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Analytical Approach to Analysis of Extremal Trajectories and Stability of Programmed Motion

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Analytical approach to analysis of optimal trajectories of a rocket and method of determining control actions for stable programmed motion are considered. It is shown that the analytical trajectory solutions can be used to test the existence of conjugate points on extremals. Control laws that can provide stable motion are discussed.

Key words and phrases: variation problem, autonomous space guidance, rocket, Lagrange multipliers, Riccati equation, Jacobi condition, conjugate points, stable motion, programmed motion, control.

1. Introduction

Analytical approach to the solution of an optimal control problem and method of determining control actions providing stable programmed motion of a rocket are considered [1]. Practical value of the problem is associated with the design and realization of a best autonomous space guidance, one of the prioritized problems of space flight and depends on the characteristics of the trajectory solutions [2, 3]. Numerically integrated trajectory solutions used in guidance problem are very sensitive to the initial conditions and do not always allow for the design of simple and reliable laws of autonomous guidance due to existence of convergence problems, unknown initial Lagrange multipliers and unknown sequence of thrust arcs on the trajectory [4]. Therefore, for the successful solution of the guidance problem, it is suggested to develop an analytical approach to the optimal control problem which would allow us to design a nominal trajectory without the convergence problems and the uncertainties mentioned above. In this paper, the optimal control problem is formulated, the first and second differentials of the performance index are analyzed, the condition of finiteness of the solutions to the Riccati equation and Jacobi condition on conjugate points are considered [5]. The classes of optimal thrust arcs are determined by the Legendre condition tests. The presented analytical method can serve as a tool of extracting the reference trajectories for the guidance problem.

Ref. [6] contains the solution to the problem of determining the mass law for a point which corresponds to a motion according to a given law or trajectory. Various statements for the inverse problems of dynamics of mechanical systems and methods of stabilization of constraints have been investigated in Ref. [7–11]. Reactive forces generated due to the change of rocket's mass and exhaust velocity allow us to realize its motion corresponding to the solution of Mayer's variation problem and to provide stability of motion with respect to the trajectory or law of motion [8].

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2. Optimal Control Problem

Let the center of mass (CM) of a spacecraft at any time can be determined by vector-function $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{x}(t) \in \Re^{(n)}$, the components of which are assumed continuous and absolutely differentiable on a time interval $[t_0, t_1]$, but their derivatives may have discontinuities. Here t_0 and t_1 are the initial and final times of motion. Then the equations of motion are given as

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_k), \tag{1}$$

The vector-function $\mathbf{u} = (u_1, \ldots, u_k)$, $\mathbf{u}(t) \in \Re^{(k)}$ is called as control vector, and its components u_r $(r = 1, ..., k)$ are defined on $[t_0, t_1]$ and considered piecewise continuous functions [12]. The functions f_i possess continuous partial derivatives of sufficiently high order with respect to all components of $\mathbf{x}$ and $\mathbf{u}$. Assume that the following equations are satisfied:

$$\Psi_l(x_{01}, x_{02}, ..., x_{0n}) = 0, \quad l = 1, ..., q_1, \quad q_1 \leqslant n. \tag{2}$$

$$F_m(x_{11}, x_{12}, ..., x_{1n}, t_1) = 0, \quad m = 1, ..., q_2, \quad q_2 < n + 1. \tag{3}$$

$$\Phi_s(u_1, u_2, ..., u_k, \alpha_1, \alpha_2, ..., \alpha_d) = 0, \quad s = 1, ..., p < k \quad d \leqslant k, \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d)$ and $\boldsymbol{\alpha} \in \Re^{(d)}$ are considered as auxiliary controls. Here and below the subscripts “0” and “1” will mean initial and final values of the variables. It is required to find $\mathbf{x}(t)$ and $\mathbf{u}(t)$ so, that (1-4) are satisfied, and the functional

$$J(x_{1,q_2+1}, x_{1,q_2+2}, ..., x_{1,n}, t_1) + \int\limits_{t_0}^{t_1} g(x, u, t)\mathrm{d}t. \tag{5}$$

is minimized. All functions Ψ_l, F_m, Φ_s, J and g are continuous and possess continuous partial derivatives of sufficiently high order with respect to all their components.

3. Differentials of Extended Functional

Consider the extended functional of the form:

$$K(\mathbf{x}_0, \mathbf{x}_1, \mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\alpha}, t_1) = G + \int\limits_{t_0}^{t_1} [H - \boldsymbol{\lambda}^T \dot{\mathbf{x}}]\mathrm{d}t. \tag{6}$$

where

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{\gamma}, t) = \boldsymbol{\lambda}^T \mathbf{f} + \boldsymbol{\gamma}^T \boldsymbol{\Phi} + g, \tag{7}$$

$$G(\mathbf{x}_0, \mathbf{x}_1, \boldsymbol{\mu}, \boldsymbol{\nu}, t_1) = J + \boldsymbol{\mu}^T \boldsymbol{\Psi} + \boldsymbol{\nu}^T \mathbf{F}, \tag{8}$$

$$\mathbf{u} = (u_1, u_2, ..., u_{k+d}), \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_d),$$

$$\boldsymbol{\Psi} = (\Psi_1, \Psi_2, ..., \Psi_{q_1}), \quad \mathbf{F} = (F_1, F_2, ..., F_{q_2}), \quad \boldsymbol{\Phi} = (\Phi_1, \Phi_2, ..., \Phi_p),$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_{q_1}), \quad \boldsymbol{\nu} = (\nu_1, \nu_2, ..., \nu_{q_2}), \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_p),$$

the vectors $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ and $\boldsymbol{\gamma}$ are considered as unknown multipliers, and $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are assumed to be constants.

It can be shown that by accepting the notation $\frac{\partial(\cdot)}{\partial x} = (\cdot)_x$, the first and second differentials of the extended functional (6) are of the form:

$$\mathrm{d}K = (G_{x_0} + \boldsymbol{\lambda}_0^T)\mathrm{d}x_0 + (G_{x_1} - \boldsymbol{\lambda}_1^T)\mathrm{d}x_1 + (G_{t_1} + H_1)\mathrm{d}t_1 +$$

$$\int_{t_0}^{t_1} [(H_x + \dot{\boldsymbol{\lambda}}^T)\delta\mathbf{x} + H_u\delta\mathbf{u} + (H_\lambda - \dot{\mathbf{x}}^T)\delta\boldsymbol{\lambda}]\mathrm{d}t. \quad (9)$$

$$\mathrm{d}^2K = [\delta\mathbf{x}_1^T \quad \mathrm{d}t_1]\begin{bmatrix} G_{x_1x_1} & \Omega_{x_1}^T \\ \Omega_{x_1} & \Omega' \end{bmatrix}\begin{bmatrix} \delta\mathbf{x}_1 \\ \mathrm{d}t_1 \end{bmatrix} + \delta\mathbf{x}_0^T G_{x_0x_0}\delta\mathbf{x}_0 + \\ + \int_{t_0}^{t_1} [\delta\mathbf{x}^T \quad \delta\mathbf{u}^T]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{u} \end{bmatrix}, \quad (10)$$

where

$$\Omega = G_{t_1} + g + \boldsymbol{\gamma}^T\boldsymbol{\Phi} + G_{x_1}\mathbf{f}, \quad \Omega' = \Omega_{t_1} + \Psi_{x_1}\dot{\mathbf{x}}_1.$$

Analysis of the condition, $\mathrm{d}K = 0$ allow us to obtain the first-order necessary conditions of optimality for weak extremals:

$$\dot{\mathbf{x}}^T = H_\lambda, \quad \dot{\boldsymbol{\lambda}}^T = -H_x, \quad (11)$$

$$H_u = 0, \quad H_\alpha = 0, \quad (12)$$

$$\boldsymbol{\Psi} = 0, \quad \mathbf{F} = 0, \quad \boldsymbol{\lambda}_0 = -G_{x_0}^T, \quad \boldsymbol{\lambda}_1 = G_{x_1}^T, \quad H_1 = -G_{t_1}. \quad (13)$$

These conditions can be used to determine $2n + k + d$ unknowns x_i, λ_i, u_r $(r = 1, ..., k+d)$ together with $2n$ constants. $2n$ constants, q_1 variables μ_l $(l = 1, ..., q_1)$, q_2 variables $\nu_m (m = 1, ..., q_2)$ and the time t_1 can be determined using q_1 conditions (2), q_2 conditions (3) and $2n+1$ conditions (13). Besides that, if $\mathbf{x}$, $\mathbf{u}$ represent an optimal trajectory, then the Weierstrass and Legendre–Clebsch conditions are satisfied on this trajectory [13]:

$$H(\mathbf{x}, \tilde{\mathbf{u}}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, t) \leqslant H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, t), \quad H_{uu} \geqslant q0, \quad (14)$$

where $\tilde{\mathbf{u}}$ is an admissible control vector [1].

Using the analysis of the auxiliary optimization problem, it can be shown that d^2K is positive definite if the conditions (11), (12), (13), $H_{uu} > 0$ and $\mathbf{D} > 0$ are satisfied, and the matrix $\bar{R}$ is finite on $[t_0, t_1)$. Here

$$\mathbf{D} = [d_{ab}], \quad [d_{ij}] = [\Xi]_{i,j}, \quad [d_{qr}] = [\bar{R} + G_{x_0x_0}]_{q,r},$$

$$\boldsymbol{\Xi} = -\boldsymbol{\Xi}_2^{-1}\boldsymbol{\Xi}_1, \quad \boldsymbol{\Xi}_1 = \frac{\partial\boldsymbol{\Psi}}{\partial q_s}, \quad \Xi_2 = \frac{\partial\boldsymbol{\Psi}}{\partial q_k}, \quad det[\boldsymbol{\Xi}_2] \neq 0,$$

$a,\ b = 1, 2, ..., n;\ i,\ j = 1, ..., q_1;\ q,\ r = q+1, ..., n;\ s = q_1+1, ..., n;\ k = 1, ..., q_1$. The elements of the matrix $\bar{R}Q$ are the known functions of the elements of R:

$$\bar{R} = R - V^T Q^{-1} P^T,$$

where $\bar{R}$, V and Q satisfy the Riccati equation and the two conditions [13]:

$$\dot{R} = C - A^T R - RA + RBR, \qquad \dot{V} = (RB - A^T)V^T, \quad \dot{Q} = VBV^T. \quad (15)$$

Here $H_{uu} > 0$, and

$$A = f_x - f_u H_{uu}^{-1} H_{xu}^T, \quad B = f_u H_{uu}^{-1} f_u^T, \quad C = (H_{xu} H_{uu}^{-1} H_{xu}^T - H_{xx}).$$

4. Conjugate Points

Let $\delta \mathbf{x}_0 = 0$ and $H_{uu} > 0$. Then at some time instance, $t = \tau$, $(\tau \in (t_0, t_1])$ the matrix $\bar{R}$ is not finite, then from $\delta \mathbf{x} = \bar{R}^{-1} \delta \boldsymbol{\lambda}$ it follows that $\delta \mathbf{x}(\tau) = 0$. The corresponding trajectory point at τ is said to be conjugate to the trajectory point at t_0 [5, 13]. It was shown that the finiteness of the matrix $\bar{R}$ on $[t_0, t_1]$ means absence of the conjugate points on $(t_0, t_1]$ with respect to the initial point at t_0. The absence of the conjugate points on (t_0, t_1) is known as the classical condition of Jacobi. When $H_{uu} > 0$, the condition $\delta \mathbf{x} \neq 0$ (or $\delta \boldsymbol{\lambda} \neq 0$) allows us to determine the presence of the conjugate points on thrust arcs using the analytical solutions, if such solutions exist.

Let the equation of an extremal, (12) contain $m(\leqslant 2n)$ constants of integration. If c is one of the constants, then its variation provides the family of solutions, $\mathbf{x} = \mathbf{x}(t, c)$, $\quad \boldsymbol{\lambda} = \boldsymbol{\lambda}(t, c)$, $\quad \mathbf{u} = \mathbf{u}(t, c)$. By varying $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, it can be shown that

$$\delta x_i = \frac{\partial x_i}{\partial c}, \quad \delta \lambda_i = \frac{\partial \lambda_i}{\partial c}. \tag{16}$$

As the necessary conditions of optimality, (11), (12) and (13) are linear and of the order $2n$, their solutions on the optimal trajectory can be written by employing the principle of superposition in the following form:

$$\delta x_i = \sum_{j=1}^{2n} N_j \frac{\partial x_i}{\partial c_j}, \quad \delta \lambda_i = \sum_{j=1}^{2n} N_j \frac{\partial \lambda_i}{\partial c_j}, \quad \delta u_r = \sum_{j=1}^{2n} U_j \frac{\partial u_r}{\partial c_j}, \tag{17}$$

where $c_j, (j = 1, ..., m)$ are the constants of integration in the solutions to the problem described in the equations (1)–(5), and $N_j,\ L_j,\ U_j$ are the constants. If the constants c_j are defined in the solution process, then $\delta x_i(c_j, t)$, $\delta \lambda_i(c_j, t)$ and $\delta u_r(c_j, t)$ will represent the analytical solutions of the auxiliary optimization problem, which can satisfy the necessary conditions of optimality. This in turn means that the first two equalities of (17) can be used to determine the presence of the conjugate points on the extremals.

5. On Satisfaction of Legendre–Clebsch Condition

The problem of optimizing the trajectory with limited power may be formulated in the context of the optimal control problem stated above [12]:

$$\dot{\mathbf{v}} = \mathbf{g}(\mathbf{r}) + \frac{2P}{I_{sp} m g_0} \mathbf{e}, \quad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{m} = -\frac{2P}{I_{sp}^2 g_0^2} \tag{18}$$

$$\Phi_1 = e_1^2 + e_2^2 + e_3^2 - 1 = 0, \quad \Phi_2 = P(P_{\max} - P) - \gamma^2 = 0,$$
$$\Phi_3 = (I_{sp,\max} - I_{sp})(I_{sp} - I_{sp,\min}) - \eta^2 = 0, \tag{19}$$

where $\mathbf{g}(\mathbf{r})$ and g_0 are the vector of gravitational acceleration and its magnitude measured on a sea level, $\mathbf{r},\ \mathbf{v},\ m$ are the radius-vector, velocity vector and mass respectively, η and γ auxiliary control variables, I_{sp} is the specific impulse, P is the power, which can be determined by the formulae, $P = \frac{1}{2} \beta I_{sp}^2 g^2$, β is the mass-flow rate. The control vector is given as $\mathbf{u} = [P, I_{sp}, e_1, e_2, e_3, \gamma, \eta]^T$.

As it was shown above, one of the sufficient conditions for $\mathrm{d}^2 K > 0$ is expressed by the conditions of strict positiveness of all main minors of the matrix H_{uu}. Consider the elements of the matrix H_{uu}. If $\mu_2 = 0, \quad \mu_3 \neq 0$, or $\mu_2 \neq 0, \quad \mu_3 = 0$, then $H_{\gamma u} = 0$ or $H_{\eta u} = 0$. This will mean that all main minors of H_{uu} are equal to zero, indicating the extremality of the trajectories for the cases $\gamma \neq 0, \quad \eta = 0$ and $\gamma = 0, \quad \eta \neq 0$.

If $\mu_2 = 0, \quad \mu_3 \neq 0$, then $\gamma \neq 0, \quad \eta = 0$, which corresponds to the case with $0 < P < P_{\max}, I_{sp} = I_{sp,\min}$ or $I_{sp} = I_{sp,\max}$, that is the motion with constant I_{sp} and variable power. If $\mu_2 \neq 0, \quad \mu_3 = 0$, then it can be shown that $\gamma = 0, \quad \eta \neq 0$, meaning that $P = P_{\max}$ or $P = 0$ and $I_{sp,\min} < I_{sp} < I_{sp,\max}$, that is the motion with variable I_{sp} and $P_{\max}$. In both cases $H_{uu} = 0$.

If $\mu_2 \neq 0, \quad \mu_3 \neq 0$, then the strict positiveness of all main minors of H_{uu} is provided by appropriate determination of the multipliers $\boldsymbol{\mu}, \boldsymbol{\lambda}$ and $\mathbf{e}$. It should be noted that the complete solutions of the canonical system of equations corresponding to this case are remaining unknown. From $H_u = 0$ it follows that $\gamma = 0$, $\eta = 0$, which mean a motion with $P = P_{\max}, \quad I_{sp} = I_{sp,\min}$ or $I_{sp} = I_{sp,\max}$ [14]. Consequently, in the optimal control problem given by (2),(3), (5), (18) and (19) with constraints on power and specific impulse, the optimal thrust arcs are those on which $P = P_{\max}$ and $I_{sp} = I_{sp,\min}$ or $I_{sp} = I_{sp,\max}$.

In the case of motion with $P = P_{\max}, \quad I_{sp} = I_{sp,\min}$ or $I_{sp} = I_{sp,\max}$, the following conditions are true: $\beta = const$ and $m = m_0 - \beta t$. It follows from these analysis that the satisfaction of the Legendre–Clebsch condition allows us to obtain optimal thrust arcs, but at the same time, it shrinks the domain of the problem parameters.

6. Class of Extremals with Free Time

The case of motion with $P = P_{\max}$ and $I_{sp} = I_{sp,\min}$ (or $I_{sp} = I_{sp,\max}$) corresponds to a case of a maximum thrust arc. It was shown that the newtonian field can be approximated by a linear central field ($\mathbf{g}(\mathbf{r}) = -k^2\mathbf{r}$), if $\varphi \approx 0$ and $(r - r_0)/r_0 \ll 1$ [3]. Here $k^2 = \mu/r_0^3$, where μ is the gravitational parameter, r_0 is the radius of a reference orbit and r is the radius vector of a center of mass. Assume that these conditions are satisfied. Then it can be shown that if the final time is not fixed ($C = 0$) and $\mathcal{J} = m_0 - m_1$, then from the transversality condition it follows that $\lambda_{\theta 1} = -\frac{\partial \mathcal{J}}{\partial \theta_1} = 0$, where the subscript "1" means the final time. The analysis of the equations (18) show that the condition $\lambda_{\theta 1} = 0$ is associated with the cases of motion, where $\varphi = 0$ and $\varphi \neq 0$.

The first case, where ($\varphi = 0$) corresponds to a motion with tangential thrust, and although it represents a practical interest, it is not considered in this paper. It can be shown that in the second case, $\psi = \psi_0, \quad \lambda = a\sin(kt + \alpha)$, where ψ_0, a, α are the constants. The equality $\dot{\psi} = 0$ means that the hodograph of the basis-vector is the straight line and the thrust direction is inertially fixed.

In the case of a free flight time and it is required to minimize the final mass, the analytical solutions of (18) for the given case are written in the form [3]:

$$
\begin{gathered}
v_1 = r\left[k\cot(kt + \alpha) + \dot{\varphi}\tan\varphi\right], \\
v_2 = -aC_2 k\frac{\sin\varphi}{\cos(kt + \alpha)} + \frac{\chi\beta}{ak}\frac{2\cos^2\varphi}{aC_2\sin 2(kt + \alpha)}, \\
r = aC_2\frac{\sin(kt + \alpha)}{\cos\varphi}, \quad \theta = \varphi + \psi_0 - \frac{\pi}{2}, \quad m = m_0 - \beta t, \\
\lambda_{v1} = a\sin(kt + \alpha)\sin\varphi, \quad \lambda_{v2} = a\sin(kt + \alpha)\cos\varphi, \\
\lambda_r = -ak\cos(kt + \alpha)\sin\varphi, \quad \lambda_\theta = 0, \quad \lambda_m = acm_0^2\left[\frac{\sin(kt + \alpha)}{m_0 - \beta t}\right] - \chi + \lambda_{m0},
\end{gathered}
\tag{20}
$$

where

$$
x = \frac{km_0}{\beta} - kt, \quad x_0 = \frac{km_0}{\beta}, \quad \alpha_0 = \alpha + \frac{km_0}{\beta},
$$

$$\tan\varphi = \frac{\tan\alpha\tan\varphi_0}{\tan(kt+\alpha)} + \frac{\chi\beta}{ak}\frac{1}{aC_2k} + \frac{cs}{aC_2k\tan(kt+\alpha)},$$
$$\chi = -\frac{akc}{\beta}[F_1(x_0,x)\sin(\alpha_0) + F_2(x_0,x)\cos(\alpha_0)],$$
$$F_1 = F_1(x_0,x) = Si(x) - Si(x_0), \quad F_2 = F_2(x_0,x) = Ci(x) - Ci(x_0),$$
$$Si(x) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}x^{2i-1}}{(2i-1)(2i-1)!}, \quad Ci(x) = C_0 + ln(x) + \sum_{i=1}^{\infty}\frac{(-1)^i x^{2i}}{(2i)(2i)!},$$

$C_0 = 0.577216$ is the Euler-Maskeroni constant, $Si(x)$ and $Ci(x)$ are the integral sinus and cosinus, and φ_0 is a new constant of integration. Note that these solutions are true in the case of a limited mass-flow rate and consequently, they do not describe an instantaneous change of velocity. As it was mentioned in Ref. [3], the assumption about the instantaneous change of velocity is not adequate to a real flight conditions in solving the guidance problem. The last expression (20) can be investigated for description of an approximate guidance law (thrust program) in a realistic gravitational field. This approach to a guidance law is the development of the idea of application of the linear-tangential law which is a consequence of the analysis of motion in a constant gravity field.

Let's test the presence of the conjugate points on MT arcs found above. The constants are $c_1 = a,\ c_2 = \alpha,\ c_3 = \psi_0,\ c_4 = \varphi_0,\ c_5 = m_0,\ c_6 = \lambda_{m0},\ c_7 = C_2$. It can be shown that

$$\delta x_= \sum_{i=1}^{7}\frac{\partial\theta}{\partial c_i} = \frac{\partial\theta}{\partial c_3} = 1, \quad \delta x_5 = \sum_{i=1}^{7}\frac{\partial m}{\partial c_i} = \frac{\partial m}{\partial c_5} = 1.$$

The equalities show that the solutions for the MT arcs do not satisfy the conditions

$$\delta x_2(t_0) = \delta x_2(t') = 0, \quad \delta x_5(t_0) = \delta x_5(t') = 0, \quad \forall t' \geqslant t_0.$$

Consequently, the MT arcs do not contain the conjugate points.

7. Stability of Programmed Motion

Dynamics of a rocket with variable mass, m in the central Newtonian field, where $\mathbf{g} = -\mu/r^3\mathbf{r}$, in spherical coordinates, $r^1 = r,\ r^2 = \theta,\ r^3 = \delta$, is described by the equations:

$$\begin{gathered}\dot r^i = v^i, \quad \dot v^i = a^i + \beta b^{ij}e_j, \quad i,j = 1,2,3, \dot m = -\beta,\\ a^1 = r\left(\dot\theta^2\cos^2\delta + \dot\delta^2\right) - \mu r^{-2},\\ a^2 = 2\dot\theta\dot\delta tg\delta - 2\dot r\dot\theta r^{-1}, \quad a^3 = -\dot\theta^2\sin\delta\cos\delta - 2\dot r\dot\delta r^{-1},\\ b^{11} = cm^{-1}, \quad b^{22} = c\left(mr\cos\delta\right)^{-1}, \quad b^{33} = cm^{-1}r^{-1}, \quad b^{ij} = 0, i \neq j,\end{gathered} \tag{21}$$

where μ is the gravitational parameter, β is the mass-flow rate, c is the exhaust velocity, $e_1 = e_r,\ e_2 = e_\theta,\ e_3 = e_\delta$ are the components of the unit thrust vector $\mathbf{e}$. In Eqs. (21) it is assumed the summation operations over the same indices. By considering $\mathbf{e}$ and β as control variables, one can represent the program of the motion by the constraint equations:

$$f^\kappa\left(r^i,t\right) = 0, \quad f_i^\kappa v^i + f_t^\kappa = 0, \quad f^\rho\left(r^i,v^j,t\right) = 0, \tag{22}$$
$$f_i^\kappa = \frac{\partial f^\kappa}{\partial r^i}, \quad f_t^\kappa = \frac{\partial f^\kappa}{\partial t}, \quad \kappa = 1,...,k, \quad \rho = k+1,...,s.s \leq 3.$$

In a particular case, Eqs. (22) can describe a law of motion of the rocket's center of mass which would correspond to the solution of the optimal control problem. The controls must be determined such that Eqs. (22) are satisfied for all $t > t_0$, if they are satisfied at initial time, t_0:

$$r^i\left(t_0\right) = r_0^i, \quad v^i\left(t_0\right) = v_0^i. \tag{23}$$

Obviously, the exact satisfaction of Eqs. (22) in the numerical solution process of Eqs. (21) may not be possible and the functions m and e_i must be determined in compliance with the conditions of stability of the constraints, Eqs. (22) [9,11]. For the stabilization of the constraints, the auxiliary variables, x^k, y^k, y^p, which represent deviations from Eqs. (22), are introduced:

$$x^\kappa = f^\kappa\left(r^i, t\right), \quad y^\kappa = f_i^\kappa v^i + f_t^\kappa, \quad y^\rho = f^\rho\left(r^i, v^j, t\right), \tag{24}$$

the change of which is determined by the system of equations of the perturbed constraints:

$$\frac{\mathrm{d}x^\kappa}{\mathrm{d}t} = y^\kappa, \quad \frac{\mathrm{d}y^\sigma}{\mathrm{d}t} = k_\nu^\sigma x^\nu + c_\eta^\sigma y^\eta, \tag{25}$$

$$k_\nu^\kappa = k_\nu^\kappa\left(r^i, v^j, t\right), \quad k_\nu^\rho = 0, \quad c_\eta^\sigma = c_\eta^\sigma\left(r^i, v^j, t\right),$$

$$\nu = 1, ..., k, \quad \sigma,\ \eta = 1, ..., s.$$

The trivial solution $x^k = 0$, $y^\sigma = 0$ of Eqs. (25) corresponds to Eqs. (22). For the constraint stabilization it is necessary to determine such controls β and e_i, which provide asymptotic stability of the trivial solution of Eqs. (25). For description of the corresponding conditions as the Lyapunov functions, one can use positive-definite quadratic form with constant coefficients:

$$2V = a_{\kappa\nu}x^\kappa x^\nu + 2b_{\kappa\sigma}x^\kappa y^\sigma + c_{\sigma\eta}y^\sigma y^\eta, \quad \nu = 1, ..., k, \quad \sigma,\ \eta = 1, ..., s.$$

The derivative $\dot{V}$ of V, computed using Eqs. (25), is also of a quadratic form:

$$\dot{V} = a'_{\kappa\nu}x^\kappa x^\nu + 2b'_{\kappa\sigma}x^\kappa y^\sigma + c'_{\sigma\eta}y^\sigma y^\eta,$$

$$a'_{\kappa\nu} = b_{\kappa\sigma}k_\nu^\sigma, \quad b'_{\kappa\sigma} = a_{\kappa\sigma} + b_{\kappa\eta}c_\sigma^\eta + c_{\sigma\eta}k_\kappa^\eta, \quad a_{\kappa\rho} = 0,$$

$$c'_{\sigma\eta} = b_{\eta\sigma} + c_{\sigma\theta}c_\eta^\theta, \quad b_{\rho\sigma} = 0,$$

$$\kappa,\ \nu = 1, ..., k, \quad \rho = k + 1, ..., s, \quad \sigma,\ \eta,\ \theta = 1, ..., s.$$

The trivial solution of Eqs. (25) is asymptotically stable, if the function V is positive definite function with respect to x^k, y^σ, and its derivative is negative definite, and the functions x^k, y^σ, determined by Eqs. (24) and the function V admit infinitely small supreme limit. The conditions of asymptotic stability can be satisfied by an appropriate selection of the coefficients of the quadratic form V and the right hand sides of Eqs. (25). In particular, they can be considered constants. The existence of infinitely small supreme limit of the functions in Eqs. (24) depends on the functions in Eqs. (22) which provide the program of motion.

8. Determination of Control Actions

If the coefficients k_v^σ, c_η^σ of Eqs. (25) are determined, then for e_i and β one can differentiate $y^\sigma = \phi^\sigma(r^i, v^j, t)$ taking into account Eqs. (21) and (25), and the expressions

$$\phi^\kappa(r^i, v^j, t) = f_i^\kappa v^i + f_t^\kappa, \quad \phi^\rho(r^i, v^j, t) = f^\rho(r^i, v^j, t). \tag{26}$$

The computations yield the system of linear algebraic equations with respect to βe_i:

$$s^{\sigma i}\beta e_i = s^\sigma, \quad s^{\sigma i} = \psi_j^\sigma b^{ji}, \quad s^\sigma = k_\nu^\sigma x^\nu + c_\eta^\sigma y^\eta - \left(\varphi_i^\sigma v^i + \psi_j^\sigma a^j + \varphi_t^\sigma\right), \tag{27}$$

$$\begin{gathered} \psi_j^\kappa = f_j^\kappa, \quad \psi_j^\rho = \frac{\partial f^\rho}{\partial v^j}, \\ x^\kappa = f^\kappa\left(r^i, t\right), \quad y^\kappa = f_i^\kappa v^i + f_t^\kappa, \quad y^\rho = f^\rho\left(r^i, v^j, t\right), \\ \varphi_i^k = f_{ij}^\kappa v^j + f_{it}^\kappa, \quad \varphi_i^\rho = \frac{\partial f^\rho}{\partial r^i}, \quad \varphi_t^k = f_{it}^\kappa v^i + f_{tt}^\kappa, \quad \varphi_t^\rho = \frac{\partial f^\rho}{\partial t} \end{gathered} \tag{28}$$

The solution to Eqs. (27) is determined depending on the number s of constraint equations. The following cases may take place:

1. $s = 1$. Eqs. (27) can be satisfied by the solution:

   $$\beta = \left(s^{\sigma i} e_i\right)^{-1} s^\sigma$$

   This means that the stable motion can be provided only by the change of β for any admissible components of the thrust vector $\mathbf{e}$.
2. $s = 2$. The general solution of Eqs. (27) takes the form:

   $$\beta e_i = e_0 s_i + s_{i\sigma} s^\sigma,$$

   where e_0 is an arbitrary quantity, s_i is computed as a determinant $s_i = det(\delta_i, s^{1i}, s^{2i})$, consisting of the unit vector δ_i and rows of the matrix of coefficients $S = \left(s^{\sigma i}\right)$ of Eqs. (27). The multipliers $s_{i\sigma}$ represent the elements of the matrix $S^+ = S^T(SS^T)^{-1}$, pseudoinverse to the matrix S. The control law depends on two parameters, for which the law of mass change β and one parameter determining the thrust direction can be selected.
3. $s = 3$. Eqs. (27) have the solution:

   $$\beta e_i = s_{i\sigma} s^\sigma, \quad (s_{i\sigma}) = S^{-1}.$$

   The control law depends on three parameters, one of which can be selected as β.

9. Conclusions

The optimal control problem of determining optimal trajectories of rocket center of mass and the stability of trajectories are considered. By testing the Legendre–Clebsch conditions, the classes of active arcs which can be optimal. It has been shown that the formulas for determining the existence of conjugate points on thrust arcs can be driven using the analytical solutions for these arcs. The proposed method can serve as an instrument of extraction of extremal trajectories for the guidance problem. The laws for mass-flow rate, corresponding to a stable programmed motion in the central newtonian field, have been determined.

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Аналитический синтез экстремальных траекторий и устойчивость программного движения

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Рассматриваются задачи аналитического построения оптимальных траекторий ракеты и соответствующих управляющих воздействий, обеспечивающих устойчивое программное движение. Доказывается возможность использования аналитического решения для определения сопряжённых точек экстремальных траекторий. Приводится анализ законов управления устойчивым движением.

Ключевые слова: вариационная задача, автономное управление в пространстве, ракета, множители Лагранжа, уравнение Риккати, условия Якоби, сопряженные точки, устойчивое движение, программное движение, управление.