the form of equality, and
b) the pointwise state constraint.

These features significantly complicate the analytical investigation of problem (P), even under the linearity of the dynamical system [8]. In the mentioned work, the problem was analyzed using the Pontryagin Maximum Principle [18]. As a result, the author of [8] gave an explicit formula for the optimal control, which depends on values of two unspecified parameters: the final time t_c and the scalar Lagrange multiplier λ which corresponds to the terminal state constraint. For a further specification of the optimal control, it was proposed to use a grid search of proper values of the unknown parameters. Then, the optimality conditions should be checked for each choice of t_c and λ.

Our paper is devoted to the numerical investigation of problem (P). We use the so-called “direct approach”, i.e., an approach to the study of optimal control problems, when the total discretization of the dynamic optimization problem is applied. Further, the obtained problem is solved by methods and tools of mathematical programming. This approach is often criticized by experts in the field of optimal control, but it often turns out to be effective in solving practical optimization problems.

At the discretization stage of problem (P), we apply the explicit Euler scheme, firstly reducing problem (P) to the Mayer form. We introduce an unessential state variable y, which derivative coincides with the integrand of the cost functional J (the initial condition for y is trivial). The problem may be rewritten as follows (problem (P_1)):

$$J = y(t_c) + 2C_S x(t_c) \to \min,$$

$$\dot{x} = u_1 - u_2,$$
$$\dot{y} = 2C_S u_2 - \alpha(t)x + \beta(t),$$
$$x(t_a) = x_a, \quad y(t_a) = 0,$$
$$-m \leqslant u_1(t) - u_2(t) \leqslant M, \quad u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0,$$
$$0 \leqslant x(t) \leqslant X,$$
$$g(t_c, x(t_c)) = x(t_c) - \frac{r(t_c)}{E(t_c - t_a)} = 0,$$
$$\alpha(t) = C_B E(t - t_a) - C_X, \quad \beta(t) = C_T + C_B r(t).$$

Note that problems (P) and (P_1) are equivalent to each other. Meanwhile, problem (P_1) contains the terminal and pointwise state conditions as well.

3. Numerical analysis

Let us consider a discrete analogue of problem (P_1) using the direct Euler scheme. Here we suppose that the final time moment t_c is given. Introduce the N-point grid of the time interval $[t_a, t_c]$: $t_a = t_0 < t_1 < ... < t_{N-1} < t_N = t_c$. As usual, the time lag is calculated as $h = \dfrac{t_N - t_0}{N}$. Discrete problem (P_d) takes the following form:

$$J = y(t_N) + 2C_S x(t_N) \to \min;$$

$$x(t_{k+1}) = x(t_k) + h\Big[u_1(t_k) - u_2(t_k)\Big],$$

$$y(t_{k+1}) = y(t_k) + h\Big[2C_S u_2(t_k) - \alpha(t_k)x(t_k) + \beta(t_k)\Big],$$

$$x(t_0) = x_a, \quad y(t_0) = 0,$$
$$-m \leqslant u_1(t_k) - u_2(t_k) \leqslant M, \quad u_1(t_k) \geqslant 0, \quad u_2(t_k) \geqslant 0,$$
$$k = 0, 1, \dots, N-1;$$
$$0 \leqslant x(t_k) \leqslant X, \quad k = 0, 1, \dots, N;$$
$$g(t_N, x(t_N)) = x(t_N) - \frac{r(t_N)}{E(t_N - t_0)} = 0,$$

where
$$\alpha(t) = C_B E(t - t_0) - C_X, \quad \beta(t) = C_T + C_B r(t).$$

Here, we use the previous notations for state and control variables and parameters of the problem. Furthermore, note that we have the sequences $\{u_1, u_2\}$, $\{x, y\}$ thought as control and state, respectively:

$$u_1 = \{u_1(t)\}, \quad u_2 = \{u_2(t)\}, \qquad t = t_0, t_1, \dots, t_{N-1},$$
$$x = \{x(t)\}, \quad y = \{y(t)\}, \quad t = t_0, t_1, \dots, t_N.$$

Notice that (P_d) is a linear programming problem. It contains $4N$ variables and $8N + 1$ conditions.

We have tested more than 30 variants of the parameters of problem (P_d). Some of them were unsuccessful. Such outcomes we associate with the absence of admissible plans of the problem. We think that the choice of values of the final time moment t_c was improper in certain trials.

Let us show the results of certain numerical experiments. Here, we present some interesting examples. Each of them is accompanied by a table indicating the values of the parameters. Illustrations are arranged as follows: on the upper graph, the trajectory component $x(t)$ is depicted, and on the lower graphs we show controls $u_1(t)$ and $u_2(t)$, from left to right, respectively.

3.1. The first group of examples: the rate of fire spread is constant

Example 1. The parameters of problem (P_d) are presented in Table 1.

Table 1

Parameters for Example 1

Parameter	t_0	t_N	N	C_s	α	β	m	M	E	r	x_0	X
Value	0	22	1000	10	1	5	30	30	1	50	2	1000

Let us give an interpretation of the found solution (see Figure 1). The corresponding value of the cost functional $J \approx 75$.

The initial attack force at the moment t_a is characterized by $x_0 = 2$. The high rate of fire spread ($r \equiv 50$) compels us to use the maximum speed dispatching of new fire fighting forces. Then, the value of forces takes a turnpike state $x \approx 32$, and we keep it up to the time of withdrawal. The initial and final time intervals are characterized by maximum values of dispatching (u_1) and withdrawal (u_2) speeds.

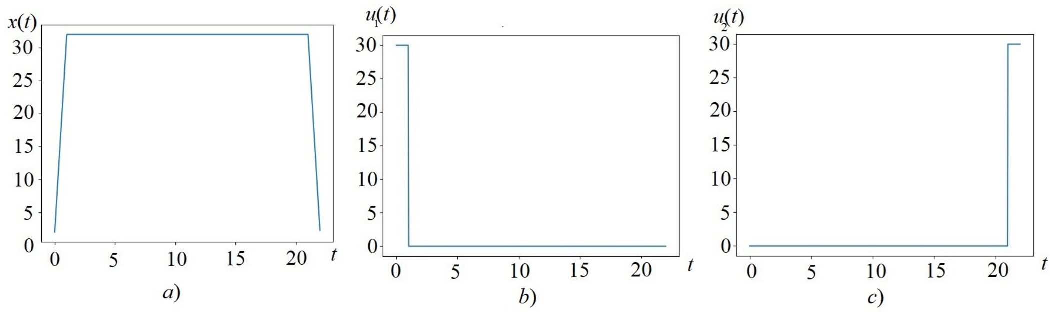

Figure 1. Graphs of optimal trajectory (a) and optimal controls (b, c) in Example 1

Note that the state constraints on $x(t)$ are inactive in this example (the contrary case is shown by the Example 3).

Example 2. We decrease some parameters: the rate of fire spread r, the maximal rate of fire-fighters withdrawal m, the final time t_c. The updated data is shown in Table 2.

Table 2

Parameters for Example 2

Parameter	t_0	t_N	N	C_s	α	β	m	M	E	r	x_0	X
Value	0	10	1000	5	1	5	1	30	1	5	2	100

The solution is presented on Figure 2. It refers to the high speed of withdrawal fire forces at the final period. The most effective attack needs only forces in the place (at the initial time). In this case the optimal value of the cost functional is $J \approx 51$.

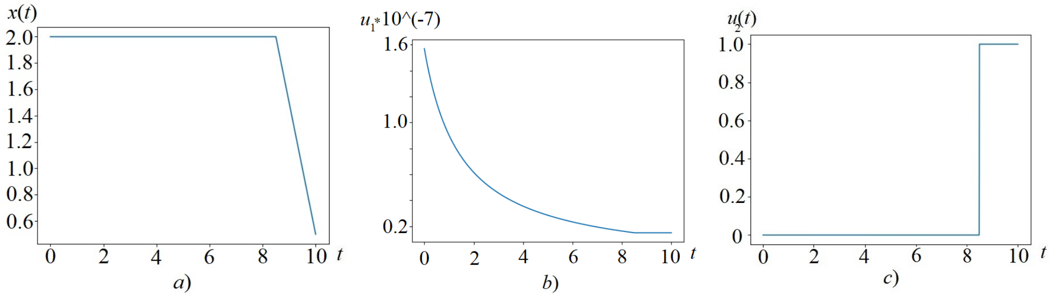

Figure 2. Graphs of optimal trajectory (a) and optimal controls (b, c) in Example 2

Note that control u_1 is rather close to zero, and we look at its graph as computational error.

Although the value of fire-fighting forces in the place is decreased to $X = 100$, the pointwise state constraints remain inactive.

Table 3

Parameters for Example 3

Parameter	t_0	t_N	N	C_s	α	β	m	M	E	r	x_0	X
Value	0	100	1000	10	10	1	30	30	1	5	2	1000

Example 3. We essentially increase the time interval and change values of α and β, and some other parameters (see Table 3).

Figure 3 shows that the pointwise state constraint becomes active. The trajectory graph means increasing of the initial value of forces $x_0 = 2$ to the maximal level $X = 1000$. Herewith, both controls u_1 and u_2 take maximum values on the initial and final time intervals, respectively.

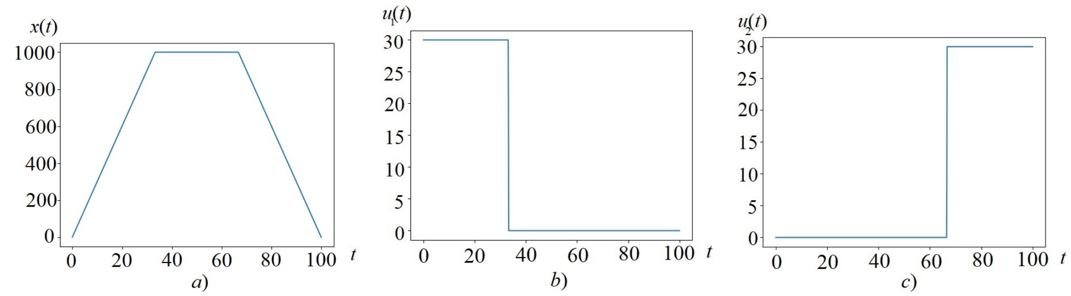

Figure 3. Graphs of optimal trajectory (a) and optimal controls (b, c) in Example 3

3.2. The second group of examples: the variable rate of fire spread

Here, we suppose that the rate of fire spread increases by the following law:

$$r(t) = \begin{cases} 1, & t \in [1, 6], \\ 15, & t \in (6, 12]. \end{cases}$$

Also, we fix a number of time-grid points $N = 300$.

Note that problem (P) was investigated in [5], where some features (advantages and shortcomings) of the model were indicated. Particularly, the author said that the optimal solution is characterized by three stages of control. The first of them corresponds to the maximum speed of fire fighting forces dispatching. On the second stage all involved forces fight with the fire. And then, fire fighters are withdrew with the maximal rate. The previous examples correspond to this consequence.

However, even linear problems of dynamic optimization admit "singular" intervals of control, where the last one can take intermediate values. In particular, the considered model is able to contain such intervals. This feature is illustrated by the following examples.

Example 4. The data of the problem is presented in Table 4.

Consider Figure 4.

Table 4

Parameters for Example 4

Parameter	t_0	t_N	C_s	C_t	C_b	C_x	m	M	E	x_0	X
Value	0	12	1	0	0	0	3	3	1	0	10

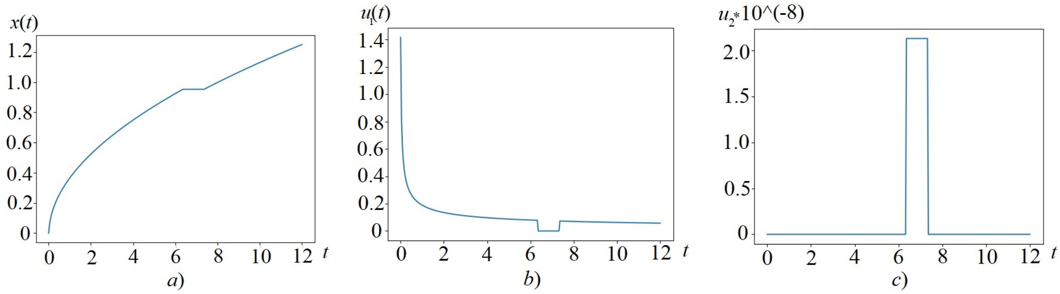

Figure 4. Graphs of optimal trajectory (a) and optimal controls (b, c) in Example 4

One can see that control u_1 takes intermediate values of admissible set $[0; 3]$. Note that the "jump" of control u_2 is inessential, and its values are close to zero, in fact.

The found solution admits the following conclusions.

On the first stage, when a fire spreading rate $r = 1$, the most effective fire extinguishing strategy was achieved at the time point $t = 6$. The fire was localized. However, a significant increase in the rate of fire spread (up to the level $r = 15$) required the use of additional forces (such a situation, for example, is due to the weather deterioration). The short time of the pause in dispatching of fire-fighters is associated with an excess of the cost of transporting fire forces in comparison with the damage of the action of fire.

In this example, the pointwise state constraint was again inactive.

Example 5. Minor changes of parameters E and x_0 entail certain changes in the controls (see Table 5 and Figure 5). Note, there are also time intervals with intermediate values of the controls.

Table 5

Parameters for Example 5

Parameter	t_0	t_N	C_s	C_t	C_b	C_x	m	M	E	x_0	X
Value	0	12	1	0	0	0	3	3	1,1	1	10

In order to localize the fire, the fire prevention forces increased gradually. It is noteworthy that the found solution contains two "turnpike" intervals [19]. These intervals are characterized by the constancy of the trajectory on the plot of the function x. Each of them corresponds to the fire spread rate levels $r = 1$ and $r = 15$, respectively. The graph of control of u_2 is close to zero (we assume, as before, the depicted "blow-ups" are computational errors). At the same time, the descending control peaks of u_1 are currently not explicable by the authors and require additional analysis.

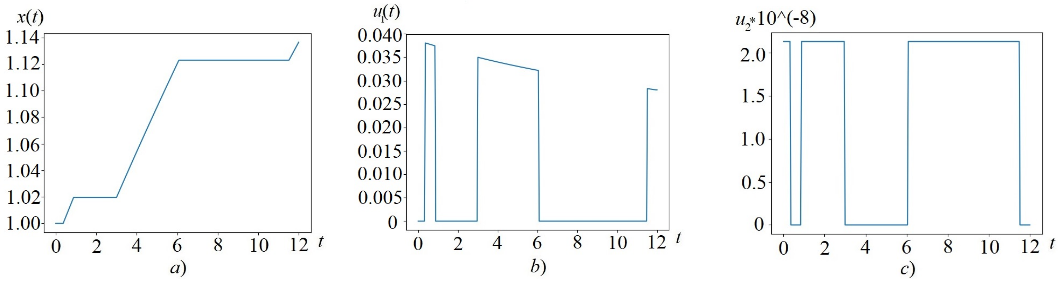

Figure 5. Graphs of optimal trajectory (a) and optimal controls (b, c) in Example 5

4. Conclusion

Modern mathematical methods and tools are currently very accessible for applied research. Of course, their use requires certain skills and understanding of the field of research. The article shows how using the Julia programming language one can numerically investigate some applied mathematical models and solve the corresponding optimization problems.

The results of the experiments are very interesting. Examples 1–3 illustrate rather obvious strategies of forest fire fighting, which applied on practice. Note that the results of calculations for example 2 correspond to the behavior of decision makers in the north of the Irkutsk region. Such tactics, of the non-attraction of additional countervailing forces, is substantiated by the economic inefficiency of them.

Examples 4, 5 also interesting from an applied point of view, are entertaining by mathematics view as well. Apparently, their solutions contain the trunk modes of dynamic systems (in the last example one can see two turnpike intervals) [19]. The noted finding requires further study with the involvement of the corresponding mathematical apparatus.

Further research will be associated with a more detailed analytical study of the obtained results, consideration of nonlinear modifications of the presented model and more complicated statements of optimization problems.

Acknowledgments

The publication has been prepared with the financial support of the Ministry of Science and Higher Education of the Russian Federation, project no. 0279-2019-0003.

References

1. A. P. Sukhodolov, A. A. Izmestiev, Economic accessibility of forest resources as rent-forming factor and assessment basis for forest raw material potential [Ekonomicheskaya dostupnost' lesnykh resursov kak rentoobrazuyushchiy faktor i osnova otsenki lesosyr'yevogo potentsiala], Bulletin of Baikal State University (6 (86)) (2012) 32–35, in Russian.

2. A. V. Panov, A. S. Prokushkin, A. V. Bryukhanov, M. A. Korets, E. I. Ponomarev, N. V. Sidenko, G. K. Zrazhevskaya, A. V. Timokhina, M. O. Andreae, A complex approach for the estimation of carbonaceous emissions from wildfires in Siberia, Russian Meteorology and Hydrology 43 (5) (2018) 295–301.
3. I. V. Tikhonova, N. V. Efimova, Prevalence of the chronic respiratory tract pathology in teenagers: role of some factors [Chastota khronicheskoy patologii verkhnikh dykhatel'nykh putey u podrostkov: rol' nekotorykh faktorov], Hygiene and Sanitation (6) (2012) 51–53, in Russian.
4. G. Dorrer, I. Buslov, S. Yarovoy, Conception of managing system for wild fire struggle [Kontseptsiya sistemy upravleniya bor'boy s prirodnymi pozharami], Siberian Fire and Rescue Bulletin (1 (1)) (2016) 38–44, in Russian.
5. A. V. Kolyada, Optimization of the process of extinguishing a forest fire using simulation [Optimizatsiya protsessa tusheniya lesnogo pozhara s ispol'zovaniyem imitatsionnogo modelirovaniya], Scientific Notes of the RSSU (8) (2010) 89–94, in Russian.
6. G. M. Parks, Development and application of a model for suppression of forest fires, Management Science 10 (4) (1964) 760–766. doi:10.1287/mnsc.10.4.760.
7. M. Parlar, R. G. Vicson, Optimal forest fire control: an extension of Park's model, Forest Science 28 (2) (1982) 345–355. doi:10.1093/forestscience/28.2.345.
8. M. Parlar, Optimal forest fire control with limited reinforcements, Optimal Control Applications Methods 4 (1983) 185–191. doi:10.1002/oca.4660040208.
9. J. Rodriguez-Veiga, M. J. Ginzo-Villamayor, B. Casas-Mendez, An integer linear programming model to select and temporally allocate resources for fighting forest fires, Forests 9 (583) (2018) 2–18. doi:10.3390/f9100583.
10. C. Artigues, E. Hébrard, Y. Pencolé, A. Schutt, P. J. Stuckey, A study of evacuation planning for wildfires, in: The Seventeenth International Workshop on Constraint Modelling and Reformulation (ModRef 2018), Lille, France, 2018.
URL `https://hal.archives-ouvertes.fr/hal-01814083`
11. J. Rodriguez-Veiga, I. Gomez-Costa, M. J. Ginzo-Villamayor, B. Casas-Mendez, J. L. Saiz-Diaz, Assignment problems in wildfire suppression: models for optimization of aerial resource logistics, Forest Science 64 (5) (2018) 504–514. doi:10.1093/forsci/fxy012.
12. E. V. Bogdanova, G. V. Davydova, Methodical approaches for forecasting forest fire in the Irkutsk region [Metodicheskiye podkhody k prognozirovaniyu lesnykh pozharov v Irkutskoy oblasti], in: S. V. Chuprova, N. N. Danilenko (Eds.), Revitalization of the Intellectual and Resource Potential of the Regions: New Challenges for the Management of Companies Materials of the 3rd All-Russian Conference, Irkutsk, 18 May 2017, BGU, Irkutsk, 2017, pp. 45–51, in Russian.
URL `http://vseup.ru/static/files/IRKUTSK_-_2017.pdf`
13. E. A. Pyanova, A. A. Faleychik, L. M. Faleychik, Forest fires in Transbaikalia: numerical simulation [Lesnyye pozhary v Zabaykal'ye:

chislennoye modelirovaniye], in: Kulagin readings: technology and technology of production processes XV International Scientific and Practical Conference: a collection of articles in 3 parts. Part 2, 2015, pp. 125–129, in Russian.
14. O. A. Belykh, G. D. Rusetskaya, Instruments effectiveness assessment to implement sustainable management principles of forest systems in Eastern Siberia [Otsenka effektivnosti instrumentov realizatsii printsipov ustoychivogo upravleniya lesnymi sistemami v vostochnoy Sibiri], Forestry Bulletin 23 (1) (2019) 5–13, in Russian.
15. J. B. Ditsevich, O. A. Belykh, G. D. Rusetskaya, Counteracting crimes in the sphere of forest resources' use: problems and perspectives [Protivodeystviye prestupnosti v sfere lesopol'zovaniya: problemy i perspektivy], Russian Journal of Criminology 11 (2) (2017) 308–317, in Russian. doi:10.17150/2500-4255.2017.11(2).308-317.
16. I. Y. Kulagina, Problems of reforestation in Russia [Problemy vosstanovleniya lesov v Rossii], in: The economic development of society in modern crisis conditions. Collection of articles of the International Scientific and Practical Conference. In 3 parts. Samara, May 13, 2017, 2017, pp. 128–131 [in Russian].
17. M. Sherrington, Learn the language of Julia, Packt Publishing, 2016.
18. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrilidze, E. F. Mishenko, Mathematical theory of optimal processes [Matematicheskaya teoriya optimal'nykh protsessov], Fizmatlit, Moscow, 1961, in Russian.
19. V. I. Gurman, Turnpike solutions in the procedures seeking optimal controls, Automation and Remote Control 64 (3) (2003) 399–408. doi:10.1023/A:1023209524049.

For citation:

A. P. Sukhodolov, P. G. Sorokina, A. P. Fedotov, Numerical analysis of ecology–economic model for forest fire fighting in Baikal region, Discrete and Continuous Models and Applied Computational Science 27 (2) (2019) 154–164. DOI: 10.22363/2658-4670-2019-27-2-154-164.

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