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# The numerical solution of the nonlinear hyperbolic-parabolic heat equation

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**Abstract.** The article discusses a mathematical model and a finite-difference scheme for the heating process of an infinite plate. The disadvantages of using the classical parabolic heat equation for this case and the rationale for using the hyperbolic heat equation are given. The relationship between the hyperbolic thermal conductivity equation and the theory of equations with the retarded argument (delay equation) is shown. The considered mixed equation has 2 parts: parabolic and hyperbolic. Difference schemes use an integro-interpolation method to reduce errors. The problem with a nonlinear thermal conductivity coefficient was chosen as the initial boundary-value problem. The heat source in the parabolic part of the equation is equal to 0, and in the hyperbolic part of the equation sharp heating begins. The initial boundary-value problem with boundary conditions of the third kind in an infinite plate with nonlinear coefficients is formulated and numerically solved. An iterative method for solving the problem is described. A visual graph of the solution results is presented. A theoretical justification for the difference scheme is given. Also we consider the case of the nonlinear mixed equation of the fourth order.

**Key words and phrases:** hyperbolic-parabolic equation, delay equations, initial boundary-value problem, finite difference schemes, equations of the high order

## 1. Introduction

In the V.N. Khankhasaev's paper [1–3], which is bound up with the problem of mathematical modeling of the process of switching off the electric arc in the flue gas flow, various mathematical models bound up with the hyperbolic equation of thermal conductivity (obtained by generalization of the Fourier hypothesis [4]) were studied both analytically and numerically. In course of investigations bound up with the transfer processes in the case of high-intensity influence of the gas, the earlier hypotheses presuming the proportionality of the flow density to the vector of the potential gradient, which are based on the known physics laws, lead to an infinite rate of distribution of the perturbations, what contradicts to fundamental laws of nature.

The set known physics laws constructed on basis of the given theory includes the following laws:

$\overline{q}(x,y,z,t) = -\lambda \operatorname{grad} T(x,y,z,t)$ – the Fourier law; $\lambda$ – thermal conductivity coefficient; $\overline{q}$ – heat flow density; $T$ – temperature;

$\overline{q}(x,y,z,t) = -D \operatorname{grad} C(x,y,z,t)$ – the Fick law; $D$– diffusion coefficient; $\overline{q}$ – flow density of diffusion; $C$ – concentration;

$\overline{q}(x,y,z,t) = -K \operatorname{grad} H(x,y,z,t)$ – the Darcy law; $K$ – filtering coefficient; $\overline{q}$ – volume flow (or the filtering rate); $H$ – pressure.

All these laws are written in the general form as follows:

$\overline{q}(x,y,z,t) = -A \operatorname{grad} U(x,y,z,t)$, i.e. the generalized law of transfer, where $A$ is the transfer coefficient; $\overline{q}$ is the flow density; $U$ is the potential.

The differential equation of transfer obtained from the given generalized law has the following form in the one-dimensional case:

$$\frac{\partial U}{\partial t} = a\frac{\partial^2 U}{\partial x^2}.$$

It is sufficient to differentiate the fundamental solution of this equation with respect to variable $t$ and tend the time to zero to see that the rate of transfer of the potential at the initial time moment equals to infinity. The approximation of solid medium used in above laws and presuming the absence of its internal molecular structure implies that it is possible to undertake a limit transition in the integral laws of conservation for this medium, when the volume tends to zero. Such a limit transition allows one to obtain the equation of energy conservation in the differential form. Meanwhile, this procedure – from the viewpoint of contemporary physics – is incorrect because the environment is known to be composed of molecules. The environment has a discrete internal structure.

In order to avoid this paradox J.C. Maxwell [5], C. Cattaneo [6], P. Vernotte [7], who worked within the frames of the theory of thermal conductivity, based his reasoning on the molecular-kinetic conception, used the hypothesis of finiteness of duration of molecular collisions and proceeded from a new conception of the molecules' length of free path, obtained a new law of thermal conductivity. There appeared an additional addend $\tau\frac{\partial q}{\partial t}$ in the law, which took account of the discreteness of the environment's molecular structure and was responsible for the inertial character of heat. In this addend, is the relaxation time, i.e. the time of reaching some thermodynamic equilibrium between the heat flow and the temperature gradient. This generalized law of transfer may be written in the following form:

$$\tau\frac{\partial q}{\partial t} + q = -A\,\mathrm{grad}\,U. \tag{1}$$

In the process of solving the differential equation obtained from this law, observed was the first-kind discontinuity of the potential, which distributes from the source. Therefore, law (1) describes the appearance of waves in case of some high-intensity influence, which leads to some local system's non-equilibrium. These effects are most frequently observed when a body is impacted with short energy impulses, in shock waves or under high temperature gradients. The local equilibrium, which is obvious in cases of application of earlier physics laws, is valid for the time moments (intervals), which are in excess of the relaxation time. Therefore, classical transfer theories are valid, when the rate of processes is substantially smaller than the rate of distribution of perturbations in the medium under scrutiny [8, 9].

## 2. The relationship between the hyperbolic thermal conductivity equation and the theory of equations with the retarded argument (delay equations)

To the end of inference of the transfer differential equation in the one-dimensional case we have used the equation of thermal balance:

$$\frac{\partial q}{\partial x} = -g\frac{\partial U}{\partial t}. \tag{2}$$

Having substituted (1) into (2), we obtain:

$$A\frac{\partial^2 U}{\partial x^2} + \tau\frac{\partial^2 q}{\partial x \partial t} = g\frac{\partial U}{\partial t}. \tag{3}$$

Now, change the order of differentiation for the second addend in (3)

$$A\frac{\partial^2 U}{\partial x^2} + \tau\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial x}\right) = g\frac{\partial U}{\partial t}.$$

On account of (2) we obtain

$$A\frac{\partial^2 U}{\partial x^2} - \tau g\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t}\right) = g\frac{\partial U}{\partial t}.$$

Finally, we obtain the following:

$$\tau \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = \frac{A}{g} \frac{\partial^2 U}{\partial x^2}. \tag{4}$$

Equation (4) belongs to the class of linear hyperbolic partial differential equations because it contains the second derivative with respect to time. If internal sources of perturbations are taken into account, then equation (2) assumes the following form:

$$-g \frac{\partial U}{\partial t} + F(U) = \frac{\partial q}{\partial x},$$

Hence equation (4) writes as follows:

$$\tau \frac{\partial^2 U}{\partial t^2} + \left(1 - \frac{\tau}{g} \frac{dF}{dU}\right) \frac{\partial U}{\partial t} = \frac{A}{g} \frac{\partial^2 U}{\partial x^2} + \frac{F(U)}{g}. \tag{5}$$

Here $\dfrac{\partial F}{\partial U}$ may have any sign.

While turning back to equation (1), one can see that $-A \operatorname{grad} U$ represents an expansion of the flow into the Tailor series with respect to the powers of $\tau$, where taken are only the first two members of the expansion. Hence if all the terms of the expansion are taken into account, then the series shall have the following form:

$$q + \tau \frac{\partial q}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 q}{\partial t^2} + \ldots = -A \operatorname{grad} U. \tag{6}$$

Having gathered the terms of series, we can rewrite expression (6) in the following form:

$$q(t + \tau) = -A \operatorname{grad} U. \tag{7}$$

Having replaced the variables $t + \tau = t^1$ in (7), and, next, again transferring to variable t, we obtain:

$$q(t) = -A \operatorname{grad} U(t - \tau). \tag{8}$$

The physics sense of expression (8) implies that the transfer process in the locally non-homogeneous media possesses inertial properties: the system reacts to the influence not at the same time moment but with a delay equal to the relaxation time $\tau$, i.e. the flow density retards from the gradient of potential. From the technical viewpoint, expression (8), unlike that for (1), allows one to take account of all the terms of the expansion with respect to $\tau$. While continuing the above inference technique with the use of (8), one can easily obtain the following equation with the retarded (with respect to time) argument[10–12]:

$$\frac{\partial U}{\partial t} = \frac{A}{g} \frac{\partial^2 U(x, t - \tau)}{\partial x^2}.$$

Therefore, the linear hyperbolic thermal conductivity equation (5) represents the second, more correct stage in the theory of mathematical modeling of heat transfer for the fast running processes with high-intensity perturbations.

The investigation of thermal conductivity processes using the generalized Fourier law is most relevant for rapidly occurring physical phenomena (for example, with nano- and fempto-second laser pulses) in the study of high-intensity processes of heating bodies (plasma, laser processing of materials, high-intensity heating of contact connections in electrical installations and etc.) [13, 14].

## 3. The nonlinear mixed equation of thermal conductivity

An unbounded plate is given in the form of an infinite strip, the size of which along the x axis is equal to the segment $[0, X]$, and the size along the y axis is equal to $(-\infty, \infty)$. We consider the properties of the plate along the y axis to be homogeneous and we will not mention y in the list of variables. The initial temperature distribution in the plate is given by some function $u(x, T_1) = u_0(x)$; at the plate boundaries the temperature of the medium is constant. Heat exchange with the environment occurs according to Newton's law (boundary conditions of the third kind). The thermophysical characteristics

$c_v, \lambda, \rho, \alpha, c$ are specified – specific heat capacity, thermal conductivity coefficient, specific density, heat transfer coefficient and heat output coefficient. It is required to find the temperature distribution over the thickness of the plate, i.e. by variable $x$, at any time $t \in [T_1, T_2]$. The differential equation and boundary conditions will then be written as:

$$k(x,t)u_{tt} + c_v(x,t)\cdot\rho(x,t)u_t = (\lambda(u,x,t)u_x)_x + c(x,t)u + f(x,t). \tag{9}$$

In the rectangular domain $G = [0,X]\times[T_1,T_2]$, $T_1 < 0$, $T_2 > 0$. Furthermore, $\forall(x,t)\in G$, $k(x,t) = 0$, $t \le 0$; $k(x,t) > 0$, $t > 0$; i.e. when $t \le 0$ — the equation(1) is parabolic, and when $t > 0$ — the equation(1) is hyperbolic. Let us formulate the problem with the following experimental data [15].

**The initial boundary-value problem.** Find the temperature field in an infinite plate homogeneous in variable $y$ with $X = \pi$ and calculation time: $T_1 = -5$, $T_2 = 20$.

Initial condition and boundary conditions:

$$u(x,t)|_{t=T_1} = u_0(x) = 10\sin(x);$$

$$\left[\mp\lambda(u,x,t)\frac{\partial u(x,t)}{\partial x} + \alpha_{0,L}(x,t)\,u(x,t)\right]_{x=0,L} = \begin{cases} q_0(t), \\ q_L(t). \end{cases} \tag{10}$$

Coefficients: $k(x,t) = 0$ for $t \le 0$, $k(x,t) = 1$ for $t > 0$; $c(x,t) = 0$; $\lambda(u,x,t) = 0.5\cdot u^2 + 2$; $c_v(x,t)\cdot\rho(x,t) = a(x,t) = 672$; $q_0(t) = 5$; $q_L(t) = 10$; $\alpha_0(x,t) = 3.5$; $\alpha_L(x,t) = 3.5$.

Heat sources $f(x,t)$ change over time. In the parabolic part $f(x,t) = f1(x,t) = 0$ and in the hyperbolic part $f(x,t) = f2(x,t) = 100000\sin(x)\sin(t)$.

In the quasi-linear scheme, the coefficients $\lambda$ are calculated from the temperatures $U_{i,j}$ of the previous time layer $j$, while in the essentially non-linear scheme, which is being implemented now:

$$\begin{aligned} k(i\cdot h, T_1 + j\cdot\tau)\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\tau^2} &+ a(i\cdot h, T_1 + j\cdot\tau)\frac{u_{i,j+1} - u_{i,j}}{\tau} = \\ &= \left(\lambda\left(u_{i+\frac{1}{2},j+1}, i\cdot h, T_1 + j\cdot\tau\right)_{i+\frac{1}{2}}\frac{u_{i+1,j+1} - u_{i,j+1}}{h} - \right. \\ &\left. - \lambda\left(u_{i-\frac{1}{2},j+1}, i\cdot h, T_1 + j\cdot\tau\right)_{i-\frac{1}{2}}\frac{u_{i,j+1} - u_{i-1,j+1}}{h}\right)\frac{1}{h} + \\ &+ c(i\cdot h, T_1 + j\cdot\tau)u_{i,j+1} + \int_{t_j}^{t_{j+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} f(x,t)dxdt. \end{aligned} \tag{11}$$

The coefficient $\lambda$ is calculated as follows:

$$\lambda(u_{i\pm\frac{1}{2},j+1}, i\cdot h, T_1 + j\cdot\tau) = \frac{[\lambda(u_{i\pm1,j}, (i\pm1)\cdot h, T_1 + j\cdot\tau) + \lambda(u_{i,j}, i\cdot h, T_1 + j\cdot\tau)]}{2}.$$

It is clear that the resulting system of equations is nonlinear, so to solve this system we will use the simple iteration method. This method is as follows—at each time step we will determine the temperature field until it stops changing with changes:

$$\frac{\max\limits_i |u_{i,s+1} - u_{i,s}|}{\max\limits_i |u_{i,s+1}|} < \epsilon, \tag{12}$$

where $s$ is the iteration number, $\epsilon$ is the calculation accuracy. When condition (12) is satisfied, then $u_{i,s+1} = u_{i,j+1}$. The following can be considered as an initial approximation: $u_{i,s=0} = u_{i,j}$.

It can be seen that the system (11) is already linear with respect to $u_{i,s+1}$, which makes it possible to use the sweep method and determine the unknown temperature field. But in this case the system is solved until the temperature field ceases to differ [16]. In such a scheme, the volume of calculations increases compared to a quasi-linear scheme, since at each time step it is necessary to solve the system of difference equations by the sweep method not once, but $s_{\max}$ times. However, the nonlinear scheme gives a smaller error in the numerical solution of the original problem (9), (10) than the quasilinear one [17]. This is explained by the fact that the coefficients in the expressions for the grid analogues of heat flows are calculated at the same time as the temperatures. To reduce the error of a quasilinear scheme, the step size should be reduced, i.e., the number of time steps in the considered interval should be increased. Therefore, in many cases it turns out to be more profitable, even from the point of view of computer time costs, to use a nonlinear scheme and take larger time steps, performing several iterations at each [18].

The fields of temperatures for the scrutinized process have been obtained at various time moments (fig.1) in the environment of Mathcad-15 having a comfortable graphic interface. Similar to the works[19, 20] the following theorems is proved:

**Theorem 1.** Let function $c(x,t) < 0$ is sufficiently large with respect to the modulus,

$$2a - |k_t| \geqslant \delta > 0.$$

Hence for any function $f \in W_2^1(G)$ there exists a unique solution $u(x,t)$ of the initial boundary-value problem (9), (10) in space $W_2^2(G)$.

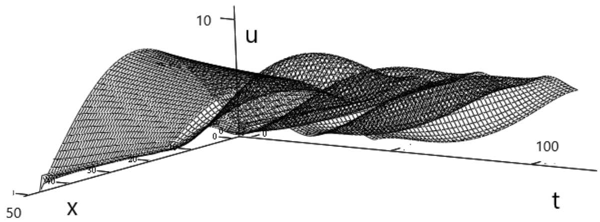


Figure 1. The result of the program solution

**Theorem 2.** Under the conditions of Theorem 1 the difference scheme (11) is stable, and interpolations $u_h^\tau(x,t)$ of solutions of this difference scheme converge weakly in $W_2^1(G)$ (when $h \to 0, \tau \to 0$) to the solution $u(x,t)$ of initial boundary-value problem (9), (10) from space $W_2^2(G)$.

## 4. The nonlinear mixed equation of the fourth order

In bounded domain $D$ from $R^n$, consider the first boundary value problem for the fourth order nonlinear mixed partial differential equation:

$$Lu \equiv \sum_{i=0}^{n+1} L_i^* A_i(x, u, u_{x_1}, \dots, u_{x_n}, Ku) = h(x), \tag{13}$$

$$u|_\Gamma = f_1(x), \frac{\partial u}{\partial \nu}|_{\Gamma/\Gamma_0} = \sum_{i,j=1}^{n} \left( a_{i,j} \frac{\partial u}{\partial x_i} \nu_j \right) \Big|_{\Gamma/\Gamma_0} = f_2(x),$$

here: $L^*$ is an operator formally Lagrange conjugate to the operator $L$; $L_0$ — identity operator; $L_i = \dfrac{\partial}{\partial x_i}$; $i = 1, \dots, n$; $L_{n+1}$ — operator $K$ of the form:

$$Ku \equiv \sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i,x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x)u$$

with sufficiently smooth coefficients, satisfying the inequality:

$$\|Ku\|_{L_m(D)} \geqslant \alpha_1 \|u\|_{W_r^1(D)}, \quad \alpha_1 > 0, \quad m \geqslant 2, \quad r \geqslant 2, \tag{14}$$

for any functions $u(x)$ from $C_K$ the class of twice continuously differentiable functions vanishing on the boundary $\Gamma$ of the domain $D$, $\Gamma_0$ is the characteristic part of the boundary of $\Gamma$ for the operator $K$,

$\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the vector of the internal normal to $\Gamma$:

$$\Gamma_0 = \left\{ x \in \Gamma : \left( \sum_{i,j=1}^{n} a_{i,j} \nu_i \nu_j \right)(x) = 0 \right\}. \tag{15}$$

As the operator $K$, we can take the linear hyperbolic-parabolic heat conduction operator described above in (9).

**Lemma 1.** For any function $u(x)$ from $C_K$ we derive inequality (14) with parameters $m = 2$ and $r = 2$ if the condition is met:

$$2a - |k_t| \geqslant \delta > 0. \tag{16}$$

**Lemma 2.** For any function $u(x)$ from $C_K$, if condition (16) is met, the following inequality with parameter $m = 2$ is deduced:

$$\|Ku\|_{L_m(D)} \geqslant \alpha_2 \left\| \frac{\partial u}{\partial N} \right\|_{L_2(\Gamma)}. \tag{17}$$

Let us define the Banach spaces $H_+$ and $H_\#$ with standards: $\|u\|_+ = \|Ku\|_{L_m(D)}$; $\|u\|_\oplus = \|Ku\|_{L_m(D)} + \|u\|_{W_e^1(D)}$, obtained by closing a set of functions from

$$C_L = \left( u \in C_K : \left. \frac{\partial u}{\partial N} \right|_{\Gamma/\Gamma_0} = 0 \right).$$

From (14) it follows that these are indeed the norms and spaces $H_+$ and $H_\#$ are obviously separable. With the help of the Clarkson's inequalities is proved the next lemma.

**Lemma 3.** Spaces $H_+$ and $H_\#$ reflective.

From the embedding theorems for Sobolev spaces it follows that functions from the spaces $H_+$ and $H_\#$ vanish on the entire boundary of $\Gamma$. Equality (15) means that on $\Gamma_0$ the derivative along the conormal is the tangent derivative to the boundary $\Gamma$ and on functions from $C_K$ vanishes on $\Gamma_0$, therefore inequality (17) actually means

$$\|u\|_+ = \|Ku\|_{L_m(D)} \geqslant \alpha_3 \left\| \frac{\partial u}{\partial N} \right\|_{L_2(\Gamma/\Gamma_0)}.$$

After introducing a continuous trace operator based on inequality (17) of Lemma 2 on functions from $C_K$ and extending it by continuity to the spaces $H_+$ and $H_\#$, we find that for functions from $H_+$ and $H_\#$ the derivative with respect to the conormal vanishes in the space $L_2(\Gamma/\Gamma_0)$.

Suppose that the functions $f_1(x)$, $f_2(x)$ admit continuation $f(x)$ inside the region $D$ from the space $W_m^2(D) \cap W_k^1(D)$, where $k = \max(r, e)$. Then a collection of functions of the form $u(x) = z(x) + f(x)$, where $z(x)$ from $H_+(H_\#)$, forms the space $H_+(f)[H_\#(f)]$.

**Definition 1.** Function $u(x)$ from $H_+(f)[H_\#(f)]$ let's call it a weak generalized solution to problem (13) if the identity holds:

$$\sum_{i=0}^{n+1} \int_D A_i(x, u, u_{x_1}, \dots, u_{x_n}), Ku) L_i \nu, \quad dD = \sum_{i=0}^{n+1} (A_i(x, L_i u), L_i \nu) = (h, \nu),$$
$$\forall \nu(x) \in C_L, \quad j = \overline{0, n+1}.$$

**Definition 2.** Function $u(x)$ from $H_+(f)[H_\#(f)]$ we call a strong generalized solution to problem (13) if there is a sequence of functions $z_i(x) \in C_L$ such that

$$\lim_{i \to \infty} \|z_i + f - u\|_{+[\oplus]} = \lim_{i \to \infty} \|L(z_i + f) - h\|_{-[\ominus]} = 0,$$

where $H_-(D)[H_\ominus(D)]$ are the negative spaces to $H_+(f)[H_\#(f)]$, constructed with respect to the Hilbert space $L_2(D)$.

Let us present a number of assumptions for various equations of the form (13), which essentially mean conditions on the behavior of nonlinear functions $A_i(x, \xi_j)$, $i, j = \overline{0, n+1}$, $\xi \in R^{n+1}$.

1. Conditions of limitation and continuity: $L : H_+(f) \to H_-(D)$.

The functions $A_i(x,\xi_j), i,j = \overline{0,n+1}$ satisfy the Caratheodory conditions, i.e. for almost all $x$ from $D$ are continuous in the set of variables $\xi_j$, for all values $\xi_j$ are measurable in $x$ and satisfy the inequalities:

$$A_i(x,\xi_j) \leqslant \alpha_4\left(a(x) + \sum_{j=0}^{n+1} |\xi_j|^{p_{ij}}\right),$$

where $p_{i,j}$ are selected indicators.

2. Condition for coercivity of the operator $Lu$. For any function $u(x)$ from $H_+(f)[H_\#(f)]$ the following inequality holds:

$$\sum_{i=0}^{n+1}(A_i(x,L_ju),L_iu) \leqslant \alpha_5\|u\|_+^m - \alpha_6, j = \overline{0,n+1},$$

$$\left[\sum_{i=0}^{n+1}(A_i(x,L_ju),L_iu) \leqslant \alpha_7(\|u\|_+^m + \|u\|^e_{W_e^1(D)}) - \alpha_8\right].$$

3. Condition for the definiteness of the variation of the operator $Lu$. For any functions $u(x)$, $v(x)$ from $H_+(f)[H_\#(f)]$ the following inequality holds:

$$(Lu - Lv, u - v) \geqslant \alpha_9\|u - v\|_+^m,$$

$$[(Lu - Lv, u - v) \geqslant \alpha_{10}(\|u - v\|_+^m + \|u - v\|^e_{W_e^1(D)})].$$

Similar to the work of Dubinsky Yu.A. the following theorem is proved:

**Theorem 3:** If assumptions 1) – 3) are met, then the first boundary value problem (13) for any function $h(x) \in H_-(D)[H_\ominus(D)]$ is set correctly, the weak solution coincides with the strong one, i.e. a mapping $Lu = h(x) \in H_-(D)[H_\ominus(D)]$ is a homeomorphism[21].

## 5. Conclusion

A program has been written to solve the mixed heat equation using the simple iteration method. A calculation was carried out with similar boundary conditions. The results coincide with the results of the first miscalculation.

There are other methods that have become widespread in practice for constructing an iterative process for solving systems of nonlinear difference equations. For example, Newton's method is based on the linearization of equations and is usually used in the case when the dependences of the coefficients on temperature are specified by analytical dependencies that can be differentiated. In further work, this method will be used to solve essentially nonlinear equations of mixed type.

Along with numerous methods for solving inverse coefficient problems for linear and nonlinear second order equations $K(u) = h$ you can also use the one proposed by Yu.A. Dubinsky. approach when this equation, which is generally unsolvable for an arbitrary right-hand side $h$, is associated with some 4th order equation $K^*\,K(u) = K^*h$, which is always solvable. Then the equation $K(u) = h$ is solvable up to the kernel of the operator $K^*$.

This construction can also be considered as a technique for describing the range of values of the mixed heat operator corresponding to an ill-posed problem with overdetermination. The presence of these additional boundary conditions, taking into account the release of some of them on the characteristic surfaces of the operator $K(u)$, is necessary for the numerical solution of the well-posed Dirichlet problem for the equation $K^*\,K(u) = K^*h$, if the operator $K^*\,K$ implements a homeomorphism.

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## References


1. Buyantuyev, S. L. & Khankhasaev, V. N. About one generalization of Navier-Stokes equation in mathematical models an electric arc. *Elektrichestvo ISSN: 0013-5380 eISSN: 2411-1333* (1996).
2. Khankhasaev, V. N. & Buyantuyev, S. L. The numerical computation of one mathematical model of electric arc in the gas flow. *Proceedings of International Conference "Energy Conserving and Nature Protecting Technologies for the Baikal Lake Territory"* (2001).
3. Khankhasaev, V. N. & Darmakheev, E. V. On certain applications of the hyperbolic heat transfer equation and methods for its solution. *Mathematical Notes NEFU.* doi:10.25587/SVFU.2018.1.12772 (2018).
4. Shashkov, A. G., Bubnov, V. A. & Yanovskiy, S. Y. *Wave phenomena of thermal conductivity. System-structural approach* 298 pp. (URSS, Moscow, 2004).
5. Maxwell, J. C. On the Dynamic theory of gases. *Philosophical Transactions* **157** (1867).
6. Cattaneo, C. Sulla conduzione de calore. *Atti del seminario matematico e fisiko dell'Universita di Modena* **3** (1948).
7. Vernotte, P. Les paradoxes de la theorie continue de l'equation de la chauleur. *Compt. Rendus.* **246** (1958).
8. Bubnov, V. A. Molecular-kinetics basis for the heat-transfer equation. *Journal of engineering physics* **28** (1975).
9. Sobolev, S. L. Local non-equilibrium transport models. *Successes of Physical Sciences* **167** (1997).
10. Formalev, V. F., Selin, I. A. & Kuznetsova, E. L. Appearance and distribution of heat waves in the nonlinear anisotropic space. *Izvestiya of Russian Academy of Sciences. Ser. "Energy Problems"* (2010).
11. Pimenov, V. G. & Lozhnikov, A. B. Numerical method for modeling controlled heat conduction equation with delay. *Bulletin of Russian Universities. Mathematics.* **18** (2013).
12. Khankhasaev, V. N., Mizhidon, G. A. & Kozhanov, A. I. On the connection between some heat and mass transfer models and the theory of equations with the relarded argument. *Proceedings of the VII International Conference "Mathematics, its Applications and Education in Mathematics" ISBN 978-5-907331-29-7* (2020).
13. Petrova, L. S. Mathematical modeling of heating processes of piecewise homogeneous bodies taking into account the relaxation of heat flow. *Bulletin of Eurasian Science* **9** (2017).
14. Scharf, G. Approach to steady state in the heat equation and the hyperbolic heat transfer equation. *Internet journal "arXiv"* (2018).
15. Voskoboinikov, Y. E. *et al. Solving engineering problems in MathCAD package* 190 pp. (State architecture-builds University (Sibstrin), Novosibirsk, 2013).
16. Kuznetsov, G. V. & Sheremet, M. A. *Difference methods for solving heat conduction problems* (2007).
17. Khankhasaev, V. N. & Bairov, S. A. Solution of a mixed equation nonlinear with respect to the thermal conductivity coefficient by the integro-interpolation method with the third boundary condition. *Mathematics, its applications and mathematical education (MAME23): Proceedings of the VIII International Conference.* doi:10.53980/9785907599970 (2023).
18. Dulnev, G. D., Parfenov, V. G. & Sigalov, A. V. *Application of computers to solve heat transfer problems* 207 pp. (High School, Moscow, 1990).
19. Terehov, A. N. Boundary value problem for a mixed type equation. *Collection "Application of methods of functional analysis to problems in mathematics, physical and calc. math."* (1979).
20. Khankhasaev, V. N. & Mestnikova, N. N. An algorithm of numerical solving of heat transfer hyperbolic-parabolic equation one-dimensional with respect to the space variable in the environment of Mathcad 15. *Proceedings of the International conference "Nano-materials and Technologies–5"* (2014).
21. Khankhasaev, V. N. On the theory of nonlinear equations of mixed type of the fourth order. *Application of functional analysis methods to non-classical equations of mathematical physics* (1988).


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# Численное решение нелинейного гиперболо-параболического уравнения теплопроводности

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**Аннотация.** В статье рассматривается математическая модель и конечно-разностная схема процесса нагрева бесконечной пластины. Приводятся недостатки использования классического параболического уравнения теплопроводности для данного случая и обоснования для использования смешанного уравнения. Показана связь гиперболического уравнения теплопроводности с теорией уравнений с запаздывающим аргументом (уравнением с запаздыванием). В смешанном уравнении присутствуют 2 части: параболическая и гиперболическая. В разностных схемах применяется интегро-интерполяционный метод для уменьшения погрешностей. В качестве краевой задачи выбрана задача с нелинейным коэффициентом теплопроводности. Источник тепла в параболической части уравнения равен 0, а в гиперболической части уравнения начинается резкий нагрев. Поставлена и численно решена начально-краевая задача с краевыми условиями третьего рода в бесконечной пластине с линейными и с нелинейными коэффициентами. Описан итерационный метод для решения задачи. Представлен наглядный график результатов решения. Дано теоретическое обоснование для разностной схемы. Также рассмотрен случай нелинейного смешанного уравнения четвертого порядка.

**Ключевые слова:** гиперболо-параболическое уравнение, уравнения с запаздыванием, начально-краевая задача, конечно-разностные схемы, уравнения высокого порядка