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On a modification of the Hamming method for summing discrete Fourier series and its application to solve the problem of correction of thermographic images

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Abstract. The paper considers mathematical methods of correction of thermographic images (thermograms) in the form of temperature distribution on the surface of the object under study, obtained using a thermal imager. The thermogram reproduces the image of the heat-generating structures located inside the object under study. This image is transmitted with distortions, since the sources are usually removed from its surface and the temperature distribution on the surface of the object transmits the image as blurred due to the processes of thermal conductivity and heat exchange. In this paper, the continuation of the temperature function as a harmonic function from the surface deep into the object under study in order to obtain a temperature distribution function near sources is considered as a correction principle. This distribution is considered as an adjusted thermogram. The continuation is carried out on the basis of solving the Cauchy problem for the Laplace equation — an ill-posed problem. The solution is constructed using the Tikhonov regularization method. The main part of the constructed approximate solution is presented as a Fourier series by the eigenfunctions of the Laplace operator. Discretization of the problem leads to discrete Fourier series. A modification of the Hamming method for summing Fourier series and calculating their coefficients is proposed.

Key words and phrases: thermogram, ill-posed problem, Cauchy problem for the Laplace equation, Tikhonov regularization method, discrete Fourier series

1. Introduction

Thermal imaging methods are widely used in medicine as a means of early diagnostics [1–4]. Visualization (thermogram) of the temperature distribution on the surface of the patient's body contains information about sources of heat inside the body associated with the functioning of internal organs. In particular, it contains information about temperature anomalies associated with pathologies of internal organs. The image on the thermogram, as a rule, is distorted due to the process of thermal conductivity, heat exchange and the relative remoteness of heat sources from the surface of the body.

Within the framework of the chosen mathematical model, it is possible to correct the image on the thermogram in order to increase the effectiveness of diagnostics. Since the evolution of the temperature distribution in the patient's body is relatively slow, it makes it possible to use stationary models, in particular, models of harmonic temperature distribution. As an adjusted thermogram, we will consider the temperature distribution near the sources obtained by the continuation of the harmonic function from the boundary (similar to the continuation of gravitational fields in geophysics problems [5]).

In [6], based on the method [7], one of the possible solutions to such a problem is proposed. The problem, as ill-posed, is solved using the Tikhonov regularization method [8]. When forming computational algorithms, discrete Fourier series [9, 10] are used, the coefficients of which are calculated from functions depending on the coefficient number [11]. To sum up such series, a modification of the Hamming method [9] is proposed here.

2. Mathematical model and inverse problem

As a mathematical model, we consider a homogeneous heat-conducting body in the form of a rectangular cylinder

$$D(F,\infty) = \{(x,y,z) :\, 0 < x < l_x,\, 0 < y < l_y,\, F(x,y) < z < \infty\} \subset \mathbb{R}^3, \quad (1)$$

limited by the surface

$$S = \{(x,y,z) :\, 0 < x < l_x,\, 0 < y < l_y,\, z = F(x,y)\}. \quad (2)$$

We'll assume that we also know that

$$a_1 < F(x,y) < a_2 < H, \quad (x,y) \in \Pi, \quad (3)$$

$$\Pi = \{(x,y) :\, 0 < x < l_x,\, 0 < y < l_y\}. \quad (4)$$

The domain $D(F,\infty)$ contains heat sources with a time-independent density function ρ, creating a stationary (harmonic) temperature distribution in the body. We associate the density function of heat sources with the anomalies under study. We assume that on side faces Γ of the cylindrical domain $D(F,\infty)$ a temperature equal to zero is maintained, and on the surface S of the form (2) there is convective heat exchange with the external environment of temperature U_0, described by Newton's law, according to which the density of the heat flux at the point of the surface S it is directly proportional to the temperature difference inside and outside.

It should be borne in mind from a physical point of view that despite the fact that the density of sources does not depend on time, the heat released by them is diverted across the boundary, the overall temperature distribution does not change over time, although the distribution gradient corresponds to stationary heat flows.

In the domain $D(F,\infty)$ of the form (1), the temperature distribution is the solution of a mixed boundary value problem for the Laplace equation

$$\begin{cases} \Delta u(M) = \rho(M), \quad M \in D(F,\infty), \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - u)\Big|_S, \\ u|_\Gamma = 0, \\ u \text{ is bounded at } \ z \to \infty. \end{cases} \tag{5}$$

We assume that the function ρ is such that the solution of the problem (5) exists in $C^2(D(F,\infty)) \bigcap C^1(\overline{D(F,\infty)})$. In particular, the solution of the problem (5) allows us to find $u|_S$, i.e. the temperature distribution u on the surface of S, which we will call a thermogram.

Now let the thermogram be obtained as a result of measurements and the density of ρ is unknown. Let us now set the inverse problem. We set the problem of continuation of the temperature distribution from the surface towards the sources in order to obtain an adjusted thermogram as the temperature distribution $u|_{z=H}$ on the plane $z = H$, closer to the density carrier than the surface S. The plane $z = H$ is related to the surface S by the condition (3).

We assume that the carrier of the function ρ is located in the domain $z > H$, then the solution of the problem (5) in the domain

$$D(F,H) = \{(x,y,z) : \, 0 < x < l_x, \, 0 < y < l_y, \, F(x,y) < z < H\} \tag{6}$$

satisfies the Laplace equation. The set of side faces of the domain $D(F,H)$ is denoted by Γ_H.

Inverse problem. Let the function be given within the framework of the model (5)

$$f = u|_S, \tag{7}$$

and the density of ρ is unknown. It is required to find $u|_{z=H}$. It is required to find $u|_{z=H}$.

Since the value of H sufficiently arbitrarily defines the plane between the support of ρ and the surface S, then in fact the inverse problem consists in obtaining a solution u in the domain $D(F,H)$ (6) of the boundary value problem

$$\begin{cases} \Delta u(M) = 0, \quad M \in D(F,H), \\ u|_S = f, \\ \dfrac{\partial u}{\partial n}\Big|_S = h(U_0 - f)\Big|_S, \\ u|_{\Gamma_H} = 0. \end{cases} \tag{8}$$

We assume that the function f in (7), (8) is taken from the set of solutions to the direct problem (5), so the solution to the inverse problem exists in $C^2(D(F,H)) \bigcap C^1(\overline{D(F,H)})$.

Note that in the problem (8) on the surface S of the form (2), Cauchy conditions are set, that is, the boundary values f of the desired function u and the values of its normal derivative are set, so the problem (8) has a unique solution. The boundary $z = H$ of the domain $D(F, H)$ is free and, thus, the problem (8) is unstable with respect to errors in the data, i.e. ill-posed.

The function $u|_{z=H}$ will be considered as an adjusted thermogram. Since the plane $z = H$ is located closer to the support of density ρ, it should be expected that the corrected thermogram more accurately conveys information about the distribution of heat sources than the original thermogram.

3. Approximate solution of the inverse problem.

Let the function f in the problem (8) be given with an error, that is, instead of f, the function f^δ is given, so that

$$\|f^\delta - f\|_{L_2(\Pi)} \leqslant \delta.$$

In [6], an approximate solution of the ill-posed problem (8) is constructed in the form

$$u_\alpha^\delta(M) = v_\alpha^\delta(M) + \Phi^\delta(M), \quad M \in D(F, H), \tag{9}$$

where function (integral over a rectangle Π of the form (4)))

$$\Phi^\delta(M) = \int\limits_{\Pi} \Big[h(U_0 - f^\delta(x_P, y_P))\varphi(M, P)\big|_{P\in S} n_1(x_P, y_P) - \\ - f^\delta(x_P, y_P)(\mathbf{n}_1, \nabla_P \varphi(M, P))\big|_{P\in S}\Big] dx_P dy_P \tag{10}$$

is calculated using the problem (8) data, the Dirichlet problem source function

$$\varphi(M, P) = \frac{2}{l_x l_y} \sum_{n,m=1}^{\infty} \frac{e^{-k_{nm}|z_M - z_P|}}{k_{nm}} \times \\ \times \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y} \sin\frac{\pi n x_P}{l_x} \sin\frac{\pi m y_P}{l_y}$$

in the cylinder

$$D^\infty = \{(x, y, z):\ 0 < x < l_x,\ 0 < y < l_y,\ -\infty < z < \infty\} \subset \mathbb{R}^3,$$

the normal to the surface S of the form (2)

$$\mathbf{n}_1 = \operatorname{grad}(F(x, y) - z) = \nabla_{xy} F - \mathbf{k}, \quad n_1 = |\mathbf{n}_1|.$$

The function v_α^δ, which is an approximation to the density potential ρ [12] was obtained in [6] using the Tikhonov regularization method [5]

$$v_\alpha^\delta(M) = -\sum_{n,m=1}^{\infty} \frac{\tilde{\Phi}_{nm}^\delta(a)\exp\{k_{nm}(z_M - a)\}}{1 + \alpha\exp\{2k_{nm}(H - a)\}} \sin\frac{\pi n x_M}{l_x} \sin\frac{\pi m y_M}{l_y}, \tag{11}$$

where $\alpha > 0$,

$$k_{nm} = \pi\left(\frac{n^2}{l_x^2} + \frac{m^2}{l_y^2}\right)^{1/2},$$

$\tilde{\Phi}^{\delta}_{nm}(a)$ — Fourier coefficients of the function $\Phi^{\delta}(M)$ of the form (10)

$$\tilde{\Phi}^{\delta}_{nm}(a) = \frac{4}{l_x l_y}\int\limits_{\Pi} \Phi^{\delta}(x,y,a)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy, \quad a < a_1. \tag{12}$$

For the Fourier coefficients $\tilde{\Phi}^{\delta}_{nm}(a)$ in [11] the expression

$$\tilde{\Phi}^{\delta}_{nm}(a) = \tilde{\Phi}^{\delta}_{1,nm}(a) + \tilde{\Phi}^{\delta}_{2,nm}(a) \tag{13}$$

is obtained, where

$$\tilde{\Phi}^{\delta}_{1,nm}(a) = \frac{4}{l_x l_y}\int\limits_{\Pi}\Big[h(U_0 - f^{\delta}(x,y))\times \\ \times\frac{e^{-k_{nm}(F(x,y)-a)}}{2k_{nm}}n_1(x,y)\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}\Big]dxdy, \tag{14}$$

$$\tilde{\Phi}^{\delta}_{2,nm}(a) = \\ = \frac{4}{l_x l_y}\int\limits_{\Pi} f^{\delta}(x,y)\frac{\pi n e^{-k_{nm}(F(x,y)-a)}}{2l_x k_{nm}}F'_x(x,y)\cos\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy+ \\ + \frac{4}{l_x l_y}\int\limits_{\Pi} f^{\delta}(x,y)\frac{\pi m e^{-k_{nm}(F(x,y)-a)}}{2l_y k_{nm}}F'_y(x,y)\sin\frac{\pi n x}{l_x}\cos\frac{\pi m y}{l_y}dxdy+ \\ + \frac{2}{l_x l_y}\int\limits_{\Pi} f^{\delta}(x,y)e^{-k_{nm}(F(x,y)-a)}\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}dxdy. \tag{15}$$

Thus, the Fourier coefficients $\tilde{\Phi}^{\delta}_{nm}(a)$ are calculated as the sum of formally calculated Fourier coefficients in accordance with (12) over orthogonal systems

$$\left\{\sin\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}\right\}_{n,m=1}^{\infty}, \quad \left\{\cos\frac{\pi n x}{l_x}\sin\frac{\pi m y}{l_y}\right\}_{n,m=1}^{\infty}, \\ \left\{\sin\frac{\pi n x}{l_x}\cos\frac{\pi m y}{l_y}\right\}_{n,m=1}^{\infty}, \tag{16}$$

of functions depending, apart from the arguments x and y, on the number nm of the Fourier coefficients.

4. Formation of an approximate solution based on discrete Fourier series

When discretizing the [13] problem (8) and performing numerical calculations using the formulas (11), (13), (14), (15) it is natural to pass to calculating the values of the approximate solution of the problem (8) on the grid of the values of the arguments x and y

$$\omega = \left\{ (x_i, y_j) : x_i = \frac{il_x}{N_x},\ i = 0, 1, \dots, N_x,\ y_j = \frac{jl_y}{N_y},\ j = 0, 1, \dots, N_y \right\}. \quad (17)$$

In this case, there is no need to use infinite Fourier series. One can pass to discrete Fourier series [9, 10], in this case two-dimensional.

The discrete Fourier series has an interpolation property, that is, the discrete Fourier series (by definition, representing a finite sum) with coefficients calculated by the corresponding formulas coincides on the grid with the values of the function. For example, if on the grid

$$x_i = i\frac{l}{N}, \quad i = 0, 1, \dots, N, \quad (18)$$

the grid function $f = (f_0, f_1, \dots, f_{N-1}, f_N)$ is given (when expanding into a discrete Fourier series in terms of sines, we assume that $f_0 = f_N = 0$). Then the function f can be represented by a discrete Fourier series in terms of sines [10]

$$f_i = \sum_{i=1}^{N-1} b_k \sin\frac{\pi k x_i}{l} = \sum_{i=1}^{N-1} b_k \sin\frac{\pi k i}{N}, \quad i = 0, 1, \dots, N, \quad (19)$$

where the coefficients b_k are calculated by the formula (equivalent to the trapezoid formula for the corresponding integral in the theory of Fourier series):

$$b_k = \frac{2}{N}\sum_{i=1}^{N-1} f_i \sin\frac{\pi k i}{N}, \quad k = 1, \dots, N-1. \quad (20)$$

In other words, if the discrete series coefficients are calculated in accordance with the formula (20), then the discrete series (19) is exactly equal to the values of the function f_i, $i = 0, 1, \dots, N$.

Applying discrete Fourier series to the approximate solution (9) on the grid (17) for each fixed z, $a_2 < z < H$, will lead to the formula for v_α^δ:

$$\left(v_\alpha^\delta\right)_{ij}(z) = -\sum_{m=1}^{N_y-1}\sum_{n=1}^{N_x-1} \frac{\tilde{\Phi}_{nm}^\delta(a)\exp\{k_{nm}(z-a)\}}{1+\alpha\exp\{2k_{nm}(H-a)\}} \sin\frac{\pi n i}{N_x} \sin\frac{\pi m j}{N_y}, \quad (21)$$
$$i = 0, \dots, N_x, \quad j = 0, \dots, N_y.$$

In this case, the integrals in calculating $\tilde{\Phi}_{nm}^\delta(a)$ by the formulas (13), (15), (14) it is natural to replace with formulas corresponding to the calculation of the coefficients of the discrete Fourier series of the form

$$\tilde{\Phi}^{\delta}_{nm}(a) = \frac{4}{N_x N_y}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} f_{ij}(n,m)\sin\frac{\pi n i}{N_x}\sin\frac{\pi m j}{N_y}+ \\ + \frac{4}{N_x N_y}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} g_{ij}(n,m)\cos\frac{\pi n i}{N_x}\sin\frac{\pi m j}{N_y}+ \\ + \frac{4}{N_x N_y}\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1} p_{ij}(n,m)\sin\frac{\pi n i}{N_x}\cos\frac{\pi m j}{N_y}, \quad (22)$$

($n = 1, 2, \dots, N_x - 1$, $m = 1, 2, \dots, N_y - 1$) on the grid ω of the form (17). To simplify the notation of integrands in integrals corresponding to systems (16), the notation f, g, p is introduced. A feature of calculating the coefficients of a discrete series in this case is that the functions f, g, p, in addition to the arguments x_i and y_j, depend on the indices n and m of the Fourier coefficients.

5. Summation of a discrete Fourier series and calculation of its coefficients by the Hamming method

Here we give some modification of the Hamming method [9] and its proof, related to the representation of a function as a discrete Fourier series in terms of sines or cosines on the interval $[0, l]$.

We now assume that the coefficients of the discrete series of some grid function w

$$w_i = \sum_{k=1}^{N-1} b_k \sin\frac{\pi k i}{N}, \quad i = 0, 1, \dots, N \quad (23)$$

are calculated formally in accordance with (20), where the values of the function f formally depend on the number k of the coefficient b_k, i.e.

$$b_k = \frac{2}{N}\sum_{i=1}^{N-1} f_i(k)\sin\frac{\pi k i}{N} = \frac{2}{N}\sum_{i=0}^{N} f_i(k)\sin\frac{\pi k i}{N}, \quad k = 1, \dots, N-1 \quad (24)$$

while maintaining the condition

$$f_0(k) = f_N(k) = 0. \quad (25)$$

Let us show that the idea of Hamming algorithm [9] for calculating coefficients b_k of a discrete Fourier series (23) is also applicable to this situation, that is, to calculating the sum (24).

We fix the number k of the Fourier coefficient. Let us denote for brevity $t_k = \pi k/N$ and consider the recurrent formulas

$$\begin{cases} U_0 = 0, \\ U_1 = f_N(k), \\ U_m = (2\cos t_k)\, U_{m-1} - U_{m-2} + f_{N-m+1}(k), \quad m = 2, 3, \dots, N. \end{cases} \quad (26)$$

Our task is to show that the Fourier coefficients b_k of the form (24) of the function w of the form (23) can be calculated by the formula

$$b_k = \frac{2}{N}\sum_{i=0}^{N} f_i(k)\sin\frac{\pi k i}{N} = \frac{2}{N}U_N \sin t_k, \tag{27}$$

where U_N is calculated by recurrent formulas (26).

For simplicity of notation, the dependence of U_m on k is not indicated. Note also that the recurrent formulas (26) use the values $f_N(k), \dots, f_1(k)$, the value $f_0(k)$ is not used when forming U_N.

Algorithm (26), (27) obviously allows to avoid calculation of sines in (24) with argument $\pi k i/N$ when changing indices i and k.

Let us represent the function $f(k)$ as a sum of functions $f^{(i)}(k)$, each of which is a vector with zero coordinates, except for the coordinate with number i equal to $f_i(k)$, that generally speaking, not equal to zero:

$$\begin{gathered} f(k) = \sum_{i=0}^{N} f^{(i)}(k), \\ f^{(i)}(k) = \left(f_0^{(i)}(k), \dots, f_N^{(i)}(k)\right) = (0, 0, \dots, 0, f_i(k), 0, \dots, 0)\,. \end{gathered} \tag{28}$$

Note that if the upper and lower indices do not coincide, the coordinate of the function $f^{(i)}(k)$ is equal to zero. Note also that when the grid function is represented by a sine series, due to (25) $f^{(0)}(k) = f^{(N)}(k) = (0, 0, \dots, 0)$.

We apply the recursive formulas (26) to each function $f^{(i)}(k)$, $i = 0, 1, 2, \dots, N$ (for a fixed k), denoting result as $U^{(i)}$:

$$\begin{cases} U_0^{(0)} = 0, \\ U_1^{(0)} = f_N^{(0)}(k), \\ U_m^{(0)} = 2\cos t_k U_{m-1}^{(0)} - U_{m-2}^{(0)} + f_{N-m+1}^{(0)}(k), \quad m = 2, 3, \dots, N, \end{cases}$$

$$\begin{cases} U_0^{(1)} = 0, \\ U_1^{(1)} = f_N^{(1)}(k), \\ U_m^{(1)} = 2\cos t_k U_{m-1}^{(1)} - U_{m-2}^{(1)} + f_{N-m+1}^{(1)}(k), \quad m = 2, 3, \dots, N, \end{cases}$$

$\dots$

$$\begin{cases} U_0^{(i)} = 0, \\ U_1^{(i)} = f_N^{(i)}(k), \\ U_m^{(i)} = 2\cos t_k U_{m-1}^{(i)} - U_{m-2}^{(i)} + f_{N-m+1}^{(i)}(k), \quad m = 2, 3, \dots, N, \end{cases}$$

$\dots$

$$\begin{cases} U_0^{(N)} = 0, \\ U_1^{(N)} = f_N^{(N)}(k), \\ U_m^{(N)} = 2\cos t_k U_{m-1}^{(N)} - U_{m-2}^{(N)} + f_{N-m+1}^{(N)}(k), \quad m = 2, 3, \dots, N. \end{cases}$$

Summing up the corresponding parts of all equalities, due to the linearity of the recursive formulas, we obtain

$$\begin{cases} \sum_{i=0}^{N} U_0^{(i)} = 0, \\ \sum_{i=0}^{N} U_1^{(i)} = \sum_{i=0}^{N} f_N^{(i)}(k), \\ \sum_{i=0}^{N} U_m^{(i)} = 2\cos t_k \sum_{i=0}^{N} U_{m-1}^{(i)} - \sum_{i=0}^{N} U_{m-2}^{(i)} + \sum_{i=0}^{N} f_{N-m+1}^{(i)}(k), \quad m = 2, 3, \dots, N. \end{cases}$$

Considering (28) for the sum $f^{(i)}(k)$ and taking the notation

$$\sum_{i=0}^{N} U_m^{(i)} = U_m, \quad m = 0, \dots, N, \tag{29}$$

we obtain the recurrent formulas (26). Thus, to prove the formula (27), it suffices to calculate $U_N^{(i)}$ for all $i = 0, \dots, N$.

Let us calculate $U_N^{(i)}$, singling out the cases $i = 0, 1, N$ separately. Applying the recurrent formulas (26) to $f^{(i)}(k)$ for $i = 0, 1$ gives

$$\begin{cases} U_0^{(0)} = 0, \\ U_1^{(0)} = f_N^{(0)}(k) = 0, \\ U_2^{(0)} = f_{N-1}^{(0)}(k) = 0, \\ \dots \\ U_{N-1}^{(0)} = f_2^{(0)}(k) = 0, \\ U_N^{(0)} = f_1^{(0)}(k) = 0 = V_0^{(0)}, \end{cases} \qquad \begin{cases} U_0^{(1)} = 0, \\ U_1^{(1)} = f_N^{(1)}(k) = 0, \\ U_2^{(1)} = f_{N-1}^{(1)}(k) = 0, \\ \dots \\ U_{N-1}^{(1)} = f_2^{(1)}(k) = 0 = V_0^{(1)}, \\ U_N^{(1)} = f_1^{(1)}(k) = f_1(k) = V_1^{(1)}. \end{cases} \tag{30}$$

Applying the recurrent formulas (26) to $f^{(i)}(k)$ for $i = N$ gives

$$\begin{cases} U_0^{(N)} = 0 = V_0^{(N)}, \\ U_1^{(N)} = f_N^{(N)}(k) = f_N(k) = V_1^{(N)}, \\ \dots \\ U_m^{(N)} = (2\cos t_k) V_{m-1}^{(N)} - V_{m-2}^{(N)}, \quad m = 2, 3, \dots, N. \end{cases} \tag{31}$$

Note that for all values of $m = 2, 3, \dots, N$ in (31) the value $f_{N-m+1}^{(N)}(k) = 0$, since the upper index is not equal to the lower one.

Applying the recurrent formulas (26) to $f^{(i)}(k)$ for $i = 2, \dots, N-1$ gives

$$\begin{cases} U_0^{(i)} = 0, \\ \dots \\ U_{N-i}^{(i)} = f_{N-(N-i)+1}^{(i)}(k) = f_{i+1}^{(i)}(k) = 0 = V_0^{(i)}, \\ U_{N-i+1}^{(i)} = f_{N-(N-i+1)+1}^{(i)}(k) = f_i^{(i)}(k) = f_i(k) = V_1^{(i)}, \\ U_{N-i+m}^{(i)} = (2\cos t_k)V_{m-1}^{(i)} - V_{m-2}^{(i)}, \quad m = 2, 3, \dots, i. \end{cases} \tag{32}$$

Here we took into account that $f_{N-(N-i+m)+1}^{(i)}(k) = f_{i+1-m}^{(i)}(k) = 0$, $m = 2, 3, \dots, i$, because the upper and lower indices do not match.

In the formulas (30), (31), (32) we introduced the notation

$$U_{N-i+m}^{(i)} = V_m^{(i)}, \quad i = 0, \dots, N, \quad m = 0, 1, 2, \dots, i. \tag{33}$$

It is easy to see that quantities $V_m^{(i)}$ are calculated using the same formulas (26), "skipping" the first $N - i - 1$ zeros for $U_m^{(i)}$. The introduction of the quantity $V_m^{(i)}$ allows us to obtain an explicit formula for it:

$$V_m^{(i)} = f_i(k)\frac{\sin mt_k}{\sin t_k}, \quad m = 0, 1, \dots, i, \tag{34}$$

for each $i = 0, 1, \dots, N$.

Let us prove it using the method of induction for m. As follows from (30), (32), this equality holds for $m = 0, 1$. Let's prove it for $m = 0, 1, \dots, i$. Let equality (34) hold for $m-2$ and $m-1$. Let's prove for m.

$$\begin{aligned} V_m^{(i)} &= (2\cos t_k)V_{m-1}^{(i)} - V_{m-2}^{(i)} = \\ &= f_i(k)\left[2\cos t_k \frac{\sin(m-1)t_k}{\sin t_k} - \frac{\sin(m-2)t_k}{\sin t_k}\right] = \\ &= f_i(k)\left[\frac{\sin mt_k + \sin(m-2)t_k - \sin(m-2)t_k}{\sin t_k}\right] = f_i(k)\frac{\sin mt_k}{\sin t_k}. \end{aligned}$$

Note that from (33), in particular, $U_N^{(i)} = V_i^{(i)}$ and from the formula (34) for $m = i$ we obtain

$$U_N^{(i)} = V_i^{(i)} = f_i(k)\frac{\sin it_k}{\sin t_k}. \tag{35}$$

Summing according to (29) with $m = N$, and using (24), we obtain

$$U_N = \sum_{i=0}^{N} U_N^{(i)} = \sum_{i=0}^{N} V_i^{(i)} = \frac{1}{\sin t_k}\sum_{i=0}^{N} f_i(k)\sin it_k = b_k \frac{1}{\sin t_k}\frac{N}{2}.$$

From here we obtain the formula (27).

Let us now proceed to calculating the coefficients of the discrete Fourier series of the grid function w in cosine expansion. On the same grid (18) consider the discrete Fourier series in cosines expansion [10]

$$w_i = \frac{a_0}{2} + \sum_{k=1}^{N-1} a_k \cos\frac{\pi k i}{N} + \frac{a_N}{2}(-1)^i, \quad i = 0, 1, \dots, N, \tag{36}$$

whose coefficients a_k are calculated on the basis of the grid function f depending on the number k by the formula

$$a_k = \frac{2}{N}\left(\frac{f_0(k)}{2} + \sum_{i=1}^{N-1} f_i(k)\cos\frac{\pi k i}{N} + \frac{f_N(k)}{2}(-1)^k\right). \tag{37}$$

We introduce the grid function

$$\tilde{f}(k) = \left(\frac{f_0(k)}{2}, f_1(k), f_2(k), \dots, f_{N-1}(k), \frac{f_N(k)}{2}\right), \tag{38}$$

that differs from f in that the current and last coordinates are divided in half. Replacing f in (26) with $\tilde{f}$, we get formulas (33), (34).

Note that the recurrent formulas (30) for $i = 0$ imply $U_N^{(0)} = 0$, $U_{N-1}^{(0)} = 0$. For $i = 1, 2, \dots, N$ from (33), (34) we obtain

$$U_{N-1}^{(i)} = V_{i-1}^{(i)} = \tilde{f}_i(k)\frac{\sin(i-1)t_k}{\sin t_k}, \quad i = 1, 2, \dots, N. \tag{39}$$

Consider the following construction (summation starts from $i = 1$, since $U_N^{(0)} = U_{N-1}^{(0)} = 0$):

$$\cos t_k U_N - U_{N-1} + \tilde{f}_0(k) = \cos t_k \sum_{i=1}^{N} U_N^{(i)} - \sum_{i=1}^{N} U_{N-1}^{(i)} + \tilde{f}_0(k) =$$
$$= \sum_{i=1}^{N}\left[\cos t_k U_N^{(i)} - U_{N-1}^{(i)}\right] + \tilde{f}_0(k).$$

Replacing $U_N^{(i)}$ and $U_{N-1}^{(i)}$ according to formulas (35) and (39), we obtain

$$\cos t_k U_N - U_{N-1} + \tilde{f}_0(k) = \sum_{i=1}^{N}\left[\cos t_k V_i^{(i)} - V_{i-1}^{(i)}\right] + \tilde{f}_0(k) =$$
$$= \sum_{i=1}^{N} \tilde{f}_i(k)\frac{\cos t_k \sin i t_k - \sin(i-1)t_k}{\sin t_k} + \tilde{f}_0(k) =$$
$$= \sum_{i=1}^{N} \tilde{f}_i(k)\frac{\cos t_k \sin i t_k - \cos t_k \sin i t_k + \sin t_k \cos i t_k}{\sin t_k} + \tilde{f}_0(k) =$$

$$= \sum_{i=1}^{N} \tilde{f}_i(k)\cos it_k + \tilde{f}_0(k) = \tilde{f}_0(k) + \sum_{i=1}^{N-1} \tilde{f}_i(k)\cos it_k + \tilde{f}_N(k)\cos Nt_k =$$

$$= \frac{f_0(k)}{2} + \sum_{i=1}^{N-1} f_i(k)\cos it_k + \frac{f_N(k)}{2}(-1)^k.$$

From here according to (37) we obtain a formula for the coefficients of the discrete Fourier series in the cosine expansion

$$a_k = \frac{2}{N}\left[\cos t_k U_N - U_{N-1} + \tilde{f}_0(k)\right] = \frac{2}{N}\left[\cos t_k U_N - U_{N-1} + \frac{f_0(k)}{2}\right], \quad (40)$$

moreover, the quantities U_N and U_{N-1} are calculated by the formulas (26), in which the grid function f is replaced by $\tilde{f}$, which is related to f by the formula (38).

Note that the formulas (26), (27) can also be used to sum the Fourier series (23), since the formulas (24) and (23) differ only by a factor. Note that when summing the series (23), we, of course, do not obtain $f(k)$, but we obtain some function w. Accordingly, the formulas (26), (40) can also be used to sum the Fourier series (36), since the formulas (37) and (36) differ only by the multiplier.

The formulas (26), (27), (40) can be used for two-dimensional discrete Fourier series both for calculating the coefficients and for summing the Fourier series (21). In this case, the formulas (26), (27) for each fixed pair nm are applied sequentially over each index i and j corresponding to the variables x and y.

The formulas (26), (40) for calculating the coefficients of the discrete Fourier series in cosine expansion, of course, are also valid in the case when $f_0 = f_N = 0$, which corresponds to the formulas (22).

6. Conclusion and discussion

Formulas (21), (22), (26), (27), (40) as a solution to the problem (8) can be used for mathematical processing of thermograms taken with a thermal imager in medicine [4] in order to correct the image on the thermogram. Note that taking into account the influence of blood flow leads to the need to use the metaharmonic equation [14], [15] in problem (8).

The thermogram, with one or another certainty, conveys an image of the structure of heat sources inside the body. However, within the framework of the task (8), the image on the thermogram can be refined. In this case, we consider the function f^δ as the original thermogram, and the function $v_\alpha^\delta|_{z=H}$ as the corrected thermogram. Since the function $v_\alpha^\delta|_{z=H}$ is the temperature distribution on a plane closer to the investigated heat sources than the original surface S, we can expect a more accurate reproduction of the source image on the calculated thermogram $v_\alpha^\delta|_{z=H}$.

The results of calculations performed on a model example show the effectiveness of the proposed method and algorithm based on the formulas (9), (10), (11), (13) that can be used for processing thermal images.

Note that the method of summation of discrete Fourier series, described in Section 5, can be used to solve other problems, the solutions of which can be obtained in the form of Fourier series in terms of eigenfunctions of the Laplace operator in a rectangle.

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Об одной модификации метода Хемминга суммирования дискретных рядов Фурье и её применение для решения задачи коррекции термографических изображений

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Аннотация. В работе рассматриваются математические методы коррекции термографических изображений (термограмм), полученных с помощью тепловизора, в виде распределения температуры на поверхности исследуемого объекта. Термограмма воспроизводит изображение тепловыделяющих структур, расположенных внутри исследуемого объекта. Это изображение передаётся с искажениями, так как источники, как правило, удалены от его поверхности и распределение температуры на поверхности объекта передаёт изображение как размытое за счёт процессов теплопроводности и теплопереноса. В работе в качестве принципа коррекции рассматривается продолжение функции температуры как гармонической функции с поверхности вглубь исследуемого объекта с целью получения функции распределения температуры вблизи источников. Такое распределение рассматривается как скорректированная термограмма. Продолжение функции температуры осуществляется на основе решения задачи Коши для уравнения Лапласа — некорректно поставленной задачи. Построение решения проводится с использованием метода регуляризации Тихонова. Основная часть построенного приближённого решения представлена в виде ряда Фурье по собственным функциям оператора Лапласа. Дискретизация задачи приводит к дискретным рядам Фурье. Для суммирования рядов Фурье и вычисления коэффициентов в работе предложена модификация метода Хемминга.

Ключевые слова: термограмма, некорректная задача, задача Коши для уравнения Лапласа, метод регуляризации Тихонова, дискретный ряд Фурье