UDC 517:519.6, 535+537.8:621.37

The Derivation of the Dispersion Equations of Adiabatic Waveguide Modes in the Thin-Film Waveguide Luneburg Lens in the Form of Non-Linear Partial Differential Equation of the First Order

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This paper presents a derivation of the dispersion equation for a three-layer integrated-optical Luneburg lens based on the method of adiabatic waveguide modes. From this equation there follows the relationship between the coefficient of phase deceleration and function, which determines the thickness of the irregular waveguide layer. The dispersion equation is represented in the form of non-linear partial differential equation of the first order with coefficients, depending on parameters. Among these parameters are regular waveguide layer thickness and optical parameters of the pending Luneburg lens. To represent the dispersion equation in the form of differential equations in partial derivatives, it is necessary to calculate a symbolic form the determinant of a matrix of 12th order, which determines the solubility of the system of linear algebraic equations, resulting from the boundary conditions. To calculate this determinant in analytical form a procedure of reduction of the system of linear algebraic equations with the use of the computer algebra system Maple is proposed.

Key words and phrases: irregular integrated optical wave guide, method of adiabatic modes, computer algebra system.

1. Introduction

The waveguide Luneburg lens is an important functional element of integrated optical devices, such as microwave, RF-devices, in particular RF spectrum analyzer [1, 2]. The task of designing such devices puts higher demands on the accuracy of calculating the parameters of the lens. One of the most promising methods used in the modeling of the waveguide propagation of radiation with the exact tangential boundary conditions, is the method of adiabatic waveguide modes. Description of the method and the results of its application are presented in the papers [3–7].

The overall objective of modeling smoothly irregular longitudinally integrated optical waveguides includes the task of finding the irregular surface of the waveguide layer by solving the dispersion equation. In the papers [3–7] computational scheme for solving this problem has been proposed. However, to develop a software module that allows to increase significantly the speed of the design of optical devices, it is proposed to build a computational solution scheme of the dispersion equation based on the continuous analog of Newton's method [8, 9]. To implement such an iterative scheme, it was necessary to present the dispersion relation in an explicit analytic form of a nonlinear differential equation of the first order. This equation can be written as a polynomial in the partial derivatives of the function describing the shape of the irregular surface of the waveguide layer, with coefficients expressed in terms of geometric and optical properties of the waveguide.

Received 9th September, 2013.

The work is partially supported by RFBR grants №11-01-00278, №13-01-00595, and JINR grant №13-603-02. The authors thank Shushanik Torosyan for useful advices.

The present work is devoted to obtaining explicit analytic form of the dispersion equation using computer algebra system Maple [10].

2. Method of Adiabatic Modes

In this section, we briefly outline the basics of the method of adiabatic waveguide modes [3–7].

The scattering of polarized light along smoothly irregular integrated optical is described by Maxwell's equations without external charges and currents:

$$\operatorname{rot}\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}, \quad \operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}; \tag{1}$$

with the coupling equations

$$\vec{D} = \varepsilon\vec{E}, \quad \vec{B} = \mu\vec{H}; \tag{2}$$

and the boundary conditions

$$\vec{H}^{\tau}\Big|_{a_i-0} = \vec{H}^{\tau}\Big|_{a_i+0}, \quad \vec{E}^{\tau}\Big|_{a_i-0} = \vec{E}^{\tau}\Big|_{a_i+0}, \quad i = 1, 2. \tag{3}$$

Here a_i are coordinates of the boundary surfaces, $\vec{D}$, $\vec{E}$ electric displacement vector and vector of the electric fields, $\vec{B}$, $\vec{H}$ are vectors of magnetic induction and magnetic field in an electromagnetic wave; ε, μ are permittivity and permeability of environment. The index τ marked tangential field components $\vec{E}$ and $\vec{H}$.

We look for solutions of Maxwell's equations in the form of adiabatic waveguide modes [3, 4]:

$$\left.\begin{matrix}\vec{E}(x;y,z,t)\\ \vec{H}(x;y,z,t)\end{matrix}\right\} = \left\{\begin{matrix}\vec{E}(x;y,z)\\ \vec{H}(x;y,z)\end{matrix}\right\} = \frac{\exp\left[i\omega t - i\varphi(y,z)\right]}{\sqrt{\beta(y,z)}}, \tag{4}$$

where ω, φ are the frequency and phase of the wave, $\beta = \sqrt{\beta_y^2 + \beta_z^2}$, $\beta_y = (\partial\varphi/\partial y)/k_0$, $\beta_z = (\partial\varphi/\partial z)/k_0$ is phase deceleration, $k_0 = \omega/c$ is wave number. Substituting (4) in (1) and taking into account that the wave propagates in the direction of the axis of Oz, for z-components of the electric and magnetic fields, we get the following ordinary differential equation of second order

$$\frac{\mathrm{d}^2 E_z}{\mathrm{d}x^2} + \chi^2 E_z = \frac{\partial(\ln\chi_z^2)}{\partial y}\left[p_y E_z + \frac{p_z}{ik_0\varepsilon}\frac{\mathrm{d}H_z}{\mathrm{d}x}\right], \tag{5}$$

$$\frac{\mathrm{d}^2 H_z}{\mathrm{d}x^2} + \chi^2 H_z = \frac{\partial(\ln\chi_z^2)}{\partial y}\left[p_y H_z - \frac{p_z}{ik_0\mu}\frac{\mathrm{d}E_z}{\mathrm{d}x}\right]. \tag{6}$$

Here

$$\chi^2 = \chi_z^2 + p_y^2 + \frac{\partial p_y}{\partial y}, \quad \chi_z^2 = k_0^2\varepsilon\mu + p_z^2 + \frac{\partial p_z}{\partial z},$$

$$p_y = -ik_0\beta_y - \frac{1}{2\beta}\frac{\partial\beta}{\partial y}, \quad p_z = -ik_0\beta_z - \frac{1}{2\beta}\frac{\partial\beta}{\partial z}.$$

However, the other components of the electromagnetic field $E_y(x;y,z)$, $H_x(x;y,z)$, $H_y(x;y,z)$, $E_x(x;y,z)$ are expressed in terms of E_z, H_z and their derivatives:

$$E_x = \frac{1}{\chi_z^2}\left[p_z\frac{\mathrm{d}E_z}{\mathrm{d}x} - ik_0\mu p_y H_z\right], \quad H_x = \frac{1}{\chi_z^2}\left[p_z\frac{\mathrm{d}H_z}{\mathrm{d}x} + ik_0\varepsilon p_y E_z\right], \tag{7}$$

$$
\begin{aligned}
E_y &= \frac{1}{\chi_z^2}\left[\left(p_y p_z + \frac{\partial p_y}{\partial z}\right) E_z + i k_0 \mu \frac{\mathrm{d} H_z}{\mathrm{d} x}\right], \\
H_y &= \frac{1}{\chi_z^2}\left[\left(p_y p_z + \frac{\partial p_y}{\partial z}\right) H_z - i k_0 \varepsilon \frac{\mathrm{d} E_z}{\mathrm{d} x}\right].
\end{aligned} \tag{8}
$$

For many smoothly irregular integrated optical waveguides, including thin-film waveguide Luneburg lens (TWL), the condition [3] is executed

$$
\max_{y,z} \frac{|(\vec{\nabla}\vec{\beta})|}{k_0 \beta^2} \equiv \delta \ll 1 \,. \tag{9}
$$

Fulfillment of this condition allows to use the method of asymptotic expansions on δ solutions of the system of relations (5)–(8). In the zero-order approximation of the asymptotic expansion from the equations (5)–(6) and relations (7)–(8), we get the system of equations

$$
\frac{\mathrm{d}^2 E_z^0}{\mathrm{d} x^2} + k_0^2(\varepsilon\mu - \beta^2) E_z^0 = 0, \quad \frac{\mathrm{d}^2 H_z^0}{\mathrm{d} x^2} + k_0^2(\varepsilon\mu - \beta^2) H_z^0 = 0, \tag{10}
$$

and relations

$$
H_y^0 = \frac{1}{k_0^2(\varepsilon\mu - \beta_z^2)}\left(-k_0^2 \beta_y \beta_z H_z^0 - i k_0 \varepsilon \frac{\mathrm{d} E_z^0}{\mathrm{d} x}\right), \tag{11}
$$

$$
E_x^0 = \frac{1}{k_0^2(\varepsilon\mu - \beta_z^2)}\left(-i k_0 \beta_z \frac{\mathrm{d} E_z^0}{\mathrm{d} x} - k_0^2 \mu \beta_y H_z^0\right), \tag{12}
$$

$$
E_y^0 = \frac{1}{k_0^2(\varepsilon\mu - \beta_z^2)}\left(i k_0 \mu \frac{\mathrm{d} H_z^0}{\mathrm{d} x} - k_0^2 \beta_y \beta_z E_z^0\right), \tag{13}
$$

$$
H_x^0 = \frac{1}{k_0^2(\varepsilon\mu - \beta_z^2)}\left(-i k_0 \beta_z \frac{\mathrm{d} H_z^0}{\mathrm{d} x} - k_0^2 \varepsilon \beta_y E_z^0\right). \tag{14}
$$

For guided modes of TWL Luneburg at the layer interfaces (except the horizontal boundary conditions) for the tangential components of the electromagnetic field $\vec{E}^\tau$, $\vec{H}^\tau$, boundary conditions on non-horizontal tangent planes

$$
\left.\vec{E}^\tau\right|_{h(y,z)-0} = \left.\vec{E}^\tau\right|_{h(y,z)+0}, \quad \left.\vec{H}^\tau\right|_{h(y,z)-0} = \left.\vec{H}^\tau\right|_{h(y,z)+0}, \tag{15}
$$

and the asymptotic conditions at infinity

$$
\left|\vec{E}^\tau\right|_{x\to\pm\infty} < +\infty, \quad \left|\vec{H}^\tau\right|_{x\to\pm\infty} < +\infty \tag{16}
$$

are also performed.

In this paper we consider TWL Luneburg with four dielectric layers (finite and semi-infinite thickness) with three boundary surfaces. For schematic view of the considered TWL Luneburg see Figure 1.

3. The Derivation of the Dispersion Equation of the Adiabatic Waveguide Modes with the Use of Computer Algebra System Maple

Since TWL Luneburg contains four dielectric layers with three boundary surface, the general solutions of equations (10) and relations (11)–(14), recorded in each of

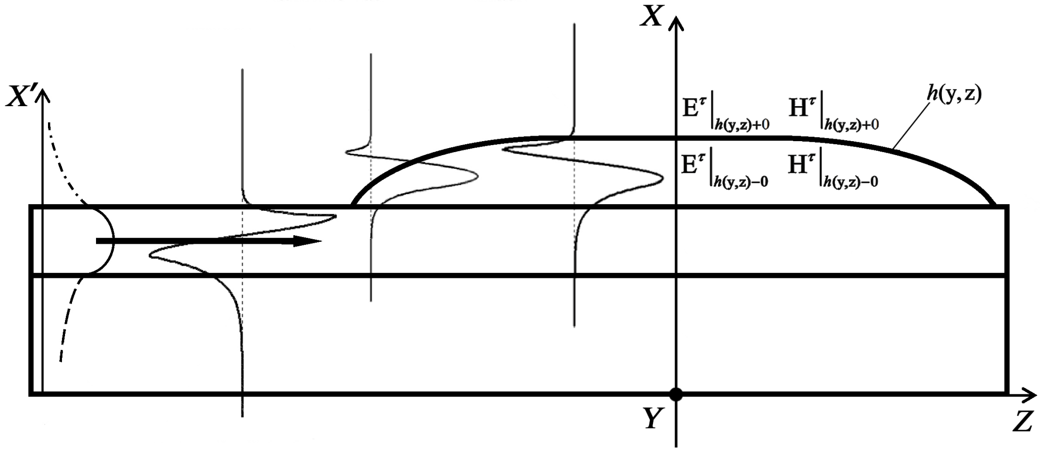

Figure 1. The cross section of the considered integrated-optical structures

the layers through some fundamental system of solutions with 12-th undetermined coefficients A_s, $A_f^{\pm}$, $A_l^{\pm}$, A_a and B_s, $B_f^{\pm}$, $B_l^{\pm}$, B_a, define the solution in the whole space in the case of satisfying the boundary conditions of the form of a homogeneous system of linear algebraic equations [4–7]

$$\hat{M}(\beta)(\vec{A},\vec{B}) = \vec{0}. \tag{17}$$

We represent the matrix of the system (17) in a form, suitable for further transformations:

$$\begin{pmatrix}
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{3,3} & 0 & m_{3,5} & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m_{4,4} & 0 & m_{4,6} & 0 & 0 & -1 & -1 & 0 & 0 \\
m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} & m_{5,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & m_{6,6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{7,3} & m_{7,4} & m_{7,5} & m_{7,6} & m_{7,7} & m_{7,8} & m_{7,9} & m_{7,10} & 0 & 0 \\
0 & 0 & m_{8,3} & m_{8,4} & m_{8,5} & m_{8,6} & m_{8,7} & m_{8,8} & m_{8,9} & m_{8,10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{9,7} & m_{9,8} & m_{9,9} & m_{9,10} & m_{9,11} & m_{9,12} \\
0 & 0 & 0 & 0 & 0 & 0 & m_{10,7} & m_{10,8} & m_{10,9} & m_{10,10} & m_{10,11} & m_{10,12} \\
0 & 0 & 0 & 0 & 0 & 0 & m_{11,7} & m_{11,8} & m_{11,9} & m_{11,10} & m_{11,11} & m_{11,12} \\
0 & 0 & 0 & 0 & 0 & 0 & m_{12,7} & m_{12,8} & m_{12,9} & m_{12,10} & m_{12,11} & m_{12,12}
\end{pmatrix}.$$

Here $m_{i,j}$ identify non-zero symbolic elements, which are detailed in Appendix.

Note that the matrix elements of $M_{ij}(\beta)$ contain $\dfrac{\partial h(y,z)}{\partial y}$ and $\dfrac{\partial h(y,z)}{\partial z}$. Due to the fact that the irregular surface of the section thickness of the waveguide layer has irregular width, and the coating layer thickness and irregular nonplanar waveguide layer $h(y,z)$ vary from point (y,z) to the point, the boundary conditions (15) consist of the terms with factors $\dfrac{\partial h(y,z)}{\partial y}$ and $\dfrac{\partial h(y,z)}{\partial z}$.

By the homogeneity of linear algebraic equation (17), it is non-trivial solvable provided that

$$\det(\hat{M}(\beta)) = 0\ , \tag{18}$$

which has the form

$$F(\partial h/\partial y, \partial h/\partial z, h, \beta_y, \beta_z, n_s, n_f, n_l) = 0 \tag{19}$$

of nonlinear partial differential equation of first order with respect to $h(y,z)$ defined by (approximately) $\beta_y(y,z)$ and $\beta_z(y,z)$.

The vector field $(\beta_y, \beta_z)^T\,(y,z)$ defines a flat projection of TWL Luneburg rays, focusing plane wave front (in the two-dimensional area) at the focal point, located on the other side of the lens. The projection rays satisfy the equations

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\beta(y,z)\frac{\mathrm{d}y}{\mathrm{d}s}\right) = \frac{\partial\beta}{\partial y}(y,z), \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\beta(y,z)\frac{\mathrm{d}z}{\mathrm{d}s}\right) = \frac{\partial\beta}{\partial z}(y,z), \tag{20}$$

with a coefficient of phase deceleration $\beta(y,z)$, which satisfy the relations

$$\frac{\beta(r)}{\beta} = \exp\left[\omega(\rho, F)\right], \quad \text{where } \rho = \frac{r\beta(r)}{\beta}, \quad \omega(\rho, F) = \frac{1}{\pi}\int\limits_{\pi}^{1}\frac{\arcsin(x/F)}{\sqrt{x^2-\rho^2}}\mathrm{d}x. \tag{21}$$

The determinant of the matrix has been using the computer algebra system Maple. It should be noted that the calculation of the determinant of the matrix with rank above 10-th in the symbolic form requires using of a large number of computing resources; therefore, we have developed the following algorithm for reducing a matrix:

1) by elementary transformations on the rows and columns the matrix can be reduced to a form where we can select blocks, in which determinant at any parameter value is not vanishing, which are will not participate in further calculations. These rows correspond to the boundary conditions at the regular layers of the waveguide;
2) determinant of the reduced matrix has been calculated on the basis of the Laplace theorem [11].

By applying the algorithm to the above matrix, we obtain the following matrix structure

$$\left(\begin{array}{c|c} M_4 & \cdots \\ \hline 0 & M_8 \end{array}\right).$$

Thereafter, by elementary transformations on rows and columns we lead unit M_4 to upper-triangular forms. The value of the determinant of this block is the product of the diagonal elements. The determinant of the block M_8 is calculated by Laplace theorem, choosing minor of the second order (pairwise combining the first four rows of the matrix) and the corresponding cofactors of order 6, which, in turn, will also rely on the Laplace theorem, choosing a minor 2nd order by combining the last four rows in pairs, and the cofactors of the 4th order.

Thus, we obtain the equations for finding $h(y,z)$:

$$\det(\hat{M}_8(\beta)) = 0. \tag{22}$$

Testing transformations. Since finding the determinants of the 12-th order for numeric matrices does not involve any computational complexity, after each stage of symbolic transformations on an arbitrary set of values of the parameters $\widetilde{\beta}$ there have been made comparisons between the primary determinant of the matrix $\det(\hat{M}_{12}(\widetilde{\beta}))$ and the determinant of the transformed $\det(\hat{M}_4(\widetilde{\beta})) \times \det(\hat{M}_8(\widetilde{\beta}))$.

The resulting equation (22) is factorized through indecomposable factors, each of which contain the lowest common denominator (over the field of rational functions), which determines the differential equation in partial derivatives relative to $h(y,z)$. After applying the Maple procedures to simplify this expression (collect, combine [10]), solvability condition (19) is reduced to the following form:

$$F = \left(\frac{\partial h}{\partial y}\right)^6 \left(a_1 \left(\frac{\partial h}{\partial z}\right)^2 + a_2 \left(\frac{\partial h}{\partial z}\right) + a_3\right) +$$
$$+ \left(\frac{\partial h}{\partial y}\right)^5 \left(a_4 \left(\frac{\partial h}{\partial z}\right)^3 + +a_5 \left(\frac{\partial h}{\partial z}\right)^2 + a_6 \left(\frac{\partial h}{\partial z}\right) + a_7\right) +$$
$$+ \left(\frac{\partial h}{\partial y}\right)^4 \left(a_8 \left(\frac{\partial h}{\partial z}\right)^4 + a_9 \left(\frac{\partial h}{\partial z}\right)^3 + a_{10} \left(\frac{\partial h}{\partial z}\right)^2 + a_{11} \left(\frac{\partial h}{\partial z}\right) + a_{12}\right) +$$
$$+ \left(\frac{\partial h}{\partial y}\right)^3 \left(a_{13} \left(\frac{\partial h}{\partial z}\right)^5 + a_{14} \left(\frac{\partial h}{\partial z}\right)^4 + a_{15} \left(\frac{\partial h}{\partial z}\right)^3 + a_{16} \left(\frac{\partial h}{\partial z}\right)^2 + a_{17} \left(\frac{\partial h}{\partial z}\right) + a_{18}\right) +$$
$$+ \left(\frac{\partial h}{\partial y}\right)^2 \left(a_{19} \left(\frac{\partial h}{\partial z}\right)^6 + a_{20} \left(\frac{\partial h}{\partial z}\right)^5 + a_{21} \left(\frac{\partial h}{\partial z}\right)^4 + a_{22} \left(\frac{\partial h}{\partial z}\right)^3 + a_{23} \left(\frac{\partial h}{\partial z}\right)^2 + \right.$$
$$\left. + a_{24} \left(\frac{\partial h}{\partial z}\right) + + a_{25}\right) + \frac{\partial h}{\partial y} \left(a_{26} \left(\frac{\partial h}{\partial z}\right)^6 + a_{27} \left(\frac{\partial h}{\partial z}\right)^5 + a_{28} \left(\frac{\partial h}{\partial z}\right)^4 + a_{29} \left(\frac{\partial h}{\partial z}\right)^3 + \right.$$
$$\left. + a_{30} \left(\frac{\partial h}{\partial z}\right)^2 + a_{31} \left(\frac{\partial h}{\partial z}\right) + a_{32}\right) + a_{33} \left(\frac{\partial h}{\partial z}\right)^6 + a_{34} \left(\frac{\partial h}{\partial z}\right)^5 +$$
$$+ a_{35} \left(\frac{\partial h}{\partial z}\right)^4 + a_{36} \left(\frac{\partial h}{\partial z}\right)^3 + a_{37} \left(\frac{\partial h}{\partial z}\right)^2 + a_{38} \left(\frac{\partial h}{\partial z}\right) + a_{39} = 0 \,, \quad (23)$$

where a_i, $i = 1, 2, \ldots, 39$ are coefficients in symbolic representation, depending on the functions $\beta_y(y,z)$, $\beta_z(y,z)$, $\beta(y,z)$ and geometrical and optical parameters of the waveguide a_1, a_2, χ_l, χ_f, γ_s, γ_a, μ, ε_s, ε_f, ε_l, ε_a, and the dependence on $h(y,z)$ is a linear combination of expressions of the form $\exp\left[i\alpha_{k,l}(y,z)h(y,z)\right]$.

The dispersion relation (23) can be written as:

$$F(h(y,z)) = \sum_{k,l} a_{k,l}\left(y, z, h(y,z)\right) \left(\frac{\partial h}{\partial y}\right)^k \left(\frac{\partial h}{\partial z}\right)^l = 0 \,, \qquad (24)$$

where $0 \leqslant k + l \leqslant 8$. This representation is convenient for the further numerical investigation.

4. Conclusion

In this paper, on the base of the method of adiabatic waveguide modes, we have received an analytical expression for the dispersion equation of adiabatic waveguide modes of TWL Luneburg, representing a non-linear partial differential equations of the first order. Using the above procedure we obtained a dispersion equation form (24) for a lens (proposed in [2]), that contains five dielectric layers with four boundary surfaces, for which a system of equations of 16-th order is obtained.

In conclusion, we note that modeling of smoothly irregular integrated optical devices is a complex computational problem. This has enhanced the importance of having a set of programs for the design of such devices with the desired properties. In particular, this complex must consist of calculation modules, a module comprising geometrical and optical parameters of the device, the module for calculating the coefficients of the phase decelerations in the form of network functions [5,7] and developing computational module of the irregular waveguide layer surface form based on the solution of the dispersion equation. For the convenience of the numerical realization, there is a need for the presentation of a numerical implementation of the dispersion equation in the form of non-linear differential equation of first order as a polynomial in the partial derivatives of the desired shape of the surface with coefficients having an explicit analytical expression in terms of geometric and optical properties of the waveguide.

The result of this work is creating a routine for calculating the coefficients of the dispersion equation, which is the basis for the calculation module of the irregular form of the surface of the waveguide layer.

The next article will present a computational scheme and its implementation for solving these equations of the form (24) based on the continuous analogue of Newton's method [8, 9].

Appendix

$$m_{3,3} = \mathrm{e}^{i\chi_f(a_2-a_1)}; \quad m_{3,5} = \mathrm{e}^{-i\chi_f(a_2-a_1)}; \quad m_{4,4} = \mathrm{e}^{i\chi_f(a_2-a_1)};$$

$$m_{4,6} = \mathrm{e}^{-i\chi_f(a_2-a_1)}; \quad m_{5,1} = -\frac{\beta_y\beta_z}{n_s^2-\beta_z^2}; \quad m_{5,2} = \frac{i\mu\gamma_s}{k_0\left(n_s^2-\beta_z^2\right)};$$

$$m_{5,3} = \frac{\beta_y\beta_z}{n_f^2-\beta_z^2}; \quad m_{5,4} = \frac{\chi_f\mu}{k_0\left(n_f^2-\beta_z^2\right)}; \quad m_{5,5} = \frac{\beta_y\beta_z}{n_f^2-\beta_z^2};$$

$$m_{5,6} = -\frac{\chi_f\mu}{k_0\left(n_f^2-\beta_z^2\right)}; \quad m_{6,1} = -\frac{i\varepsilon_s\gamma_s}{k_0\left(n_s^2-\beta_z^2\right)}; \quad m_{6,2} = -\frac{\beta_y\beta_z}{n_s^2-\beta_z^2};$$

$$m_{6,3} = -\frac{\chi_f\varepsilon_f}{k_0\left(n_f^2-\beta_z^2\right)}; \quad m_{6,4} = \frac{\beta_y\beta_z}{n_f^2-\beta_z^2}; \quad m_{6,5} = \frac{\chi_f\varepsilon_f}{k_0\left(n_f^2-\beta_z^2\right)};$$

$$m_{6,6} = \frac{\beta_y\beta_z}{n_f^2-\beta_z^2}; \quad m_{7,3} = -\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_f(a_2-a_1)}}{n_f^2-\beta_z^2}; \quad m_{7,4} = -\frac{\chi_l\mu\,\mathrm{e}^{i\chi_f(a_2-a_1)}}{k_0\left(n_f^2-\beta_z^2\right)};$$

$$m_{7,5} = -\frac{\beta_y\beta_z\mathrm{e}^{-i\chi_f(a_2-a_1)}}{n_f^2-\beta_z^2}; \quad m_{7,6} = -\frac{\chi_l\mu\,\mathrm{e}^{-i\chi_f(a_2-a_1)}}{k_0\left(n_f^2-\beta_z^2\right)}; \quad m_{7,7} = \frac{\beta_y\beta_z}{n_l^2-\beta_z^2};$$

$$m_{7,8} = \frac{\beta_y\beta_z}{n_l^2-\beta_z^2}; \quad m_{7,9} = \frac{\chi_l\mu}{k_0\left(n_l^2-\beta_z^2\right)}; \quad m_{7,10} = -\frac{\chi_l\mu}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{8,3} = \frac{\chi_f\varepsilon_f\,\mathrm{e}^{i\chi_f(a_2-a_1)}}{k_0\left(n_f^2-\beta_z^2\right)}; \quad m_{8,4} = -\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_f(a_2-a_1)}}{n_f^2-\beta_z^2}; \quad m_{8,5} = -\frac{\chi_f\varepsilon_f\,\mathrm{e}^{-i\chi_f(a_2-a_1)}}{k_0\left(n_f^2-\beta_z^2\right)};$$

$$m_{8,6} = -\frac{\beta_y\beta_z\,\mathrm{e}^{-i\chi_f(a_2-a_1)}}{n_f^2-\beta_z^2}; \quad m_{8,7} = \frac{\beta_y\beta_z}{n_l^2-\beta_z^2}; \quad m_{8,8} = \frac{\beta_y\beta_z}{n_l^2-\beta_z^2};$$

$$m_{8,9} = \frac{\chi_l\mu}{k_0\left(n_l^2-\beta_z^2\right)}; \quad m_{8,10} = -\frac{\chi_l\mu}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{9,7} = \frac{\partial h}{\partial z}\frac{\chi_l\beta_z\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right]\mathrm{e}^{i\chi_l h};$$

$$m_{9,8} = -\frac{\partial h}{\partial z}\frac{\chi_l\beta_z\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right]\mathrm{e}^{-i\chi_l h};$$

$$m_{9,9} = -\frac{\partial h}{\partial z}\frac{\mu\beta_y\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\chi_l\mu\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{9,10} = -\frac{\partial h}{\partial z}\frac{\mu\beta_y\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\chi_l\mu\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{9,11} = -\frac{\partial h}{\partial z}\frac{i\beta_z\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z\gamma_a}{n_a^2-\beta_z^2} - \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right];$$

$$m_{9,12} = \frac{\partial h}{\partial z}\frac{\mu\beta_y}{n_a^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{i\mu\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)};$$

$$m_{10,7} = \frac{\partial h}{\partial z}\frac{\varepsilon_l\beta_y\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\chi_l\varepsilon_l\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{10,8} = \frac{\partial h}{\partial z}\frac{\varepsilon_l\beta_y\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\chi_l\varepsilon_l\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{10,9} = \frac{\partial h}{\partial z}\frac{\chi_l\beta_z\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right]\mathrm{e}^{i\chi_l h};$$

$$m_{10,10} = -\frac{\partial h}{\partial z}\frac{\chi_l\beta_z\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right]\mathrm{e}^{-i\chi_l h};$$

$$m_{10,11} = \frac{\partial h}{\partial z}\frac{\varepsilon_a\beta_y\gamma_a}{n_a^2-\beta_z^2} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{i\varepsilon_a\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)};$$

$$m_{10,12} = -\frac{\partial h}{\partial z}\frac{i\beta_z\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\frac{\beta_y\beta_z}{n_a^2-\beta_z^2} - \left[1+\left(\frac{\partial h}{\partial y}\right)^2\right];$$

$$m_{11,7} = \frac{\partial h}{\partial y}\frac{\chi_l\beta_z\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\mathrm{e}^{i\chi_l h};$$

$$m_{11,8} = -\frac{\partial h}{\partial y}\frac{\chi_l\beta_z\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\beta_y\beta_z\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\mathrm{e}^{-i\chi_l h};$$

$$m_{11,9} = -\frac{\partial h}{\partial y}\frac{\mu\beta_y\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\chi_l\mu\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{11,10} = -\frac{\partial h}{\partial y}\frac{\mu\beta_y\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\chi_l\mu\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{11,11} = -\frac{\partial h}{\partial y}\frac{\beta_z\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\beta_y\beta_z\gamma_a}{n_a^2-\beta_z^2} + \frac{\partial h}{\partial y}\frac{\partial h}{\partial z};$$

$$m_{11,12} = \frac{\partial h}{\partial y}\frac{\mu\beta_y}{n_a^2-\beta_z^2} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\mu\gamma_a}{k_0\left(n_a^2-\beta_z^2\right)};$$

$$m_{12,7} = \frac{\partial h}{\partial y}\frac{\varepsilon_l\beta_y\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} + \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\chi_l\varepsilon_l\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{12,8} = \frac{\partial h}{\partial y}\frac{\varepsilon_l\beta_y\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\chi_l\varepsilon_l\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)};$$

$$m_{12,9} = \frac{\partial h}{\partial y}\frac{\chi_l\beta_z\,\mathrm{e}^{i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\beta_y\beta_z\,\mathrm{e}^{i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\mathrm{e}^{i\chi_l h};$$

$$m_{12,10} = -\frac{\partial h}{\partial y}\frac{\chi_l\beta_z\,\mathrm{e}^{-i\chi_l h}}{k_0\left(n_l^2-\beta_z^2\right)} - \left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]\frac{\beta_y\beta_z\,\mathrm{e}^{-i\chi_l h}}{n_l^2-\beta_z^2} - \frac{\partial h}{\partial y}\frac{\partial h}{\partial z}\mathrm{e}^{-i\chi_l h};$$

$$m_{12,11} = \frac{\partial h}{\partial y} \frac{\varepsilon_a \beta_y \gamma_a}{n_a^2 - \beta_z^2} - \left[1 + \left(\frac{\partial h}{\partial z}\right)^2\right] \frac{i \varepsilon_a \gamma_a}{k_0 \left(n_a^2 - \beta_z^2\right)};$$

$$m_{12,12} = -\frac{\partial h}{\partial y} \frac{i \beta_z \gamma_a}{k_0 \left(n_a^2 - \beta_z^2\right)} + \left[1 + \left(\frac{\partial h}{\partial z}\right)^2\right] \frac{\beta_y \beta_z \gamma_a}{n_a^2 - \beta_z^2} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial z};$$

On each layer $\chi_f^2 = k_0^2 \left(\varepsilon_f \mu - \beta^2\right)$, $\chi_l^2 = k_0^2 \left(\varepsilon_l \mu - \beta^2\right)$, $\gamma_s^2 = -k_0^2 \left(\varepsilon_s \mu - \beta^2\right)$, $\gamma_a^2 = -k_0^2 \left(\varepsilon_a \mu - \beta^2\right)$, where $\beta = \beta(y, z)$.

References

1. *Southwell W. H.* Inhomogeneous Optical Waveguide Lens Analysis // JOSA. — 1977. — Vol. 67, No 8. — Pp. 1004–1009.
2. *Di Falco A., Kehr S. C., Leonhardt U.* Luneburg Lens in Silicon Photonics // Optics Express. — 2011. — Vol. 19, No 6. — Pp. 5156–5162.
3. *Севастьянов Л. А., Егоров А. А.* Теоретический анализ волноводного распространения электромагнитных волн в диэлектрических плавно-нерегулярных интегральных структурах // Оптика и спектроскопия. — 2008. — Т. 105, № 4. — С. 650–658. [Sevastyanov L. A., Egorov A. A. Theoretical Analysis of the Waveguide Propagation of Electromagnetic Waves in Dielectric Smoothly-Irregular Integrated Structures // Optics and Spectroscopy. — 2010. — Vol. 105, No 4. — Pp. 650–658.]
4. *Егоров А. А., Севастьянов Л. А.* Структура мод плавно-нерегулярного интегрально-оптического четырёхслойного трёхмерного волновода // Квантовая электроника. — 2009. — Т. 39, № 6. — С. 566–574. [Egorov A. A., Sevastyanov L. A. Structure of Modes of a Smoothly Irregular Integrated-Optical Four-Layer Three-Dimensional Waveguide // Quantum Electron, — 2009. — Vol. 39, No 6. — Pp. 566–574.]
5. Адиабатические моды плавно-нерегулярного оптического волновода: нулевое приближение векторной теории / А. А. Егоров, А. Л. Севастьянов, Э. А. Айрян и др. // Математическое моделирование. — 2010. — Т. 22, № 8. — С. 42–54. [Zero Approximation of Vector Model for Smoothly-Irregular Optical Waveguide / A. A. Egorov, A. L. Sevastyanov, E. A. Ayryan et al. // Matem. Mod. — 2010. — Vol. 22, No 8. — Pp. 42–54.]
6. Mathematical Modeling of Irregular Integrated Optical Waveguides / E. A. Ayryan, A. A. Egorov, L. A. Sevastianov et al. // Lecture Notes in Computer Science. — 2012. — Vol. 7125. — Pp. 136–147.
7. Расчёт и проектирование тонкоплёночной обобщённой волноводной линзы Люнеберга методом адиабатических мод / А. А. Егоров, А. Л. Севастьянов, Э. А. Айрян и др. // Вестник Тверского государственного университета. — 2012. — Т. Прикладная математика, № 26. — С. 35–47. [Application of Adiabatic Modes Method for Calculation and Design of Thin-Film Waveguide Generalized Luneburg Lens / A. A. Egorov, A. L. Sevastyanov, E. A. Ayryan et al. // Vestnik Tverskogo gosudarstvennogo universiteta. Prikladnaja matematika. — 2012. — Vol. 3, No 26. — Pp. 35–47.]
8. *Жидков Е. П., Макаренко Г. И., Пузынин И. В.* Непрерывный аналог метода Ньютона в нелинейных задачах физики // ЭЧАЯ. — 1973. — Т. 4, № 1. — С. 127–166. [Zhidkov E. P., Makarenko G. I., Puzynin I. V. A Continuous Analogue of Newton's Method in Nonlinear Problems of Physics // Physics of Elementary Particles and Atomic Nuclei. — 1973. — Vol. 4, No 1. — Pp. 127–166.]
9. Обобщённый непрерывный аналог метода Ньютона для численного исследования некоторых нелинейных квантово-полевых моделей / И. В. Пузынин, И. В. Амирханов, Е. В. Земляная и др. // ЭЧАЯ. — 1999. — Т. 30, № 1. — С. 210–265. [Generalized Continuous Analogue of Newton's Method for the

Numerical Investigation of Some Nonlinear Quantum-Field Models / I. V. Puzynin, I. V. Amirkhanov, E. V. Zemlyanaya et al. // Physics of Elementary Particles and Atomic Nuclei. — 1999. — Vol. 30, No 1. — Pp. 210–265.]
10. Maplesoft Online Help. — `http://www.maplesoft.com/support/help/`.
11. *Ильин В. А., Позняк Э. Г.* Линейная алгебра. 4-е изд. — М.: Физматлит, 1999. [Il'in V. A., Poznjak E. G. Linear Algebra. 4-th edition. — Moscow: Fizmatlit, 1999.]

УДК 517:519.6, 535+537.8:621.37

Вывод дисперсионного уравнения для трехслойной интегрально-оптической линзы Люнеберга в виде дифференциального уравнения в частных производных

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В работе представлен вывод дисперсионного уравнения для трёхслойной интегрально-оптической линзы Люнеберга на основе метода адиабатических волноводных мод. Из этого уравнения следует связь между коэффициентом фазового замедления и функцией, определяющей толщину нерегулярного волноводного слоя. Дисперсионное уравнение представляется в виде нелинейного дифференциального уравнения в частных производных первого порядка с коэффициентами, зависящими от параметров. В число таких параметров входят как толщины регулярных волноводных слоёв, так и оптические параметры рассматриваемой линзы Люнеберга. Для представления дисперсионного уравнения в виде дифференциального уравнения в частных производных возникает необходимость вычисления в символьном виде определителя матрицы 12-го порядка, определяющего разрешимость системы линейных алгебраических уравнений, следующих из граничных условий. Для вычисления данного определителя в аналитической виде предлагается процедура редуцирования системы линейных алгебраических уравнений с применением системы компьютерной алгебры Maple.

Ключевые слова: нерегулярный интегрально-оптический волновод, метод адиабатических мод, системы компьютерной алгебры.