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On summation of Fourier series in finite form

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Abstract. The problem of summation of Fourier series in finite form is formulated in the weak sense, which allows one to consider this problem uniformly both for classically convergent and for divergent series. For series with polynomial Fourier coefficients $a_n, b_n \in \mathbb{R}[n]$, it is proved that the sum of a Fourier series can be represented as a linear combination of 1, $\delta(x)$, $\cot \frac{x}{2}$ and their derivatives. It is shown that this representation can be found in a finite number of steps. For series with rational Fourier coefficients $a_n, b_n \in \mathbb{R}(n)$, it is shown that the sum of such a series is always a solution of a linear differential equation with constant coefficients whose right-hand side is a linear combination of 1, $\delta(x)$, $\cot \frac{x}{2}$ and their derivatives. Thus, the issue of summing a Fourier series with rational coefficients is reduced to the classical problem of the theory of integration in elementary functions.

Key words and phrases: mathematical physics, Fourier series, elementary functions

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1. Introduction

The problem of summing a functional series in elementary functions naturally arises when solving problems in mathematical physics [1–6]. If desired, even d'Alembert's method of solving the wave equation can be considered as a method of summing a Fourier series [7, 8]. Frequently, results on summation in the final form arose as surprising side effects, for example, when accelerating the convergence of series by A.N. Krylov's method [9–12]. However, the authors of the past avoided considering divergent series, the summation of which, as it seemed then, could yield anything [13, p. 641], [14, Ch. 12, Sect. 4].

With the advent of the theory of generalized functions [15], a reliable basis for considering divergent functional series arose. The surprising fact is that divergent series are usually summed up in a finite form much more easily than convergent ones, and, moreover, the summation of convergent series in a finite form is conveniently reduced to the summation of divergent series. In this paper we illustrate this statement using the example of one-dimensional Fourier series. The possibility of interpreting Krylov's method in terms of generalized functions was mentioned in [16, p. 32].

2. Statement of the Problem

Definition 1. A periodic function is called piecewise elementary if its period can be divided into a finite number of segments, on each of which an elementary expression in the Liouville sense can be specified for it.

We understand the equality between the sum of a Fourier series and a piecewise elementary function in the weak sense [15], which allows a further uniform consideration of the series summation in elementary functions separate from the issue of its pointwise convergence.

Definition 2. The Fourier series

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{1}$$

is called a piecewise elementary function v in the strong sense if the equality

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = v \tag{2}$$

is satisfied at almost all points of the real axis.

Definition 3. The Fourier series (1) is said to be a piecewise elementary function v in the weak sense if the equality

$$\frac{a_0}{2} \int\limits_{x=-\pi}^{\pi} w dx + \sum_{n=1}^{\infty} a_n \int\limits_{x=-\pi}^{\pi} \cos nx w dx + b_n \int\limits_{x=-\pi}^{\pi} \sin nx w dx = \int\limits_{x=-\pi}^{\pi} v w dx \tag{3}$$

is true on any smooth function w with period 2π.

The Fourier coefficients of a smooth function w converge to zero faster than any power of n, so the numerical series on the left in Eq. (3) always converges. If this does not lead to misunderstandings, instead of Eq. (3) we will write Eq. (2), implying that it is satisfied in the weak sense.

Example 1. For example, by virtue of the classical Dirichlet theorem, the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges at all points of the interval $0 < x < 2\pi$ and its sum is equal to $\frac{\pi - x}{2}$. Therefore, this series is not only piecewise elementary, but also piecewise polynomial in the strong sense (Def. 2).

Example 2. Although the series

$$\sum_{n=1}^{\infty} \sin nx$$

diverges by Euler's test, the equality

$$\sum_{n=1}^{\infty} \int\limits_{x=-\pi}^{\pi} \sin nx \cdot w dx = \text{v.p.} \int\limits_{x=-\pi}^{\pi} \frac{1}{2} \cot \frac{x}{2} \cdot w dx \tag{4}$$

is true for all smooth w with period 2π. The integrand has a pole at zero, so we specify that the integral is understood in the sense of the Cauchy principal value. Therefore, this series, diverging in the classical sense, is also a piecewise elementary function in the weak sense (Def. 3). In this case, we consider the equality

$$\sum_{n=1}^{\infty} \sin nx = \frac{1}{2} \cot \frac{x}{2}$$

only as a short notation for Eq. (4).

Based on Definitions 1 and 3, we formulate the problem under consideration.

Problem 1. The coefficients of the Fourier series a_n, b_n are given as rational functions of number n:

$$a_n, b_n \in \mathbb{R}(n).$$

It is required to find out whether this series is a piecewise elementary function and, if the answer is affirmative, indicate this function.

3. Summation of series with polynomial coefficients

Problem 1 is solved very simply for the polynomial case when a_n and $b_n \in \mathbb{R}[n]$. However, this interesting case escaped the attention of authors of the 19th century, since in this case the general term of the Fourier series does not tend to zero, and therefore the series diverges. This difficulty is removed in Definition 3.

Indeed, let

$$a_n = \sum_{m=0}^{M} \alpha_m n^m, \quad b_n = \sum_{m=0}^{M} \beta_m n^m,$$

then the Fourier series under consideration can be rewritten as

$$u = \frac{a_0}{2} + \sum_{m=0}^{M} \alpha_m \sum_{n=1}^{\infty} n^m \cos nx + \sum_{m=0}^{M} \beta_m \sum_{n=1}^{\infty} n^m \sin nx.$$

The series that arise here are derivatives of the two main series

$$s(x) = \sum_{n=1}^{\infty} \sin nx$$

and

$$c(x) = \sum_{n=1}^{\infty} \cos nx.$$

For example,

$$\sum_{n=1}^{\infty} n^{2m} \cos nx = (-1)^m D^{2m} c(x).$$

We understand series in the sense of Definition 3, therefore

$$s(x) = \sum_{n=1}^{\infty} \sin nx = \frac{1}{2} \cot \frac{x}{2}$$

and

$$c(x) = \sum_{n=1}^{\infty} \cos nx = -\frac{1}{2} + \pi\delta(x)$$

for $-\pi < x < \pi$.

Theorem 1. *If $a_n, b_n \in \mathbb{R}[n]$, then the sum of the Fourier series* (1) *can be represented as a linear combination of 1, $\delta(x)$, $\cot\frac{x}{2}$ and their derivatives, this representation can be found in a finite number of steps.*

4. Summation of series with rational coefficients

Let us now turn to the solution of Problem 1 in the case when $a_n, b_n \in \mathbb{R}(n)$. Differentiation of $a_n \cos nx$ and $b_n \sin x$ reduces to multiplication by $\pm n$ and permutation of sine and cosine. Therefore, there always exists a linear differential operator L such that

$$L[u] = \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx, \quad A_n, B_n \in \mathbb{R}[n]. \tag{5}$$

We will say that the operator L annihilates the denominator of the Fourier coefficients of the original series, and A_n and B_n are the Fourier coefficients obtained after the annihilation. The divergent series in the right-hand side of Eq. (5) has polynomial coefficients and is summed as described in the previous Section.

Example 3. Consider the Fourier series

$$u = \sum_{n=1}^{\infty} \frac{\sin nx}{1+n^2}.$$

We have

$$(-D^2+1)u = \sum_{n=1}^{\infty} \sin nx = \frac{1}{2}\cot\frac{x}{2}.$$

By Theorem 1, Problem 1 is reduced to the following problem.

Problem 2. A linear differential operator L and a linear combination f of functions 1, $\delta(x)$, $\cot\frac{x}{2}$ and their derivatives are given. It is required to find out whether the equation

$$L[u] = f$$

has a solution in piecewise elementary functions.

Since the coefficients of the operator L are constant, the general solution of the equation $L[u] = f$ can be written in quadratures using the method of variation of constants. Quadratures containing the δ-function and its derivative are always taken.

Theorem 2. *If the given series converges and after annihilation the coefficient A_n is an even function of n, and the coefficient B_n is an odd function of n, then Problem 1 is solvable.*

Numerous examples illustrating this theorem were considered in classical studies of accelerating the summation of Fourier series [9, 10]. In the general case, the solution $L[u] = f$ will contain quadratures of the form

$$\int x^p e^{\lambda x} D^q \cot\frac{x}{2}\,dx.$$

The conditions found in Liouville theory [17] under which integrals of this type are taken in elementary functions provide sufficient conditions for the solvability of Problem 1. Thus, Problem 1 is reduced to the classical problem of computer algebra [18].

Example 4. Returning to Example 3, we see that u is a solution of the linear differential equation

$$u'' - u = -\frac{1}{2}\cot\frac{x}{2}$$

whose general solution is given by the quadrature

$$u = \frac{e^{-x}}{4}\int e^{x}\cot\frac{x}{2}dx - \frac{e^{x}}{4}\int e^{-x}\cot\frac{x}{2}dx.$$

Thus, the solution of Problem 1 for the series from Example 3 is reduced to the study of the elementariness of this expression.

5. Results

The sum of the Fourier series

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

whose coefficients are rational functions of n, is always a solution of some linear differential equation

$$L[u] = f,$$

whose right-hand side is the sum of a divergent Fourier series with polynomial coefficients. This series can always be represented as a linear combination of 1, $\delta(x)$, $\cot\frac{x}{2}$ and their derivatives, so the original series can be represented as a quadrature of piecewise elementary functions. The conditions under which these quadratures are taken in elementary functions provide sufficient conditions for the summation of the Fourier series in piecewise elementary functions.

6. Discussion

In this paper, we propose a simple approach to summation of a certain class of trigonometric series. Its distinctive feature is the term-by-term differentiation of Fourier series, which inevitably leads to the appearance of divergent series. We believe that working with them can serve as the basis for symbolic algorithms for summation of eigenfunction series, and significantly supplement the generally accepted methods for summation, see [19, 20]. From the point of view of computer algebra, the approach under consideration allows us to establish a connection between the problem of summation of a certain class of series and the classical problem of integration in elementary functions. This is achieved by adding the Dirac δ-function and other distributions to the set of elementary functions.

7. Conclusions

The transition from convergent to divergent series using an annihilation operator allows us to divide the problem of summing a convergent Fourier series into two simpler ones: summing a series with polynomial coefficients, which is solved explicitly, and integrating LDEs with constant coefficients. The development of an algorithm for constructing an annihilation operator for a given Fourier series with rational coefficients and its implementation in computer algebra systems will allow a wide class of Fourier series to be summed in a finite form.

Thus, the transition to summation of Fourier series in the weak sense allows reducing the problem of summation of series in a finite form (Problem 1) to calculating integrals of elementary functions in this form, i.e., a classic problem of computer algebra.

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О суммировании рядов Фурье в конечном виде

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Аннотация. Задача о суммировании рядов Фурье в конечном виде сформулирована в слабом смысле, что позволяет единообразно рассматривать эту задачу как для сходящихся в классическом смысле рядов, так и для расходящихся. Для рядов с полиномиальными коэффициентами Фурье $a_n, b_n \in \mathbb{R}[n]$ доказано, что сумма ряда Фурье может быть представлена как линейная комбинация 1, $\delta(x)$, $\cot\frac{x}{2}$ и их производных. Показано, что это представление может быть найдено за конечное число действий. Для рядов с рациональными коэффициентами Фурье $a_n, b_n \in \mathbb{R}(n)$ показано, что сумма такого ряда всегда является решением линейного дифференциального уравнения с постоянными коэффициентами, правая часть которого является линейной комбинацией 1, $\delta(x)$, $\cot\frac{x}{2}$ и их производных. Тем самым вопрос о суммировании рядов Фурье с рациональными коэффициентами сведен к классическому вопросу теории интегрирования в элементарных функциях.

Ключевые слова: математическая физика, ряды Фурье, элементарные функции