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Well-posedness of the microwave heating problem

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Abstract. A number of initial boundary-value problems of classical mathematical physics is generally represented in the linear operator equation and its well-posedness and causality in a Hilbert space setting was established. If a problem has a unique solution and the solution continuously depends on given data, then the problem is called well-posed. The independence of the future behavior of a solution until a certain time indicates the causality of the solution. In this article, we established the well-posedness and causality of the solution of the evolutionary problems with a perturbation, which is defined by a quadratic form. As an example, we considered the coupled system of the heat and Maxwell's equations (the microwave heating problem).

Key words and phrases: evolutionary problems, nonlinear perturbation, Lipschitz continuous, quadratic form, coupled problems

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1. Introduction

Here we consider a non-linear, coupled system in thermoelectricity. Thermoelectric effects are viewed as the result of the mutual interference of heat flow and electric flow in a system. The interaction of thermal and electric processes is modeled by the heat equation

$$\rho C_\rho \partial_0 \vartheta + \operatorname{div} q = Q$$

and Maxwell's equations

$$\begin{aligned} -\operatorname{curl} H + J + \partial_0 D &= J_1 \\ \operatorname{curl} E + \partial_0 B &= 0. \end{aligned}$$

Here q is the thermal current flux, ρ is the volumetric mass density, C_ρ is the specific heat density, ϑ is the absolute temperature, J is the electric current flux, E, H are the electric and magnetic fields, respectively, D is the displacement current, B is the magnetic induction and J_1 is the given electric source. Q describes the production of internal energy by various mechanisms, such as the Joule heating, radioactive decay, etc. In our system the Joule heating $Q = \langle E \mid J \rangle$ produces the internal energy. This term governs the non-linearity in the system and, moreover, it couples the heat and Maxwell's equations. The system of these equations has to be supplemented by so-called constitutive equations, which describe the material's properties and effects. As constitutive equations, we deal with the following thermoelectric material relations

$$J = \sigma E$$

$$
\begin{aligned}
q &= -\lambda \operatorname{grad}\vartheta \\
D &= \varepsilon E \\
B &= \mu H.
\end{aligned}
$$

Here σ is the electric conductivity, λ is the thermal conductivity, ε is the electric permittivity, μ is the magnetic permeability. The coupled system of the heat and Maxwell's equations with these constitutive equations becomes the microwave heating problem. The microwave heating problem has wide industrial applications and it has been studied theoretically and numerically in various situations (see e.g. [1–3] and the references therein). We study the coupled systems in the three-dimensional case. Moreover, we consider this system with the physical coefficients defined as 3-by-3—matrix-valued functions depending on the spatial variables only. We assume (homogeneous) Dirichlet boundary conditions for ϑ, (homogeneous) electric boundary conditions for E and non-vanishing initial values. We say that a problem is *well-posed* if the problem has a unique solution and the solution continuously depends on the given data. The independence of the future behavior of a solution until a certain time indicates the *causality* of the solution. In our solution theory the well-posedness and causality of a given problem are discussed.

The idea of tackling well-posedness and causality of the problem just discussed is to frame the above system in the theory of evolutionary equations: In [4, 5] it has been found that a number of initial boundary-value problems of classical mathematical physics is represented by the following general form

$$\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right) u = F. \tag{3}$$

Here ∂_0 is the (continuously invertible) derivative with respect to time in a suitable weighted Hilbert space, A is a skew-selfadjoint operator in a suitable Hilbert space; the mapping $(z \mapsto M(z))$ is bounded operator valued and holomorphic in an open ball $B_{\mathbb{C}}(r, r)$ with some positive radius r centered at r. The operator $M\left(\partial_0^{-1}\right)$ is interpreted in the sense of a function calculus by establishing ∂_0 as a normal operator in a suitable Hilbert space. The solution theory associated to (3) was established in [4, 5] and many diverse problems were studied there. For applications, we focus on a particular case of $M\left(\partial_0^{-1}\right)$, namely,

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1.$$

Here M_0 is a selfadjoint, bounded, linear operator with $M_0|_{N(M_0)} \geqslant c_0 > 0$, M_1 is a bounded, linear operator satisfying $\Re M_1|_{R(M_0)} \geqslant c_1 > 0$. In the next section we establish the solution theory of the following problem

$$\left(\partial_0 M_0 + M_1 + A\right) u + \tilde{F}(u) = F, \tag{4}$$

which covers the aforementioned non-linear coupled system. Here $\tilde{F}$ is a quadratic form. The non-linear problem (1) yields a fixed point problem. In our approach the well-posedness of (4) is based on the strict positive definiteness of the operators $\Re\left(\partial_0 M_0 + M_1 + A\right)$ and $\Re\left(\partial_0 M_0 + M_1 + A\right)^*$ and a Lipschitz continuous approximation of $\tilde{F}$. Due to the strict positive definiteness result, the inverse operator $\left(\partial_0 M_0 + M_1 + A\right)^{-1}$ becomes Lipschitz continuous in a suitable Hilbert space. Thus, (4) amounts to be an evolutionary problem in the sense of (3) with a Lipschitz continuous perturbation, which is eventually solved by the contraction mapping principle. As an application we shall consider the microwave heating problem in the third section.

2. Solution theory

We start by establishing time differentiation ∂_0 as a normal operator. It is initially considered on $\overset{\circ}{C}_\infty(\mathbb{R})$, which is the set of infinitely often differentiable, complex-valued functions defined on the real line $\mathbb{R}$ having compact support. Hence ∂_0 is a densely defined, closed linear operator on $L^2(\mathbb{R})$, moreover, it is an essentially skew-selfadjoint operator on $L^2(\mathbb{R})$. We define the following weighted L^2-space

$$H_{\nu,0}(\mathbb{R}) := L^2(\mathbb{R}, \exp(-2\nu x)\, dx) := \left\{\varphi \in L^1_{loc}(\mathbb{R}) \mid \exp(-\nu m_0)\,\varphi \in L^2(\mathbb{R})\right\}$$

equipped with the norm

$$|\varphi|_{\nu,0} := \sqrt{\int_{\mathbb{R}} |\varphi(x)|^2 \exp(-2\nu x)\, dx},\ \varphi \in H_{\nu,0}(\mathbb{R}).$$

Here m_0 is the closure of the following operator

$$\begin{array}{rcl} \overset{\circ}{C}_\infty(\mathbb{R}) \subseteq L^2(\mathbb{R}) & \to & L^2(\mathbb{R}) \\ \varphi & \mapsto & (x \mapsto x\varphi(x)). \end{array}$$

The operator m_0 is called the multiplication by argument operator and it is densely defined, Hermitian and moreover, it is self-adjoint. Let $\nu \in \mathbb{R}$. We also define the operator $\exp(-\nu m_0)$ such that $\exp(-\nu m_0)\,\varphi := (x \mapsto \exp(-\nu x)\,\varphi(x))$ for $\varphi \in L^1_{loc}(\mathbb{R})$. Note that $\exp(-\nu m_0)\left[\overset{\circ}{C}_\infty(\mathbb{R})\right] = \overset{\circ}{C}_\infty(\mathbb{R})$. Due to the density of $\overset{\circ}{C}_\infty(\mathbb{R})$ in both the spaces $L^2(\mathbb{R})$ and $H_{\nu,0}(\mathbb{R})$, $\exp(-\nu m_0)$ can be extended to a unitary operator from $L^2(\mathbb{R})$ onto $H_{\nu,0}(\mathbb{R})$ and the unitary extension is denoted again by $\exp(-\nu m_0)$. The inverse of $\exp(-\nu m_0)$ is

$$\exp(\nu m_0) : L^2(\mathbb{R}) \to H_{\nu,0}(\mathbb{R}).$$

Note that we may utilize the notation $H_{0,0}(\mathbb{R})$ for the space $L^2(\mathbb{R})$ with the inner product $\langle\cdot \mid \cdot\rangle_{0,0}$ and the norm $|\cdot|_{0,0}$. The following operator

$$\partial_\nu := \exp(\nu m_0)\,\partial_0 \exp(-\nu m_0)$$

is unitarily equivalent to ∂_0 on $H_{\nu,0}(\mathbb{R})$. The operator $\partial_\nu + \nu$ is the time derivative on $H_{\nu,0}(\mathbb{R})$ and we denote it again by ∂_0. Moreover, for all $\nu \in \mathbb{R}_{>0}$, $\partial_0 : D(\partial_0) \subseteq H_{\nu,0}(\mathbb{R}) \to H_{\nu,0}(\mathbb{R})$ is continuously invertible on $H_{\nu,0}(\mathbb{R})$, that is

$$\left\|\partial_0^{-1}\right\|_{L(H_{\nu,0}(\mathbb{R}))} \leq \frac{1}{\nu}$$

and a normal operator for all $\nu \in \mathbb{R}\setminus\{0\}$ on $H_{\nu,0}(\mathbb{R})$. Furthermore, $\partial_0^{-1} : H_{\nu,0}(\mathbb{R}) \to H_{\nu,0}(\mathbb{R})$ is a normal operator (see e.g. [6, Theorem 5.42]) and there is the Sobolev chain

$$H_{\nu,k+1}(\partial_0) \hookrightarrow H_{\nu,k}(\partial_0),\ k \in \mathbb{N}$$

with respect to ∂_0, where $H_{\nu,k}(\partial_0) := D\left(\partial_0^k\right)$ is the Hilbert space with the norm

$$|\cdot|_{\nu,k} = \left|\partial_0^k \cdot\right|_{\nu,0}$$

for each $k \in \mathbb{N}$. Furthermore, we have

$$H_{\nu,-k}(\partial_0) \hookrightarrow H_{\nu,-k-1}(\partial_0),\ k \in \mathbb{N},$$

where $H_{\nu,-k}(\partial_0)$ are completions of $H_{\nu,0}(\mathbb{R})$ for all $k \in \mathbb{N}$ with the norms $|\cdot|_{\nu,-k} := |\partial_0^{-k}\cdot|_{\nu,0}$. Note that we can unitarily extend the following operator

$$\begin{array}{rcl} H_{\nu,0}(\mathbb{R}) \subseteq H_{\nu,-1}(\partial_0) & \to & H_{\nu,0}(\mathbb{R}) \\ \varphi & \mapsto & \partial_0^{-1}\varphi. \end{array}$$

We denote its extension again by ∂_0^{-1}. This motivates the unitary extension of ∂_0 from $H_{\nu,0}(\mathbb{R})$ onto $H_{\nu,-1}(\partial_0)$ for each $\nu \in \mathbb{R}\backslash\{0\}$ and we denote the extension again by ∂_0. In the same manner we obtain unitary operators

$$\begin{array}{rcl} H_{\nu,k+1}(\partial_0) & \to & H_{\nu,k}(\partial_0) \\ \varphi & \mapsto & \partial_0\varphi \end{array}$$

for $k \in \mathbb{Z}$, as appropriate unitary extension/restriction of the originally discussed operator ∂_0 defined on $H_{\nu,0}(\mathbb{R})$.

2.1. On skew-selfadjoint operator

Let H_1 and H_2 be Hilbert spaces. For a densely defined, closed linear operator $C : D(C) \subseteq H_1 \to H_2$ and a block operator matrix B defined as follows

$$B = \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix} \tag{5}$$

is skew-selfadjoint and so is the following diagonal operator matrix

$$A = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_n \end{pmatrix},$$

where each B_i, $i = \overline{1.n}$ is defined as in (5).

In a coupled system of the heat and Maxwell's equations with Dirichlet boundary condition and electric boundary condition A has the following form

$$\begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \overset{\circ}{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{curl} \\ 0 & 0 & \overset{\circ}{\operatorname{curl}} & 0 \end{pmatrix}$$

where div, $\overset{\circ}{\operatorname{grad}}$, curl and $\overset{\circ}{\operatorname{curl}}$ are defined as follows. Let $\Omega \subseteq \mathbb{R}^3$ be an open set. Consider the following vector analytical differential operators

$$\operatorname{grad}_c : \overset{\circ}{C}_\infty(\Omega) \subseteq L^2(\Omega) \to \bigoplus_{k=1}^{n} L^2(\Omega)$$

$$\phi \quad \mapsto \quad (\partial_k \phi)_{k\in\{1,\ldots,n\}}$$

and

$$\begin{aligned} \operatorname{div}_c \; : \; \bigoplus_{k=1}^{n} \overset{\circ}{C}_\infty (\Omega) \subseteq \bigoplus_{k=1}^{n} L^2 (\Omega) \quad &\to \quad L^2 (\Omega) \\ (\varphi_k)_{k\in\{1,\ldots,n\}} \quad &\mapsto \quad \sum_{k=1}^{n} \partial_k \varphi_k . \end{aligned}$$

The operators grad_c and $-\operatorname{div}_c$ are formally adjoint to each other and closable. Denoting

$$\overset{\circ}{\operatorname{grad}} := \overline{\operatorname{grad}_c}, \; \overset{\circ}{\operatorname{div}} := \overline{\operatorname{div}_c}$$

and

$$\operatorname{grad} := \left(-\overline{\operatorname{div}_c}\right)^*, \; \operatorname{div} := \left(-\overline{\operatorname{grad}_c}\right)^*,$$

we can construct the following skew-selfadjoint operator

$$A_1^D := \begin{pmatrix} 0 & \operatorname{div} \\ \overset{\circ}{\operatorname{grad}} & 0 \end{pmatrix},$$

where $\overset{\circ}{\operatorname{grad}}$, div and grad, $\overset{\circ}{\operatorname{div}}$ are all together densely defined, closed linear operators. The operator A_1^D is not only skew-selfadjoint but also encode Dirichlet boundary condition, that is, φ being in $D\left(\overset{\circ}{\operatorname{grad}}\right)$ means that φ satisfies a generalized homogeneous Dirichlet boundary condition.

Due to the skew-selfadjointness of A, we have a long Sobolev chain with respect to $A + 1$. Since $\pm 1 \in \rho (A)$, the domains of the operators A and $A + 1$ coincide. There is the Sobolev chain

$$H_{k+1} (A + 1) \hookrightarrow H_k (A + 1) \text{ for } k \in \mathbb{Z}.$$

Here $H =: H_0 (A + 1)$, $H_k (A + 1)$ is the domain of $(A + 1)^k$ and it is a Hilbert space with the norm $|\cdot|_k := \left|(A + 1)^k \cdot\right|_{0,0}$ for each $k \in \mathbb{N}$ and $H_{-k} (A + 1)$ is the completion of H for each $k \in \mathbb{N}$ under the norm $|\cdot|_{-k} := \left|(A + 1)^{-k} \cdot\right|_{0,0}$. For the sake of brevity, we also denote $H_{k,A} := H_k (A + 1)$. Now we are in the position to construct the Sobolev lattices

$$\left(H_{\nu,k} \otimes H_{n,A}\right)_{k,n\in\mathbb{Z}}$$

for the chains $\left(H_{\nu,k} (\partial_0)\right)_{k\in\mathbb{Z}}$ and $(H_n (A + 1))_{n\in\mathbb{Z}}$ with respect to the operators $\partial_0 \otimes I_H$ and $I_{H_{\nu,0}} \otimes A$. Here $I_H : H \to H$ and $I_{H_{\nu,0}} : H_{\nu,0} (\mathbb{R}) \to H_{\nu,0} (\mathbb{R})$ are the identity operators. Note that $H_{\nu,k} \otimes H$ can be interpreted as the completion of the linear space generated by $H-$valued functions of the special form

$$t \mapsto \psi (t) w =: (\psi \otimes w) (t)$$

for each $k \in \mathbb{N}$, where $\psi \in \overset{\circ}{C}_\infty (\mathbb{R})$, $w \in H$. In fact, $H_{\nu,k} \otimes H$ is unitarily equivalent to $H_{\nu,k} (\mathbb{R}, H)$ for each $k \in \mathbb{N}$.

The operators $\partial_0 \otimes I_H$ and $I_{H_{\nu,0}} \otimes A$ are well-defined and have essentially the same properties as the operators ∂_0 and A, respectively. Therefore, we also write A and ∂_0 for their canonical extensions $A \otimes I_H$ and $I_{H_{\nu,0}} \otimes \partial_0$ in $H_{\nu,0} (\mathbb{R}, H)$.

2.2. The material law operator

The Fourier-Laplace transform

$$\mathcal{L}_\nu := \mathcal{F}\exp(-\nu m_0) : H_{\nu,0}(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$$

given as a composition of the (temporal) Fourier transform $\mathcal{F}$ and the unitary weight operator $\exp(-\nu m_0)$, is a spectral representation associated with ∂_0. It is

$$\partial_0 = \mathcal{L}_\nu^*(\mathrm{i}m_0 + \nu)\mathcal{L}_\nu.$$

This observation allows us to consistently define an operator function calculus associated with ∂_0 in a standard way and we can even extend this calculus to operator-valued functions by letting

$$M(\partial_0^{-1}) := \mathcal{L}_\nu^* M\left(\frac{1}{\mathrm{i}m_0 + \nu}\right)\mathcal{L}_\nu.$$

Here the linear operator $M\left(\frac{1}{\mathrm{i}m_0+\nu}\right) : L^2(\mathbb{R}, H) \to L^2(\mathbb{R}, H)$ is determined uniquely via

$$\left(M\left(\frac{1}{\mathrm{i}m_0 + \nu}\right)\varphi\right)(\lambda) := M\left(\frac{1}{\mathrm{i}m_0 + \nu}\right)\varphi(\lambda) \text{ in } H$$

for every $\lambda \in \mathbb{R}$, $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$ by an operator-valued function M. For a material law the operator-valued function M needs to be bounded and an analytic function $z \mapsto M(z)$ in an open ball $B_\mathbb{C}(r, r)$ with some positive radius r centered at r. Here we will concentrate on the following particular form of the material law

$$M(\partial_0^{-1}) = M_0 + \partial_0^{-1}M_1,$$

where M_0 is selfadjoint, bounded linear and $M_0 \geqslant c_0 > 0$ on the range $M_0[H]$, the null space $[\{0\}]M_0$ is non-trivial and $M_1 \in L(H)$ with $\Re M_1 \geqslant c_1 > 0$ on $[\{0\}]M_0$.

This is not an artificial assumption, rather a necessary constraint enforced by the requirement of causality and strictly positive definite condition

$$\Re\left\langle u \mid \left(\partial_0 M(\partial_0^{-1})\right)u\right\rangle_{H_{\nu,0}(\mathbb{R},H)} \geqslant c\langle u \mid u\rangle_{H_{\nu,0}(\mathbb{R},H)}$$

for $c \in \mathbb{R}_{>0}$ and all sufficiently large $\nu \in \mathbb{R}_{>0}$ and all $u \in D(\partial_0)$. The strict positive definite condition implies

$$\Re\langle u \mid (\partial_0 M_0 + M_1 + A)u\rangle_{H_{\nu,0}(\mathbb{R},H)} \geqslant c\langle u \mid u\rangle_{H_{\nu,0}(\mathbb{R},H)}$$

for $c \in \mathbb{R}_{>0}$ and all sufficiently large $\nu \in \mathbb{R}_{>0}$ and all $u \in D(\partial_0)$. Moreover,

$$\partial_0 M_0 + M_1 + A$$

has dense range in $H_{\nu,0}(\mathbb{R}, H)$. For all sufficiently large $\nu \in \mathbb{R}_{>0}$, we have

$$\left\|\left(\overline{\partial_0 M_0 + M_1 + A}\right)^{-1}\right\|_{L(H_{\nu,0}(\mathbb{R},H))} \leq \frac{1}{c_\nu},\ 0 < c_\nu < c_1$$

and this also implies the solution theory of the following evolutionary problems

$$(\partial_0 M_0 + M_1 + A)u = S(u) + f,$$

where S is a suitable Lipschitz mapping.

2.3. Well-posedness of evolutionary problems with a non-linear perturbation term

After making some reformulations in the microwave heating problem, the problem gets the following shape

$$(\partial_0 M_0 + M_1 + A)\, u + \tilde{F}(u) = F,$$

where $\tilde{F}$ is a quadratic form and it is not Lipschitz continuous. For a Lipschitz continuous approximation of the quadratic form we recall the following Lemma and Theorem in [7].

Lemma 1. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be differentiable, and such that $\left(z \mapsto \left|\sqrt{z} f'(z)\right|\right)$ is bounded. Let $\mathcal{E} \in \mathbb{C}^{n\times n}$ be selfadjoint with $\mathcal{E} \geq 0$. Then there exists $C > 0$ such that*

$$\left| f\left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right) - f\left(\langle v \mid \mathcal{E}v\rangle_{\mathbb{C}^n}\right)\right|_{\mathbb{R}} \leqslant C\, |u - v|_{\mathbb{C}^n}$$

for all $u, v \in \mathbb{C}^n$.

Theorem 4. *Let (Ω, μ) be a σ–finite measure space. Let $\mathcal{E} \in (L^\infty(\Omega))^{n\times n}$ and*

$$\begin{aligned} \tilde{F} : D(\tilde{F}) &\to H_{\nu,0}(\mathbb{R}) \otimes L^2(\Omega) \\ u &\mapsto \left(\mathbb{R} \times \Omega \ni (t,\omega) \mapsto \langle u(t,\omega) \mid \mathcal{E}(\omega)\, u(t,\omega)\rangle_{\mathbb{C}^n}\right) \end{aligned}$$

with maximal domain. Here $D(\tilde{F}) \subseteq H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^n$. We assume that $\mathcal{E}(\omega) \in \mathbb{C}^{n\times n}$ is selfadjoint and positive for a.e. $\omega \in \Omega$. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be differentiable with $f(0) = 0$ and such that

$$z \mapsto \left|\sqrt{z} f'(z)\right|$$

is bounded. Define

$$\begin{aligned} F_f : D(F_f) \subseteq H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^n &\to H_{\nu,0}(\mathbb{R}) \otimes L^2(\Omega) \\ u &\mapsto \left(\mathbb{R} \times \Omega \ni (t,\omega) \mapsto f\left(\tilde{F}(u)(t,\omega)\right)\right) \end{aligned}$$

with maximal domain. Then $D(F_f) = H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^n$ and F_f is Lipschitz continuous.

Hence, it suffices to find a specific function f which satisfies all the assumptions in Lemma 1 and approximates the quadratic form.

Example 1. We consider the following function

$$\begin{aligned} f_\xi : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \frac{2}{\xi}\left(\sqrt{1 + \xi x} - 1\right) \end{aligned}$$

for $\xi \in \mathbb{R}_{>0}$. The function is infinitely differentiable for all $x > -\frac{1}{\xi}$ and as $\xi \to 0+$, it is approximated by the argument, that is,

$$f_\xi(x) \approx x.$$

The mapping $\left(z \mapsto \left|\sqrt{z} f'_\xi(z)\right|\right)$ is uniformly bounded. Indeed,

$$\left(z \mapsto f'_\xi(z)\sqrt{z}\right) = \left(z \mapsto \frac{\sqrt{z}}{\sqrt{1 + \xi z}}\right)$$

$$\leqslant \frac{1}{\sqrt{\xi}}.$$

Let $\mathcal{E}(\omega) \in \mathbb{C}^{n \times n}$ be selfadjoint and positive for a.e. $\omega \in \Omega$. By Lemma 1, the following holds

$$\left| f_\xi\left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right) - f_\xi\left(\langle v \mid \mathcal{E}v\rangle_{\mathbb{C}^n}\right)\right| \leqslant C\,|u - v|_{\mathbb{C}^n}$$

for $C \in \mathbb{R}_{>0}$. Since $f_\xi(0) = 0$ for all $\xi \in \mathbb{R}_{>0}$, the mapping defined by

$$\begin{aligned} F_{f_\xi} : H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^n &\to H_{\nu,0}(\mathbb{R}) \otimes L^2(\Omega) \\ u &\mapsto \left((t,\omega) \mapsto f_\xi\left(\langle u(t,\omega) \mid \mathcal{E}(\omega)\, u(t,\omega)\rangle_{\mathbb{C}^n}\right)\right) \end{aligned}$$

is Lipschitz continuous for all $\xi \in \mathbb{R}_{>0}$ and $(t,\omega) \in \mathbb{R} \times \Omega$ (Theorem 4) and the Lipschitz constant of the mapping F_{f_ξ} is $\frac{2}{\sqrt{\xi}}\left\|\sqrt{\mathcal{E}}\right\|_\infty$. Furthermore, the following holds

$$F_{f_\xi}(u) = f_\xi\left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right) \approx \langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}$$

for sufficiently small $\xi \in \mathbb{R}_{>0}$. We have obtained the Lipschitz continuous mapping $F_{f_\xi} = \left(u \mapsto f_\xi\left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right)\right)$ which approximates $\tilde{F} = \left(u \mapsto \left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right)\right)$ as $\xi \to 0+$.

Hence, the solution theory of the perturbed problem

$$(\partial_0 M_0 + M_1 + A)\, u + F_{f_\xi}(u) = F$$

provides an approximate solution of (4).

Theorem 5. *Let (Ω, μ) be a σ–finite measure space. Let $H = L^2(\Omega)^n$. Let $M_0 \in L(H)$ be selfadjoint, positive definite and $M_0|_{M_0[H]} \geqslant c_0 > 0$, and $M_1 \in L(H)$ with $\Re M_1|_{[\{0\}]M_0} \geqslant c_1 > 0$. Let $A : H_{1,A} \subseteq H \to H$ be a skew-selfadjoint operator. Assume that $\mathcal{E} \in (L^\infty(\Omega))^{n \times n}$ is selfadjoint and positive definite, and $0 \leq \frac{2}{c_\nu\sqrt{\xi}}\left\|\sqrt{\mathcal{E}}\right\|_\infty < 1$ for some $\xi \in \mathbb{R}_{>0}$. Let $u_0 \in D(A)$ and $F \in \chi_{\mathbb{R}_{\geq 0}}(m_0)\left[H_{\nu,0} \otimes H\right]$ be given data. Then there exists a unique solution $u \in H_{\nu,0} \otimes H$ of*

$$\left(\overline{(\partial_0 M_0 + M_1 + A)\,|_{D(\partial_0) \cap D(A)}}\right) u = F_{f_\xi}(u) + F + \delta \otimes M_0 u_0$$

for all $\nu \geq \nu_0$ for some $\nu_0 \in \mathbb{R}_{>0}$. The solution depends continuously and causally on the data. Moreover, the initial condition

$$(M_0 u)(0+) = M_0 u_0$$

is attained in $H_{-1,A}$.

3. The microwave heating problem

The microwave heating problem has wide industrial applications and it has been studied theoretically and numerically in various situations (see e.g. [1–3] and the references therein). In the study of the microwave heating problem the electric conductivity and/or thermal conductivity are considered as an operator, which may depend on the temperature (see e.g. [3]). But we will study the microwave heating problem with temperature independent thermal conductivity, electric conductivity, magnetic permeability and electric permittivity. These material coefficients are defined as 3×3 matrix-valued functions depending only on the spatial variables. This may describe material properties more substantially. The equations are introduced in the introduction. The set of originally given equations turns into the following equations

$$\rho C_\rho \partial_0 \vartheta + \mathrm{div} q = \langle E \mid \sigma E\rangle_{\mathbb{C}^3}$$

$$\begin{aligned}&\operatorname{grad}\vartheta + \lambda^{-1} q = 0\\ &-\operatorname{curl} H + \sigma E + \partial_0 \varepsilon E = J_1\\ &\operatorname{curl} E + \partial_0 \mu H = 0.\end{aligned}$$

A formal reformulation of these equations yields

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \operatorname{grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} \\ 0 & 0 & \operatorname{curl} & 0 \end{pmatrix}\right)\begin{pmatrix} \vartheta \\ q \\ E \\ H \end{pmatrix} = F + \begin{pmatrix} \langle E \mid \sigma E\rangle_{\mathbb{C}^3} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{7}$$

where

$$M_0 = \begin{pmatrix} \rho C_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \; M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\langle E \mid \sigma E\rangle_{\mathbb{C}^3} = \langle u \mid \mathcal{E} u\rangle_{\mathbb{C}^{10}}$ with

$$\mathcal{E} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now we reformulate (7) to the proper evolutionary problem. Assume that ϑ satisfies the (generalized) Dirichlet boundary condition and E satisfies the (generalized) homogeneous electric boundary condition. Then we have the following skew-selfadjoint operator

$$A := \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} \\ 0 & 0 & \mathring{\operatorname{curl}} & 0 \end{pmatrix}$$

in $H_{1,A} := H\left(\mathring{\operatorname{grad}}, \Omega\right) \oplus H(\operatorname{div}, \Omega) \oplus H\left(\mathring{\operatorname{curl}}, \Omega\right) \oplus H(\operatorname{curl}, \Omega)$, where $\Omega \subseteq \mathbb{R}^3$ is an open set. As in the preceding application, we assume that $\rho C_\rho : L^2(\Omega) \to L^2(\Omega)$, $\varepsilon : \left(L^2(\Omega)\right)^3 \to \left(L^2(\Omega)\right)^3$ and $\mu : \left(L^2(\Omega)\right)^3 \to \left(L^2(\Omega)\right)^3$ are selfadjoint, bounded linear and strictly positive definite operators. Hence, M_0 is selfadjoint, bounded linear and strictly positive definite in $M_0\left[\left(L^2(\Omega)\right)^{10}\right]$. Let λ^{-1} be in $L\left(\left(L^2(\Omega)\right)^3\right)$ and $\Re\lambda^{-1}$ strictly positive definite. Furthermore, assume that σ is selfadjoint, positive definite and it is in $(L^\infty(\Omega))^{3\times 3}$. Then, M_1 is in $L\left(\left(L^2(\Omega)\right)^{10}\right)$ and $\Re M_1$ is strictly positive definite on $[\{0\}]\, M_0 = \{0\} \oplus \left(L^2(\Omega)\right)^3 \oplus \{0\}^6$. Since $\sigma \in (L^\infty(\Omega))^{3\times 3}$ is selfadjoint and positive definite, so is $\mathcal{E} \in (L^\infty(\Omega))^{10\times 10}$. Moreover, there exists a selfadjoint, positive definite operator $\sqrt{\mathcal{E}}$. Hence, the quadratic form $\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^{10}}$ is approximated by the Lipschitz continuous mapping $F_{f_\xi} = \left(u \mapsto f_\xi\left(\langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^{10}}\right)\right)$ as $\xi \to 0+$, where f_ξ is defined in Example 1. Let $\vartheta_0 \in H\left(\mathring{\text{grad}}, \Omega\right)$, $E_0 \in H\left(\mathring{\text{curl}}, \Omega\right)$, $H_0 \in H(\text{curl}, \Omega)$ and $V_0 := \left(\rho C_\rho \vartheta_0, 0, \varepsilon E_0, \mu H_0\right)$. Then $u_0 := (\vartheta_0, 0, E_0, H_0) \in D(A)$, $V_0 = M_0 u_0$ and $V_0 \in M_0\left[D(A)\right]$. The initial boundary-value problem with respect to the microwave heating problem is presented in our framework as follows

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & \mathring{\text{div}} & 0 & 0 \\ \text{grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{curl} \\ 0 & 0 & \mathring{\text{curl}} & 0 \end{pmatrix}\right)\begin{pmatrix} \vartheta \\ q \\ E \\ H \end{pmatrix} = F + \begin{pmatrix} \langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^{10}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \otimes \begin{pmatrix} \rho C_\rho \vartheta_0 \\ 0 \\ \varepsilon E_0 \\ \mu H_0 \end{pmatrix}. \tag{8}$$

This problem yields the following initial value evolutionary problem with the Lipschitz continuous perturbation

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & \text{div} & 0 & 0 \\ \mathring{\text{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{curl} \\ 0 & 0 & \mathring{\text{curl}} & 0 \end{pmatrix}\right)\begin{pmatrix} \vartheta \\ q \\ E \\ H \end{pmatrix} = F + \begin{pmatrix} F_{f_\xi}(u) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \delta \otimes \begin{pmatrix} \rho C_\rho \vartheta_0 \\ 0 \\ \varepsilon E_0 \\ \mu H_0 \end{pmatrix} \tag{9}$$

for sufficiently small $\xi \in \mathbb{R}_{>0}$. In the next theorem we sum up the solution theory of (9), which concerns the approximation solution of (8).

Theorem 6. *Let $\lambda^{-1} \in L\left(\left(L^2(\Omega)\right)^3\right)$, and $\Re\lambda^{-1}$ be strictly positive definite. Let $\varepsilon, \mu \in L\left(\left(L^2(\Omega)\right)^3\right)$ be selfadjoint and strictly positive definite operators. Let $\rho C_\rho \in L\left(L^2(\Omega)\right)$ be selfadjoint and strictly positive definite. Let $\sigma \in (L^\infty(\Omega))^{3\times 3}$ be selfadjoint, positive definite. Let $(\vartheta_0, 0, E_0, H_0) \in D(A)$ and $F \in \chi_{\mathbb{R}_{\geq 0}}(m_0)\left[H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^{10}\right]$ be given data. Let $\nu_0 \in \mathbb{R}_{>0}$. Furthermore, assume that $0 \leq \frac{2}{c_\nu\sqrt{\xi}}\left\|\sqrt{\mathcal{E}}\right\|_\infty < 1$ for all $\nu \geq \nu_0$ and for some parameter $\xi \in \mathbb{R}_{>0}$ and some $c_\nu > 0$. Then there exists a unique solution $u \in H_{\nu,0}(\mathbb{R}) \otimes \left(L^2(\Omega)\right)^{10}$ of (8) for all $\nu \geq \nu_0$. The solution depends continuously and causally on the given data.*

4. Conclusions

We have obtained a Lipschitz continuous function approximating the quadratic form

$$\left(u \mapsto \langle u \mid \mathcal{E}u\rangle_{\mathbb{C}^n}\right)$$

for a selfadjoint, positive definite operator $\mathcal{E}$ in $(L^\infty(\Omega))^{n\times n}$, $n \in \mathbb{N}$. This gives us an opportunity to conclude the well-posedness and causality of the evolutionary problems with non-linear term consisting of the quadratic form with the help of the solution theory associated to the evolutionary problems with a Lipschitz continuous perturbation.

The quadratic form can be found in the heat equation coupled with Maxwell's equations. One of these coupled systems is the microwave heating problem. Here we assumed that the physical coefficients describing the properties of the underlying material, $\varepsilon, \mu \in L\left(\left(L^2(\Omega)\right)^3\right)$, $\lambda, \sigma, \alpha \in (L^\infty(\Omega))^{3\times 3}$ are 3×3 matrix-valued functions depending only on the spatial variables.

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Корректность задачи о микроволновом нагреве

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Аннотация. Ряд начально-краевых задач классической математической физики формулируется в виде линейного операторного уравнения, а его корректность и причинность в гильбертовом пространстве были установлены ранее. Если задача имеет единственное решение и решение постоянно зависит от заданных параметров, то задача называется корректной. Независимость дальнейшего поведения решения до определенного момента указывает на причинность решения. В данной работе установлены корректность и причинность решения эволюционных задач с возмущением, определяемым квадратичной формой. В качестве примера рассмотрена связанная система уравнений теплопроводности и Максвелла (задача микроволнового нагрева).

Ключевые слова: Эволюционные задачи, нелинейное возмущение, Липшицева непрерывность, квадратичная форма, связанные задачи