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Asymptotic diffusion analysis of the retrial queuing system with feedback and batch Poisson arrival

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Abstract. The mathematical model of the retrial queuing system $M^{[n]}/M/1$ with feedback and batch Poisson arrival is constructed. Customers arrive in groups. If the server is free, one of the arriving customers starts his service, the rest join the orbit. The retrial and service times are exponentially distributed. The customer whose service is completed leaves the system, or reservice, or goes to the orbit. The method of asymptotic diffusion analysis is proposed for finding the probability distribution of the number of customers in orbit. The asymptotic condition is growing average waiting time in orbit. The accuracy of the diffusion approximation is obtained.

Key words and phrases: retrial queuing system, batch arrival, feedback, asymptotic diffusion analysis

1. Introduction

There are situations in practice where an arriving customer that sees the server being occupied temporarily leaves the system or goes to orbit. In some random time customer retries to occupy a server again. These situations are modeled as retrial queuing systems. In addition, there are queuing systems in which a customer that has already received service requires a second service. It depends on the quality of the received service or external factors. Classical examples are communication networks in which erroneously transmitted data is retransmitted. The functioning of such systems is described by retrial queuing systems with feedback.

There are many reviews on the study of queuing systems with repeated calls, for example [1, 2]. Models with feedback, instantaneous and delayed, have also been intensively studied in the last two decades [3–5]. At the same time, classical methods do not allow us to evaluate the characteristics of such systems. The application of asymptotic analysis methods makes it possible to obtain the asymptotic characteristics of the system under various limiting conditions. For example, in [6], a stationary probability distribution of the number of customers in orbit was obtained under conditions of a large delay of customers in orbit. To perform more detailed and accurate analysis of the model a method of asymptotic diffusion analysis is applied [7].

In this paper, we study retrial queuing systems with single server, batch Poisson arrival process, instantaneous and delayed feedback. The retrial and service times are exponentially distributed. A diffusion approximation of the probability distribution of the number of customers in orbit is constructed. It is shown that the accuracy of the diffusion approximation is higher then the accuracy of Gaussian approximation obtained in [6].

2. System description

We consider the queuing system $M^{[n]}/M/1$ with repeated calls (see figure 1) with Poisson batch input flow with a parameter λ and given probabilities q_ν of occurrence of ν customers in the group $\left(\nu > 0,\, q_0 = 0, \sum_{\nu=1}^{\infty} q_\nu = 1\right)$. If the server is free, then one customer receive service, the rest of customers go to the orbit. If the server is busy, the arriving customers join the orbit. The service time is exponentially distributed with parameter μ. A customer whose service is completed leaves the system with probability r_0, receives service again with probability r_1 or goes to the orbit with probability r_2, thus $r_0 + r_1 + r_2 = 1$. In orbit, customers wait for a time distributed exponentially with parameter σ, after which they repeat an attempt to occupy the server. In case of an unsuccessful attempt, the customers remain in orbit.

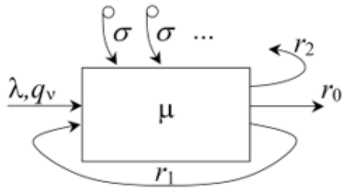

Figure 1. Queuing system model with retrial calls and feedback

We denote by $i(t)$ the number of customers in orbit at time t, the process $n(t)$ determines the state of the server as follows:

$$n(t) = \begin{cases} 0, & \text{if the server is idle;} \\ 1, & \text{if the server is busy.} \end{cases}$$

The two-dimensional process $\{i(t), n(t)\}$ is a continuous-time Markov chain. It is required to find the probability distribution of the number of customers in orbit, taking into account the state of the server

$$P_n(i,t) = P\{n(t) = n,\ i(t) = i\}, \quad n = 0,1; \quad i = \overline{0,\infty}.$$

We compose a system of Kolmogorov differential equations for the probability distribution $P_n(i,t)$

$$\begin{cases} \dfrac{\partial P_0(i,t)}{\partial t} = -(\lambda + i\sigma)P_0(i,t) + \mu r_0 P_1(i,t) + \mu r_2 P_1(i-1,t); \\ \dfrac{\partial P_1(i,t)}{\partial t} = (i+1)\sigma P_0(i+1,t) + (\mu r_1 - \mu - \lambda)P_1(i,t) + \\ \qquad + \sum\limits_{\nu=1}^{i+1} \lambda q_\nu P_0(i-\nu+1,t) + \sum\limits_{\nu=1}^{i} \lambda q_\nu P_1(i-\nu,t). \end{cases} \tag{1}$$

We consider the partial characteristic functions of the number of customers in the orbit $H_n(u,t) = \sum\limits_{i=0}^{\infty} e^{jui} P_n(i,t)$ and the characteristic function for the number of customers in the batch $h(u) = \sum\limits_{\nu=1}^{\infty} e^{ju\nu} q_\nu$, where $j = \sqrt{-1}$. Then we take into account that

$$\frac{\partial H_n(u,t)}{\partial u} = \sum_{i=0}^{\infty} ije^{jui} P_n(i,t),$$

$$\sum_{i=0}^{\infty} \sum_{\nu=1}^{i} q_\nu e^{jui} P_1(i-\nu,t) = h(u)H_1(u,t),$$

$$\sum_{i=0}^{\infty} \sum_{\nu=1}^{i+1} q_\nu e^{jui} P_0(i-\nu+1,t) = e^{-ju} h(u) H_0(u,t),$$

and rewrite system (1) as

$$\begin{cases} \dfrac{\partial H_0(u,t)}{\partial t} = \sigma j \dfrac{\partial H_0(u,t)}{\partial u} - \lambda H_0(u,t) + \left(\mu r_0 + \mu r_2 e^{ju}\right) H_1(u,t); \\ \dfrac{\partial H_1(u,t)}{\partial t} = -\sigma j e^{-ju} \dfrac{\partial H_0(u,t)}{\partial u} + \lambda e^{-ju} h(u) H_0(u,t) + \\ \qquad + \left(\lambda h(u) - \mu r_0 - \mu r_2 - \lambda\right) H_1(u,t). \end{cases} \tag{2}$$

The total characteristic function $H(u,t)$ of the number of customers in orbit is $H(u,t) = H_0(u,t) + H_1(u,t)$. We summarize the equations of system (2) and write

$$\frac{\partial H(u,t)}{\partial t} = \sigma j \left(1 - e^{-ju}\right) \frac{\partial H_0(u,t)}{\partial u} + \lambda \left(e^{-ju} h(u) - 1\right) H_0(u,t) + \\ + \left(\mu r_2 \left(e^{ju} - 1\right) + \lambda \left(h(u) - 1\right)\right) H_1(u,t). \tag{3}$$

3. The first stage of asymptotic. A transfer coefficient

We solve the equations for the characteristic function (2) under the asymptotic condition of the growing average waiting time in orbit, that is, we assume that $\sigma \to 0$. We denote $\sigma = \varepsilon$ and make the following substitutions in system (2)

$$\tau = \varepsilon t, \quad u = \varepsilon w, \quad H_n(u,t) = F_n(w,\tau,\varepsilon), \quad n = 0,1,$$

$$\frac{\partial H_0(u,t)}{\partial u} = \frac{1}{\varepsilon}\frac{\partial F_0(w,\tau,\varepsilon)}{\partial w}, \quad \frac{\partial H_n(u,t)}{\partial t} = \varepsilon\frac{\partial F_n(w,\tau,\varepsilon)}{\partial \tau}, \quad n = 0,1,$$

then we get a system of equations

$$\begin{cases} \varepsilon\dfrac{\partial F_0(w,\tau,\varepsilon)}{\partial \tau} = j\dfrac{\partial F_0(w,\tau,\varepsilon)}{\partial w} - \lambda F_0(w,\tau,\varepsilon) + \\ \qquad\qquad + \left(\mu r_0 + \mu r_2 e^{jw\varepsilon}\right) F_1(w,\tau,\varepsilon); \\ \varepsilon\dfrac{\partial F_1(w,\tau,\varepsilon)}{\partial \tau} = -je^{-jw\varepsilon}\dfrac{\partial F_0(w,\tau,\varepsilon)}{\partial w} + \lambda e^{-jw\varepsilon}h(w,\varepsilon)F_0(w,\tau,\varepsilon) + \\ \qquad\qquad + \left(\lambda h(w,\varepsilon) - \mu r_0 - \mu r_2 - \lambda\right) F_1(w,\tau,\varepsilon). \end{cases} \tag{4}$$

We look for a solution to the equations in the form $F_n(w,\tau,\varepsilon) = R_n e^{jwx(\tau)}$, then

$$\begin{cases} \varepsilon jwx'(\tau)R_0 = -x(\tau)R_0 - \lambda R_0 + \left(\mu r_0 + \mu r_2 e^{jw\varepsilon}\right) R_1; \\ \varepsilon jwx'(\tau)R_1 = e^{-jw\varepsilon}x(\tau)R_0 + \lambda e^{-jw\varepsilon}h(w,\varepsilon)R_0 + \\ \qquad\qquad + \left(\lambda h(w,\varepsilon) - \mu r_0 - \mu r_2 - \lambda\right) R_1. \end{cases} \tag{5}$$

As $\varepsilon \to 0$, we have $\lim\limits_{\varepsilon\to 0} h(w,\varepsilon) = 1$ and system (5) reduces to a single equation

$$-(x(\tau) + \lambda)R_0 + (\mu r_0 + \mu r_2)R_1 = 0. \tag{6}$$

Equation (6) with the normalization condition $R_0 + R_1 = 1$ give R_0 and R_1 as functions of x

$$R_0(x) = \frac{\mu r_0 + \mu r_2}{x + \lambda + \mu r_0 + \mu r_2}, \; R_1(x) = \frac{x + \lambda}{x + \lambda + \mu r_0 + \mu r_2}. \tag{7}$$

We summarize the equations of system (5) and obtain

$$jwx'(\tau) = \left(x(\tau)\frac{e^{-jw\varepsilon} - 1}{\varepsilon} + \lambda\frac{e^{-jw\varepsilon}h(w,\varepsilon) - 1}{\varepsilon}\right) R_0(x) + \\ + \left(\mu r_2\frac{e^{jw\varepsilon} - 1}{\varepsilon} + \lambda\frac{h(w,\varepsilon) - 1}{\varepsilon}\right) R_1(x).$$

As $\varepsilon \to 0$ we get $x'(\tau) = \left(-x(\tau) + \lambda\left(\bar{\nu} - 1\right)\right) R_0(x) + \left(\mu r_2 + \lambda\bar{\nu}\right) R_1(x)$.

We denote by $a(x)$ the right side of the last equality

$$a(x) = \left[\lambda\left(\bar{\nu} - 1\right) - x\right] R_0(x) + \left(\lambda\bar{\nu} + \mu r_2\right) R_1(x). \tag{8}$$

It will be shown below that the function $a(x)$ is the transfer coefficient of some diffusion process approximating the number of customers in orbit.

4. The second stage of asymptotic. A diffusion coefficient

We substitute

$$H_n(u,t) = H_n^{(2)}(u,t)\exp\left\{j\frac{u}{\sigma}x(\sigma t)\right\},\quad n=0,1$$

in the system (2) and the equation (3). Here $H_n^{(2)}(u,t)$ is the characteristic function of the centered random variable $i(t)-x(\sigma t)/\sigma$. Then we obtain a system of equations for $H_n^{(2)}(u,t)$ in the form

$$\frac{\partial H_0^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H_0^{(2)}(u,t) = \sigma j\frac{\partial H_0^{(2)}(u,t)}{\partial u} - \\ -(x(\sigma t)+\lambda)H_0^{(2)}(u,t) + \left(\mu r_0 + \mu r_2 e^{ju}\right)H_1^{(2)}(u,t);$$

$$\frac{\partial H_1^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H_1^{(2)}(u,t) = -\sigma je^{-ju}\frac{\partial H_0^{(2)}(u,t)}{\partial u} + \\ +(x(\sigma t)+\lambda h(u))\,e^{-ju}H_0^{(2)}(u,t) + \\ +(\lambda h(u)-\lambda-\mu R_0-\mu R_2)\,H_1^{(2)}(u,t);$$

$$\frac{\partial H^{(2)}(u,t)}{\partial t} + jux'(\sigma t)H^{(2)}(u,t) = \sigma j\left(1-e^{-ju}\right)\frac{\partial H_0^{(2)}(u,t)}{\partial u} + \\ +\left(x(\sigma t)\left(e^{-ju}-1\right)+\lambda\left(h(u)e^{-ju}-1\right)\right)H_0^{(2)}(u,t) + \\ +\left(\mu r_2\left(e^{ju}-1\right)+\lambda\left(h(u)-1\right)\right)H_1^{(2)}(u,t).$$

We denote $\sigma=\varepsilon^2$ and make a replacement

$$\tau=\varepsilon^2 t,\quad u=\varepsilon w,\quad H_n^{(2)}(u,t)=F_n^{(2)}(w,\tau,\varepsilon),$$

then we get the system

$$\begin{cases} \varepsilon^2\dfrac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x)F_0^{(2)}(w,\tau,\varepsilon) = j\varepsilon\dfrac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} - \\ \quad -(x+\lambda)F_0^{(2)}(w,\tau,\varepsilon) + \left(\mu r_0+\mu r_2 e^{j\varepsilon w}\right)F_1^{(2)}(w,\tau,\varepsilon); \\ \varepsilon^2\dfrac{\partial F_1^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x)F_1^{(2)}(w,\tau,\varepsilon) = -j\varepsilon e^{-j\varepsilon w}\dfrac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} + \\ \quad +(x+\lambda h(\varepsilon w))\,e^{-j\varepsilon w}F_0^{(2)}(w,\tau,\varepsilon) + \\ \quad +(\lambda h(\varepsilon w)-\lambda-\mu r_0-\mu r_2)\,F_1^{(2)}(w,\tau,\varepsilon) \end{cases} \tag{9}$$

and equation

$$\varepsilon^2 \frac{\partial F^{(2)}(w,\tau,\varepsilon)}{\partial \tau} + j\varepsilon w a(x,\tau) F^{(2)}(w,\tau,\varepsilon) = \\ = \left(x(\tau)\left(e^{-j\varepsilon w}-1\right) + \lambda\left(h(\varepsilon w)e^{-j\varepsilon w}-1\right)\right) F_0^{(2)}(w,\tau,\varepsilon) + \\ + j\varepsilon\left(1-e^{-j\varepsilon w}\right)\frac{\partial F_0^{(2)}(w,\tau,\varepsilon)}{\partial w} + \\ + \left(\mu r_2\left(e^{j\varepsilon w}-1\right) + \lambda\left(h(\varepsilon w)-1\right)\right) F_1^{(2)}(w,\tau,\varepsilon). \quad (10)$$

We write the solution $F_n^{(2)}(w,\tau,\varepsilon)$, $n = 0,1$ in the form

$$F_n^{(2)}(w,\tau,\varepsilon) = \Phi(w,\tau)\left(R_n + j\varepsilon w F_n\right) + O(\varepsilon^2) \quad (11)$$

and expand $e^{\pm j\varepsilon w}$, $h(\varepsilon w)$ in Taylor series up to the first order of ε in system (9), and up to the second order in equation (10). We substitute (11) into (9), (10) and take into account equations (6), (8), then we can write

$$\begin{aligned} j\varepsilon w a(x) R_0 &= -j\varepsilon w(x+\lambda) f_0 + j\varepsilon w \mu r_2 R_1 + j\varepsilon w\left(\mu r_0 + \mu r_2\right) f_1 + \\ &\quad + j\varepsilon R_0 \frac{1}{\Phi(w,\tau)}\frac{\partial \Phi(w,\tau)}{\partial w} + O(\varepsilon^2); \\ j\varepsilon w a(x) R_1 &= j\varepsilon w\left(\lambda\bar{\nu} - \lambda - x\right) R_0 + j\varepsilon w (x+\lambda) f_0 + j\varepsilon w \lambda\bar{\nu} R_1 - \\ &\quad - j\varepsilon w\left(\mu r_0 + \mu r_2\right) f_1 - j\varepsilon R_0 \frac{1}{\Phi(w,\tau)}\frac{\partial \Phi(w,\tau)}{\partial w} + O(\varepsilon^2); \end{aligned} \quad (12)$$

and

$$\varepsilon^2 \frac{\partial \Phi(w,\tau)}{\partial \tau} + (j\varepsilon w)^2 a(x)\Phi(w,\tau) f = (j\varepsilon w)^2\left[\left(\lambda\nu_2 - 2\lambda\bar{\nu} + \lambda + x\right)\frac{R_0}{2} + \\ + \left(\lambda\bar{\nu} - \lambda - x\right) f_0 + \left(\lambda\nu_2 + \mu r_2\right)\frac{R_1}{2} + \left(\lambda\bar{\nu} + \mu r_2\right) f_1\right]\Phi(w,\tau) + \\ + (j\varepsilon w)^2 \frac{R_0}{w}\frac{\partial \Phi(w,\tau)}{\partial w} + O(\varepsilon^3), \quad (13)$$

where $\nu_2 = \sum\limits_{\nu=1}^{\infty} \nu^2 q_\nu$.

After simple transformations using (8), two equations of system (12) are reduced to a single equation

$$-(x+\lambda) f_0 + \left(\mu r_0 + \mu r_2\right) f_1 = a(x) R_0(x) - \mu r_2 R_1(x) - \frac{R_0(x)}{w\Phi(w,\tau)}\frac{\partial \Phi(w,\tau)}{\partial w}.$$

Solution we find in the form

$$f_n = C \cdot R_n(x) + g_n - \varphi_n \frac{1}{w\Phi(w,\tau)}\frac{\partial \Phi(w,\tau)}{\partial w}. \quad (14)$$

Here $C \cdot R_n(x)$ is the general solution of the homogeneous equation due to (6), g_n is the solution of the equation

$$-(x+\lambda)g_0 + (\mu r_0 + \mu r_2)\, g_1 = a(x)R_0(x) - \mu r_2 R_1(x), \tag{15}$$

and φ_n satisfies the equation

$$-(x+\lambda)\frac{(-\varphi_0)}{w\Phi(w,\tau)}\frac{\partial\Phi(w,\tau)}{\partial w} + (\mu r_0 + \mu r_2)\frac{(-\varphi_1)}{w\Phi(w,\tau)}\frac{\partial\Phi(w,\tau)}{\partial w} = \\ = -\frac{R_0(x)}{w\Phi(w,\tau)}\frac{\partial\Phi(w,\tau)}{\partial w}.$$

or

$$-(x+\lambda)\varphi_0 + (\mu r_0 + \mu r_2)\, \varphi_1 = R_0(x). \tag{16}$$

Differentiating (6) with respect to x and comparing with (16), we note that $\varphi_0 = \dfrac{\partial R_0(x)}{\partial x}$, $\varphi_1 = \dfrac{\partial R_1(x)}{\partial x}$, $\varphi_0 + \varphi_1 = 0$.

Then, taking into account (7), we obtain

$$\varphi_0 = -\frac{\mu(r_0+r_2)}{(x+\lambda+\mu(r_0+r_2))^2}, \quad \varphi_1 = -\varphi_0. \tag{17}$$

Similarly, we set $g_0 + g_1 = 0$, then from equation (15)

$$g_1 = \frac{aR_0(x) - \mu r_2 R_1(x)}{x+\lambda+\mu r_0 + \mu r_2}, \quad g_0 = -g_1. \tag{18}$$

The equation (13) can be written as

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} + (jw)^2 a(x)\Phi(w,\tau) f = (jw)^2\Bigg[(\lambda\nu_2 - 2\lambda\bar{\nu} + \lambda + x)\frac{R_0(x)}{2} + \\ + (\lambda\bar{\nu} - \lambda - x)\, f_0 + (\lambda\nu_2 + \mu r_2)\frac{R_1(x)}{2} + (\lambda\bar{\nu} + \mu r_2)\, f_1\Bigg]\Phi(w,\tau) + \\ + (jw)^2\frac{R_0(x)}{w}\frac{\partial\Phi(w,\tau)}{\partial w}.$$

We substitute solution (14) into it and, taking into account (8), (17), (18), we obtain

$$\frac{\partial\Phi(w,\tau)}{\partial\tau} = w\frac{\partial\Phi(w,\tau)}{\partial w}\left[(\lambda\bar{\nu} - \lambda - x)\,\varphi_0 + (\lambda\bar{\nu} + \mu r_2)\,\varphi_1 - R_0(x)\right] + \\ + (jw)^2\Bigg[(\lambda\bar{\nu} - \lambda - x)\, g_0 + (\lambda\bar{\nu} + \mu r_2)\, g_1 + (\lambda\nu_2 - 2\lambda\bar{\nu} + \lambda + x)\frac{R_0(x)}{2} + \\ + (\lambda\nu_2 + \mu r_2)\frac{R_1(x)}{2}\Bigg]\Phi(w,\tau). \tag{19}$$

We denote

$$b(x) = (\lambda\nu_2 - 2\lambda\bar{\nu} + \lambda + x)\, R_0(x) + (\lambda\nu_2 + \mu r_2)\, R_1(x) + \\ + 2\,(\mu r_2 + \lambda + x)\, g_1(x). \quad (20)$$

It will be shown below that the function $b(x)$ is the diffusion coefficient of some diffusion process approximating the number of customers in orbit. Then, taking into account (8), (17), equation (19) can be written in the form

$$\frac{\partial \Phi(w,\tau)}{\partial \tau} = w\frac{\partial \Phi(w,\tau)}{\partial w} a'(x) + \frac{(jw)^2}{2}\Phi(w,\tau) b(x). \quad (21)$$

5. The third stage of asymptotic. A diffuse approximation

The inverse Fourier transform $\Phi(w,\tau) = \int\limits_{-\infty}^{\infty} e^{jwy} P(y,\tau)dy$ converts an equation (21) for the characteristic functions to the equation for the probability density $P(y,\tau)$.

Given the relationship

$$w\frac{\partial \Phi(w,\tau)}{\partial w} = -\int\limits_{-\infty}^{\infty} e^{jwy}(yP(y,\tau))'dy,$$

$$(jw)^2\Phi(w,\tau) = \int\limits_{-\infty}^{\infty} e^{jwy}\frac{\partial^2 P(y,\tau)}{\partial y^2}dy,$$

we obtain the equation

$$\frac{\partial P(y,\tau)}{\partial \tau} = -a'(x)\frac{\partial\,(yP(y,\tau))}{\partial y} + \frac{b(x)}{2}\frac{\partial^2 P(y,\tau)}{\partial y^2}.$$

The resulting equation is the Fokker–Planck equation for the probability density of some diffusion process $y(\tau)$ with transfer coefficient $a'(x)y$ and diffusion coefficient $b(x)$. Thus, the process $y(\tau)$ is a solution of the stochastic differential equation

$$dy(\tau) = a'(x)y(\tau)d\tau + \sqrt{b(x)}d\omega(\tau),$$

where $\omega(\tau)$ is a Wiener process.

We introduce a diffusion process $z(\tau) = x(\tau) + \varepsilon y(\tau)$, where the function $x(\tau)$ is a solution of the ordinary differential equation $dx(\tau) = a(x)d\tau$. Then the diffusion process $z(\tau)$ is a solution of the following stochastic differential equation

$$dz(\tau) = [a(x) + \varepsilon a'(x)y(\tau)]\, d\tau + \varepsilon\sqrt{b(x)}d\omega(\tau).$$

We consider the right hand side of the resulting stochastic differential equation

$$a(x) + \varepsilon a'(x)y = a(x + \varepsilon y) + O(\varepsilon^2) = a(z) + O(\varepsilon^2),$$

$$\varepsilon\sqrt{b(x)} = \varepsilon\sqrt{b(x + \varepsilon y - \varepsilon y)} = \varepsilon\sqrt{b(z - \varepsilon y)} = \varepsilon\sqrt{b(z)} + O(\varepsilon^2)$$

and assume that the terms $O(\varepsilon^2)$ do not contribute significantly to the solution and can be neglected. Then we obtain a stochastic differential equation of the form

$$dz(\tau) = a(z)d\tau + \varepsilon\sqrt{b(z)}d\omega(\tau).$$

We denote by the probability density of the diffusion process $z(\tau)$ as

$$\Pi(z,\tau) = \frac{\partial P\{z(\tau) < z\}}{\partial z}$$

and write the Fokker–Planck equation for this distribution

$$\frac{\partial \Pi(z,\tau)}{\partial \tau} = -\frac{\partial a(z)\Pi(z,\tau)}{\partial z} + \frac{\varepsilon^2}{2}\frac{\partial^2 b(z)\Pi(z,\tau)}{\partial z^2}.$$

The inverse replacement $\sigma = \varepsilon^2$ leads to the equation for stationary probability distribution of diffusion process $z(\tau)$

$$-(a(z)\Pi(z))' + \frac{\sigma}{2}(b(z)\Pi(z))'' = 0,$$

$$(b(z)\Pi(z))' = \frac{2}{\sigma}a(z)\Pi(z).$$

To solve this equation we introduce replacement of variables

$$G(z) = b(z)\Pi(z),$$

and obtain the equation

$$G'(z) = \frac{2}{\sigma}\frac{a(z)}{b(z)}G(z),$$

then the solution is written in the form

$$G(z) = C\exp\left\{\frac{2}{\sigma}\int\limits_0^z \frac{a(x)}{b(x)}dx\right\}.$$

Inverse replacement leads to

$$\Pi(z) = \frac{C}{b(z)}\exp\left\{\frac{2}{\sigma}\int\limits_0^z \frac{a(x)}{b(x)}dx\right\}.$$

On the basis of obtained probability density function we construct the diffusion approximation by formula

$$PD(i) = \frac{\Pi(i\sigma)}{\sum\limits_{n=0}^{\infty} \Pi(n\sigma)}. \tag{22}$$

6. Numerical results

We determine the applicability of the obtained approximation by comparing the asymptotic distribution (22) with the steady state distribution $P(i)$ obtained when solving the system (1) by the matrix method. We consider different values of the parameter σ. To compare two probability distributions, we use the Kolmogorov distance

$$\Delta_1 = \max_{0 \le n < \infty} \left| \sum_{i=0}^{n} \left(P(i) - PD(i)\right) \right|. \tag{23}$$

We consider following system parameters $\lambda = 1$, $r_0 = 0.5$, $r_1 = 0.3$, $r_2 = 0.2$, $q_1 = 0.5$, $q_2 = 0.3$, $q_3 = 0.1$, $q_4 = 0.1$. We introduce the system loading parameter $\rho = \frac{\lambda \bar{\nu}}{\mu r_0}$. It defines the value of the parameter μ. We take $\Delta = 0.05$ as a threshold value.

Table 1 presents Kolmogorov distances Δ_1 calculated by formula (23), table 2 presents Kolmogorov distances Δ_2 calculated for the Gaussian approximation obtained in [6]. Bold in the tables are the values that correspond to a satisfactory approximation accuracy. It can be concluded that the accuracy of diffusion approximation increases with decreasing the parameter σ and increasing the system load ρ, and the accuracy of Gaussian approximation decreases with high system load. In addition, the accuracy of the diffusion approximation is higher than the accuracy of the Gaussian approximation.

Table 1

Kolmogorov distance Δ_1

Δ_1	$\sigma = 2$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.1$	$\sigma = 0.05$
$\rho = 0.2$	0.135	0.089	0.050	**0.016**	**0.012**
$\rho = 0.5$	0.094	0.060	**0.035**	**0.013**	**0.009**
$\rho = 0.7$	0.059	**0.036**	**0.021**	**0.009**	**0.006**
$\rho = 0.9$	**0.019**	**0.011**	**0.007**	**0.003**	**0.002**

Table 2

Kolmogorov distance Δ_2

Δ_2	$\sigma = 2$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.1$	$\sigma = 0.05$
$\rho = 0.2$	0.221	0.152	0.086	**0.018**	**0.013**
$\rho = 0.5$	0.162	0.105	**0.047**	**0.027**	**0.019**
$\rho = 0.7$	0.175	0.108	**0.045**	**0.039**	**0.027**
$\rho = 0.9$	0.187	0.109	0.084	0.057	**0.040**

7. Conclusions

The mathematical model of the system $M^{[n]}/M/1$ with an incoming batch Poisson flow and feedback is constructed. The system of equations for probability distribution of the number of customers in orbit is present.

A diffusion approximation of the probability distributions of the number of customers in orbit is obtained. The asymptotic condition is growing average waiting time in orbit. The accuracy of the approximation is determined using the Kolmogorov distance in comparison with the steady state probability distribution obtained by the matrix method. Numerical examples are given for different values of the system parameters, the accuracy of the diffusion approximation and the Gaussian approximation is compared. It is shown that the accuracy of the diffusion approximation is higher than the accuracy of the Gaussian approximation.

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Асимптотически диффузионный анализ RQ-системы с обратными связями и неординарным входящим потоком

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Аннотация. В работе исследована $M^{[n]}/M/1$ RQ-система с неординарным пуассоновским входящим потоком. Заявки на вход системы поступают группами. В каждый момент времени обслуживается не более одной заявки, остальные попадают на орбиту. Заявка, обслуживание которой завершено, либо покидает систему, либо повторно поступает на обслуживание, либо переходит на орбиту. Методом асимптотически диффузионного анализа при асимптотическом условии растущего среднего времени ожидания на орбите построена аппроксимация распределения вероятностей числа заявок на орбите. Показано, что точность диффузионной аппроксимации превышает точность гауссовской аппроксимации.

Ключевые слова: система массового обслуживания, RQ-система, неординарный поток, обратная связь, асимптотически-диффузионный анализ