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Calculation of the Magnetic System by the Solution of Inverse Problem

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In this paper we study the problem of searching for the design of the magnetic system for creation of a magnetic field with the required characteristics in the given area. On the basis of analysis of the mathematical model of the magnetic system rather a general approach is proposed to the solving of the inverse problem, which was written by the Fredholm equation:

$$H(z) = \int\limits_S J(s)G(z,s)\mathrm{d}s, \quad z \in U, \quad s \in S.$$

It was necessary to define the current density distribution function $J(s)$ and the existing winding geometry for creation of a required magnetic field $H(z)$. In the paper a method of solving those by means of regularized iterative processes is proposed. On the base of the concrete magnetic system we perform the numerical study of influence of different factors on the character of the magnetic field being designed.

Key words and phrases: magnet systems, inverse problem, Fredholm equation, regularized iterative processes.

1. Introduction

When designing magnetic system it's necessary to solve the inverse problem, that is, via a given magnetic field to define current parameters or, its geometrical characteristics, or all that simultaneously.

The definition of the beam density distribution in the magnetic system, in which the geometry is known, is a linear inverse problem for the given field.

When the required field must be created with the help of conductors, the value of the current which varies similar to the coordinates of their position providing the current in all the conductors is the same, we come to the solving of the inverse problem [1].

In this paper we consider the construction of a mathematical model of the magnetic system for this kind of the problem and the methods and numerical algorithms for their solution by using the Tikhonov regularization methods. Because a magnetic field is supposed to be given by one of its components (H_x, H_y, H_z) depending on a specific problem, so further H sample will be used for notation.

2. Mathematical Model of the Magnetic System

Let in a region U with the help of the sources of current distributed in the region S a field H should be created with the given characteristics (for example, the whole homogeneous field in the region U). It is known, that the field in any point z of set U is defined by the expression

$$H(z) = \int\limits_S J(s)G(z,s)\mathrm{d}s, \quad z \in U, \quad s \in S \tag{1}$$

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where $J(s)$ is a distributed density function of the current in the system, $G(z,s)$ is a Green function, that analytically depends both on the geometry of the source of the magnetic system and on the point $z \in U$.

The inverse problem, namely, a definition over the given density of distribution of current in the magnetic system with the known geometry is a linear inverse problem (model 1).

Then the mathematical problem reduces to the solution of the Fredholm linear integral equation of the first order with unknown function $J(s)$.

If the composition of the magnetic field includes not only variant density of current and arrangement source of current, then we must solve the nonlinear inverse problem (model 2) with unknown $J(s)$ and $s \in S$.

3. Method of Solution of the Inverse Problem (Model 1)

It is known that the problem of solution of the first order Fredholm integral equation (1) is related to the non-correct defined class of the problems, because the large changing in the $J(s)$ solution can correspond the small changing of the input data $H(z)$. To obtain a stable solution of the non-correct defined problem, A. N. Tikhonov developed a regularized algorithm [1].

Here we will use the second order method of a regularization in order to solve the problem.

For this, we construct a smooth parametric functional

$$F^{\alpha}[J(s),H(z)] = \Phi[J(s),H(z)] + \alpha\ \Omega[J(s)], \tag{2}$$

where

$$\Phi[J(s),H(z)] = \int\limits_U [H(z) - \int\limits_S J(s)G(z,s)\mathrm{d}s]^2 \mathrm{d}z \tag{3}$$

is the quadratic deviation of the operator $A[z,J(s)] = \int\limits_S J(s)G(z,s)\mathrm{d}s$ of function $H(z)$,

$$\Omega[J(s)] = \int\limits_S J^2(s)\mathrm{d}s \tag{4}$$

is the regularizational functional, or a stable one, and α is a numeric parameter of the regularization ($\alpha > 0$).

Theorem 1. *For any function $H(z) \in L_2$ and for any $\alpha > 0$ there exists one and only one $2(n+1)$ differencial function $J_n^{\alpha}(s)$, which realizes the minimum of the smooth functional $F^{\alpha}[J(s),H(z)]$ of the form (2).*

Theorem 2. *If $H(z) = A[z,J(s)]$, $J(s) \in C^{(n+1)}$, then for any $\epsilon > 0$ and auxiliary values $0 < \gamma_1 < \gamma_2$ there exists $\delta(\epsilon,\gamma_1,\gamma_2,J)$ so that, if*

1. – $\|\tilde{H}_\delta(z) - {*H}(z)\|_{L_2} \leqslant \delta$, *where* $\tilde{H}_\delta(z) \in L_2$;
2. – $\alpha = \alpha(\delta)$ *has the order* δ^2;
3. – $\gamma_1 \leqslant \dfrac{\delta^2}{\alpha(\delta)} \leqslant \gamma^2$,

then $J^{\alpha}_{\delta,n}(s)$ is a minimum of $F_n^{\alpha}[J_n^{\alpha}(s),\tilde{H}_\delta(z)]$ and
$\|J^{\alpha}_{\delta,n}(s)^{(i)} - J^*(s)^{(i)}\| \leqslant \epsilon, \quad s \in S,\ i = 1,2,\cdots,n$ *with* $\delta < \delta_0(\epsilon,\gamma_1,\gamma_2,J)$.

For this theorem we infer that there exists a function J_n^{α}, that is a minimum of functional F_n^{α} in the form (2) which reduces to the solution of the equation (1) $J(s)$. The complete demostration of this and other conditions one can find in [1].

When applying the regularizational method, the selection of parameter α is one of the main problems.

The point is that not always for the obtained smooth solution the discrepancy principle is being fulfilled, i.e. the inequality

$$\|\tilde{H}_\delta - H^*\| \leqslant \delta, \tag{5}$$

where δ is precision of the approximate input data $\tilde{H}_\delta$, H^* is the precise value of the input data, holds.

In practice, for the solution of the non-correct and correct problem it is necessary to find the solution, that satisfies the required precision. In [2] V. A. Morozov suggested, as the main quality criteria to select, the regularization parameter of the deflection principle.

Discrepancy Principle. Set any $0 < \delta \leqslant \delta_0$ and any $0 < h \leqslant h_0$ with condition

$$\chi(h, \delta, J) = (\|A_h J - AJ\|_0 + \delta) \times (1 + \beta(\delta, h))^{1/2} < \|A_h J - \tilde{H}_\delta\|_0,$$

where $\beta(\delta, h)$ is a positive function, such that $\lim_{\delta,h\to 0} \beta(\delta, h) = 0$, and J^α is the solution obtained for the minimum of functional $F^\alpha(J, H)$.

Then there exists at least one value of the regulartization parameter $\alpha = \alpha(\delta, h) > 0$, so that $\rho_{\delta h}(\alpha(\delta, h)) = \chi^2(h, \delta, J)$, $\lim_{\delta,h\to 0} J^\alpha_{\delta h} = J^\alpha$ and $J^\alpha_{\delta h} = J^{\alpha(\delta,h)}_{\delta h}$.

4. Numerical Algorithm for the Solution of the Problem of Model 1

In the expression (2), if presenting the integral in the form of sums, we obtain

$$F^\alpha = \sum_{j=1}^{N} [H_j(z_j) - \sum_{i=1}^{M} J_i(s_i) K_{ij}(z_j, s_i)]^2 \Delta z_j + \alpha \sum_{i=1}^{M} J_i^2(s_i) \Delta s_i, \tag{6}$$

where N is a number of points from the set U, M is a number of points from the set S, $M \leqslant N$ and $K_{ij} = \int\limits_{\Delta s_i} G(z_j, s) \mathrm{d}s$.

Suppose $\Delta s_i = \Delta s = \text{const}$, $\Delta z_j = \Delta z = \text{const}$.

The condition of the minimum of the functional F^α is

$$\frac{\partial F^\alpha}{\partial J_1} = 0, \quad \frac{\partial F^\alpha}{\partial J_2} = 0, \cdots, \frac{\partial F^\alpha}{\partial J_M} = 0. \tag{7}$$

Taking into account (7), we obtain

$$\frac{\partial F^\alpha}{\partial J_l} = -\sum_{j=1}^{N} H_j K_{lj} \Delta z + \sum_{j=1}^{N} \sum_{i=1}^{M} J_i K_{lj} K_{ij} \Delta z + \alpha J_l \Delta s = 0, \quad l = 1 \div M. \tag{8}$$

In such a way we have a system M linear algebraic equations with the unknowns N of the J_l form:

$$\sum_{i=1}^{M} J_i \sum_{j=1}^{N} K_{lj} K_{ij} \Delta z + \alpha J_l \Delta s = \sum_{j=1}^{N} H_j K_{lj} \Delta z, \quad l = 1 \div M. \tag{9}$$

Supposing $\alpha\Delta s = \alpha'\Delta z$, we obtain

$$\sum_{i=1}^{M} J_i \sum_{j=1}^{N} K_{lj}K_{ij} + \alpha' J_l = \sum_{j=1}^{N} H_j K_{lj}, \quad l = 1 \div M. \tag{10}$$

Obviously, α' serves the meaning of arbitrary coefficient α. Therefore the system of equations for J_l can be written finally in the form

$$\sum_{i=1}^{M} J_i \sum_{j=1}^{N} K_{lj}K_{ij} + \alpha J_l = \sum_{j=1}^{N} H_j K_{lj}, \quad l = 1 \div M. \tag{11}$$

If the magnetic system is a discrete set of coils, then the field $H(z)$ in any point $z \in U$ is defined in the following way:

$$H(z) = \sum_{i=1}^{M} J_i \int_{\Delta s_i} G(z, s)\mathrm{d}s, \tag{12}$$

where M is a number of coils, J_i is the current density in the i-st coil, Δs_i is the selection of the i-th coil, $G(z, s)$ is a Green function.

Having solved the system of equations (12), we obtain a discrete set J_l, $l = 1 \div M$, that is a solution of the problem (1). Similar to that, we define the distribution of the current density in the magnetic system for creation of field $H(z_j), j = 1 \div N, \quad z_j \in U$.

5. Particular Case of the Mathematical Model 2

Let in some region S with a disposition M of conductors with the same current I_0, the field H be created.

In the system $H(z)$, $z \in U$ we have

$$H(z) = I_0 \sum_{i=1}^{M} G(s_i, z), \tag{13}$$

where $G(s_i, z)$ is the Green function for the i-th conductor.

Both a current I_0 and the coordinates s_i of the conductors, which would provide the given field $H(z)$, $z \in U$ in a best way, should be defined.

The function $G(s_i, z)$ is usually a nonlinear one concerning the coordinate of the conductors s_i, therefore the analysed problem is a non-linear inverse problem.

Additional difficulty in the solving of the inverse problem is the restriction in the parameters [3]. However in any particular case one can efficiently find a solution for the given condition of the problem. Let us consider this case.

Let a parameter of the conductors disposition in the region S be only one coordinate, for example x, the region of disposition of the conductors in the axis x being known, $x_1 \leqslant x_i \leqslant x_2$.

Then the equation (14) will has the form:

$$H(z) = I_0 \sum_{i=1}^{M} G(x_i, z), z \in U, x_1 \leqslant x_i \leqslant x_2. \tag{14}$$

We must define I_0 , x_i to create the field $H(z), z \in U$ in the magnetic system. The problem (15) is a non-linear inverse problem.

6. Numerical Algorithm of the Solution of the Problem in Model 2

The solution of the problem (15) was divided in two steps. In the first step, the current density in the twisters is continuously distributed in the range of the given problem. The equation (14) has the form

$$H(z) = \int_{x_1}^{x_2} J(x)G(x,z)\mathrm{d}x.$$

This problem and the algorithm of its solution was analysed in the points 2, 3. To select the solution $J^\alpha(x)$, (α is the regularization parameter) we calculate the following conditions of the problem:

1. The precision of the calculation of $H(z)$ cannot be worse than the required precision of the magnetic field in the created magnetic system;
2. For all permissible interval $[x_1, x_2]$, a function $J^\alpha(x)$ must keep the sign;
3. $|J^\alpha(x)| \leqslant J_{dop}$ is the permissible current density.

Suppose, that there exists a continuous solution $J^\alpha(x)$, that satisfies all three conditions.

In the second step, we divide the interval $[x_1, x_2]$ in M subintervals $[x_1^i, x_2^i]$, $i = 1 \div M$.

Then

$$H(z) = \sum_{i=1}^{M} \int_{x_1^i}^{x_2^i} J^\alpha(x)G(x,z)\mathrm{d}x, \quad z \in U. \tag{15}$$

For each subinterval $[x_1^i, x_2^i]$ satisfies the conditions of the theorems about the mean value (as chosen function $J^\alpha(x)$), therefore

$$H(z_j) = \sum_{i=1}^{M} G(x_i^j, z_j) \int_{x_1^i}^{x_2^i} J^\alpha(x)\mathrm{d}x, \quad j = 1 \div N, \tag{16}$$

where N is a number of points in the region U, in which analyses field H, x_i^j is a point in the i-th intervals. The limits x_1^i, x_2^i chosen such that

$$\int_{x_1^i}^{x_2^i} J^\alpha(x)\mathrm{d}x = \int_{x_1^{i+1}}^{x_2^{i+1}} J^\alpha(x)\mathrm{d}x = I_0, \tag{17}$$

i.e.

$$I_0 = \frac{1}{M} \int_{x_1}^{x_2} J^\alpha(x)\mathrm{d}x,$$

then

$$H(z_j) = I_0 \sum_{i=1}^{M} G(x_i^j, z_j) \tag{18}$$

Obviously that for different z_j there exists its point x_i^j, but based on the theorem of the mean values, it is always in the intervals $[x_1^i, x_2^i]$. This means, that unknown coordinate x_i is also in the i-th interval and it is defined from the condition of minimum of the functional

$$\varphi(x_i) = \sum_{j=1}^{N} \left[\int\limits_{x_1^i}^{x_2^i} J^{\alpha}(x) G(x, z_j) \mathrm{d}x - I_0 G(x_i, z_j) \right]^2 m, \tag{19}$$

$$\frac{\partial \varphi(x_i)}{\partial x_i} = \sum_{j=1}^{N} \left[\int\limits_{x_1^i}^{x_2^i} J^{\alpha}(x) G(x, z_j) \mathrm{d}x - I_0 G(x_i, z_j) \right] \frac{\partial G(x_i, z_j)}{\partial x_i} = 0, \; x_1^i \leqslant x_i \leqslant x_2^i. \tag{20}$$

Then the solution of the problem is reduced to the solving of M sequential nonlinear equations in the form (20) with one unknown, moreover, the limits of the existence of the solution are known.

Note that we have analyzed the algorithms for creating a magnetic system with infinite thin conductor.

It's easy to demonstrate that for the finite size of the conductor, the algorithms completely persist, but in this case the Green function is under the sign of integration by the section of the conductor.

When the permissible geometrical region of arranging the conductor is defined as the nonlinear one, the density does not involve particular difficulties too and can be described by the similar algorithm.

7. Example of a Numerical Calculation of Real Magnetic System

Let's consider an example of practically developed application of the algorithm to create non-metallic superconductor (SP) of the bipolar magnet, that was composed by triangular winding of excitement, its geometry is showed in Fig. 1.

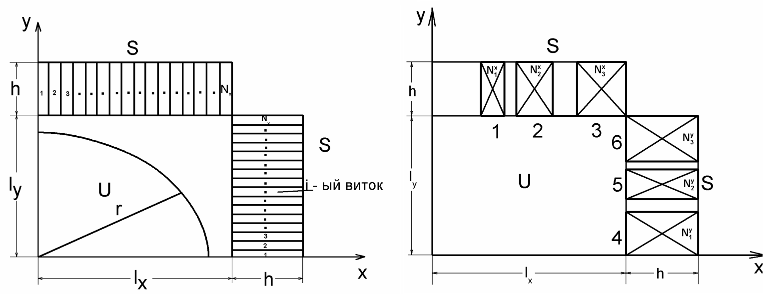

Figure 1. The bipolar SP configuration in the plane of the winding; b-one of the possible real SP configuration

From Fig. 1 it is obvious that the magnetic system was composed by triangular winding and it has a perimeter dimension of the aperture of magnet.

Using the developed numerical algorithm for the nonlinear inverse solution we calculate the mathematical model of the system, with a homogeneous field in which 80% of the aperture represent $10^{-5} \div 10^{-6}$ for the magnitude of file 4–5 Tl.

The mathematical problem was set in the following mode.

Let in some region U (see Fig. 1a), an halogenous field $H(z), z \in U$ be created, using an arrangement M of the conductors of the triangular section in the given limited region S with condition that the current I_0 for all conductors is the same. For this magnetic system

$$H(z) = I_0 \sum_{i=1}^{M} G(s_i, z), \quad s_i \in S, z \in U. \tag{21}$$

In the Descartes'system of carotenes $s_i = \{x_i, y_i\}, z = \{x, y\}$

$$G(s_i, z) = \frac{y - y_i + b}{2} \ln \frac{(x - x_i + a)^2 + (y - y_i + b)^2}{(x - x_i - a)^2 + (y - y_i + b)^2} + \\ + \frac{y - y_i - b}{2} \ln \frac{(x - x_i - a)^2 + (y - y_i - b)^2}{(x - x_i + a)^2 + (y - y_i - b)^2} + \\ + (x - x_i + a) \left(\operatorname{arctg} \frac{x - x_i + a}{y - y_i - b} - \operatorname{arctg} \frac{x - x_i + a}{y - y_i + b} \right) + \\ + (x - x_i - a) \left(\operatorname{arctg} \frac{x - x_i - a}{y - y_i + b} - \operatorname{arctg} \frac{x - x_i - a}{y - y_i - b} \right), \tag{22}$$

where a is the half-dimension of the tire along x, b is the half-dimension of the tire along y, $G(s_i, z)$ is the Green function for the triangular tire in the Descartes system of coordinates.

$$M = N_x + N_y,$$

N_x is the number of the creep tire of axis x, N_y is the number of the creep tire of axis y, or

$$M = \sum_{l=1}^{k} N_l,$$

k is the number of the winding blocks, and N_l is the number of the creeps in the 1-th bloks.

We must define not only I_0, but undetermine the block configuration that forms the homogeneus field $H(z)$ for every point $z \in U$ with a precision non less than $10^{-5} - 10^{-6}$.

Fig. 2 shows a continuous a continuous distribution J_x^α and J_y^α for $M = 48$ creep and its approximation by continous "blocks" function for each subinterval.

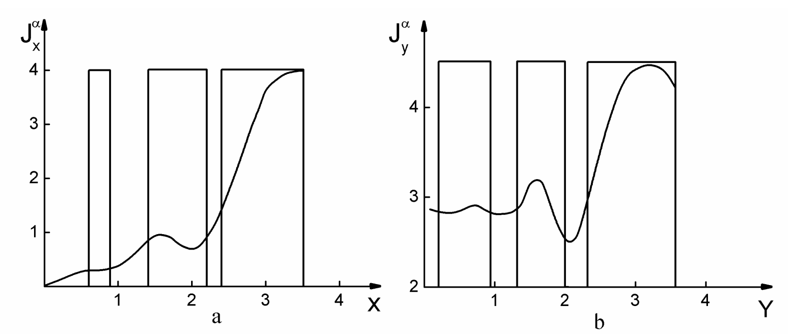

Figure 2. The continuous distribution $J^\alpha(s)$ and its approximation $J^\alpha(s)$ by constant subinterval function "blocks"

Table 1 contains the numerical calculation for the optimal variant of the magnet; the scheme of this magnet is presented in Fig. 1b.

Table 1

Values of the coordinates of these “blocks” center for x and y, given by the best relative precision of lower harmonics in the homogenous field H

		x=const=3.8					y=const=3.8			
N_y	N_{k_1} ⋮ N_2 N_1	y_1	y_2	y_3	N_x	N_{k_2} ⋮ N_2 N_1	x_1	x_2	x_3	Averaged relative precision of the lower harmonics c_2, c_4, c_6
20	10 5 5	0.509	1.5972	2.8588	9	6 2 1	1.0017	2.2937	3.1406	$0.3 \times 10^{-6} \div 0.5 \times 10^{-6}$
20	12 8	0.8421	2.6494		9	6 2 1	1.6024	2.2468	3.225	$0.9 \times 10^{-5} \div 0.45 \times 10^{-6}$

8. Conclusion

1. In this paper we analyze the method of solving the nonlinear inverse problems which are necessary for the description of the mathematical model of magnetic system of some class.
2. The developed numerical algorithm, based on the method of regularization of the solution of non-correct problems with restrictions in the searched parameters, is reduced to the nonlinear type of problem (15) for the solution of M sequential nonlinear equations with one incognita. It permits to avoid difficulties, related to the solution of the system of the nonlinear equations. This solution is frequently reduced to the inverse problem.
3. To realize the proposed method in computer a numerical algorithm was developed and Fortran programs package was written.
4. Using this complex program, some practical problems [4] were solved, one of those was analyzed as an example in section 7.

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Расчёт магнитной системы при помощи решения обратной задачи магнитостатики

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В данной работе решается задача поиска конструкции магнитной системы для создания магнитного поля с требуемыми характеристиками в заданной области. На основе анализа математической модели магнитной системы предлагается достаточно общий подход к решению нелинейной обратной задачи, которая описывается уравнением Фредгольма:

$$H(z) = \int_S J(s)G(z,s)\mathrm{d}s, \quad z \in U, \quad s \in S.$$

Необходимо определить распределение плотности тока $J(s)$, а также расстановку источников тока для создания поля $H(z)$. В работе предлагается метод решения этих задач с помощью регуляризованных итерационных процессов. На примере конкретной магнитной системы проводится численное исследование влияния различных факторов на характер создаваемого магнитного поля.

Ключевые слова: магнитные системы, уравнение Фредгольма, метод регуляризации.