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Физика

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On Nonlinear Generalization of Yukawa Meson Theory of Nuclear Forces

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We consider a nonlinear generalization of the Yukawa meson theory of nuclear forces that belongs to the class of theories with the Lagrangians proposed by L. Shiff. Relying on this basis, we study nonlinear equations for a nuclear potential and dynamic equations for nuclear particles which contain one unknown function of this potential. To find it, we suggest a principle which implies the proportionality of the nuclear field potential to the potential energy of nucleons in the field. The found equations are applied to study the dynamics of nuclear particles moving under the simultaneous action of nuclear and electromagnetic forces. It is shown that in the case of sufficiently intensive interaction of nucleons, the suggested equations give the value 14.9 for the dimensionless constant of strong interaction close to the experimental value.

Key words and phrases: Yukawa's theory, Shiff's Lagrangians, nonlinear theory of nuclear forces, strong interaction.

As is well known, the meson theory of nuclear forces proposed by H. Yukawa can give an adequate description of them only in the case of comparatively small values of nuclear potentials. Because of linearity, this theory does not allow one to describe such nonlinear effects as the saturation of nuclear forces, alteration of their sign at sufficiently short distances between nucleons and other effects [1, 2]. That is why various attemts were made to construct nonlinear generalizations of the Yukawa theory [3, 4]. One of them was an interesting program of investigation of nonlinear nuclear interactions proposed by L. Shiff [4].

Before considering the Shiff's method, let us turn to the classical Yukawa's theory of nuclear forces. We represent the well-known Yukawa's equation [2] in the following form:

$$\partial^n \partial_n \psi + \left(\frac{m_\pi c}{\hbar}\right)^2 \psi = -4\pi G \vartheta(\mathbf{r}, t), \tag{1}$$

where ψ is a scalar potential of nuclear forces, $\vartheta(\mathbf{r}, t)$ is the density of distribution of the nucleonic source of the nuclear field, $\mathbf{r}$ and t are the radius vector and time, respectively, $\partial_n = \partial/\partial x^n$, x^n are rectangular coordinates of the Minkowski space-time, m_π is the pion rest mass and G is a constant of nuclear interaction.

In the Yukawa's theory the constant G is regarded as a nucleon's "nuclear charge" and the value $G\psi$ presents the potential energy of a nucleon moving in the considered nuclear field.

Let us put

$$\varphi = \frac{G\psi}{m_p}, \quad \vartheta = \frac{\rho_0(\mathbf{r}, t)}{m_p}, \tag{2}$$

where m_p is the rest mass of the proton and ρ_0 is the density of the rest mass of the source of the nuclear field. Then the Yukawa's equation (1) can be rewritten as

$$\partial^n \partial_n \varphi + \left(\frac{m_\pi c}{\hbar}\right)^2 \varphi = -4\pi \left(\frac{G}{m_p}\right)^2 \rho_0(\mathbf{r}, t). \tag{3}$$

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Since the value $m_p\varphi \equiv G\psi$ is the potential energy of a nucleon moving in the Yukawa field, the value $m\varphi$ presents the potential energy of a nuclear particle with rest mass m.

It should be noted that an individual pion is a pseudoscalar particle. That is why the scalar potential φ can describe only classical nuclear fields consisting of a sufficiently large number of virtual pions. We will further study such nuclear fields.

Let us consider the nuclear field of a nucleon that is approximated by a material point. Then it follows from (1) and (2) that outside the nucleon its nuclear potential φ at distance r from its center is determined by the Yukawa's solution [2]

$$m_p\varphi = -\frac{G^2}{r}\exp(-m_\pi r/c\hbar). \tag{4}$$

The experimental value of the constant G obtained by the investigation of the deuteron's field is given by the formula [2]

$$\frac{G^2}{\hbar c} = 0.080. \tag{5}$$

The dimensionless value (5) characterizes the case of low-energy interaction of nucleons in nuclei. As to the case of high-energy interaction of nucleons, the corresponding dimensionless constant is approximately equal to the number 15 [5]. This implies that the linear Yukawa's equation becomes incorrect when the nuclear potential φ is sufficiently large and in such cases it should be replaced by its nonlinear generalization.

Let us now turn to the nonlinear Shiff's model of nuclear interaction [4]. In this model a classical nonlinear nuclear field is considered which consists of a sufficiently large number of virtual pions and can be described by a classical scalar nuclear potential φ. In Shiff's model the sources of such a nuclear field are considered in the classical approximation and described by their rest mass density ρ_0.

To obtain his nonlinear generalization of Yukawa's equation, Shiff used the class of Lagrangians L which we represent in the form

$$L = -\rho_0\Phi(\varphi) + \frac{m_p^2}{8\pi G^2}\left[\partial^n\varphi\partial_n\varphi - \left(\frac{m_\pi c}{\hbar}\right)^2\varphi^2 - \frac{k}{2}\varphi^4\right], \tag{6}$$

where $\Phi(\varphi)$ is some function describing the nonlinear interaction of the nuclear field with its source and k is some constant which determines the self-action of this field [4].

Let us use the principle of least action for Lagrangian (6):

$$\delta S = 0, \quad S = \frac{1}{c}\int L\,\mathrm{d}^4x. \tag{7}$$

Then the variation of the action S with respect to the potential φ gives the Euler-Lagrange equation

$$\frac{\partial L}{\partial\varphi} - \partial_n\left(\frac{\partial L}{\partial_n\varphi}\right) = 0. \tag{8}$$

Substituting expression (6) for the Lagrangian L into equation (8), we obtain

$$\partial^n\partial_n\varphi + \left(\frac{m_\pi c}{\hbar}\right)^2\varphi + k\varphi^3 = -4\pi\left(\frac{G}{m_p}\right)^2\rho_0 f(\varphi), \quad f(\varphi) = \Phi'(\varphi). \tag{9}$$

Let us now consider the dynamics of nuclear matter in the classical approximation using Lagrangian (6). For this purpose we can apply the well-known formula [6] for the following 4-invariant of a small part of the field source relative to inertial frames:

$$\rho_0\,\mathrm{d}^4x = \rho_0\,\mathrm{d}V_0\,\mathrm{d}s = \mathrm{d}m_0\,\mathrm{d}s, \tag{10}$$

where $\mathrm{d}V_0$ and $\mathrm{d}s/c$ are the small spatial volume and proper time in the local inertial frame comoving to a small part of the field source with the density ρ_0 and rest mass $\mathrm{d}m_0$ in the comoving frame.

Then, using (6) and (10), the action S in (7) can be represented in the form

$$S = -\frac{1}{c}\int \mathrm{d}m_0 \int \Phi(\varphi)\mathrm{d}s + \frac{m_p^2}{8\pi c G^2}\int \left[\partial^n\varphi\partial_n\varphi - \left(\frac{m_\pi c}{\hbar}\right)^2\varphi^2 - \frac{k}{2}\varphi^4\right]\mathrm{d}^4x, \quad (11)$$

where

$$\mathrm{d}m_0 = \rho_0\mathrm{d}V_0, \quad \mathrm{d}s = \sqrt{\mathrm{d}x^n\mathrm{d}x_n}, \quad x^n = x^n(\tau). \quad (12)$$

Here $\mathrm{d}m_0$ is the rest mass of a small part of the nuclear matter under consideration, $x^n(\tau)$ are space-time coordinates of this small part and τ is an arbitrary timelike parameter.

Since when $\varphi = 0$ expression (11) should coincide with the classical expression for action, the value $\Phi(0)$ should be equal to the following [6]:

$$\Phi(0) = c^2. \quad (13)$$

Let us apply the principle of least action to expression (11), using formulas (12), with respect to the trajectory $x^n(\tau)$ of a small part of the nuclear field source with mass $\mathrm{d}m_0$. Then we obtain the following Euler-Lagrange equations:

$$\frac{\partial\overline{L}}{\partial x^n} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\overline{L}}{\partial(\mathrm{d}x^n/\mathrm{d}\tau)}\right) = 0, \quad \overline{L} = \Phi(\varphi)\sqrt{\overline{g}_{mn}\frac{\mathrm{d}x^m}{\mathrm{d}\tau}\frac{\mathrm{d}x^n}{\mathrm{d}\tau}}, \quad (14)$$

where $\overline{g}_{mn}$ is the Minkowski tensor.

From (14) we readily obtain after dividing it by $\mathrm{d}s/\mathrm{d}\tau$:

$$\Phi'(\varphi)\frac{\partial\varphi}{\partial x^n} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi(\varphi)\frac{\mathrm{d}x_n}{\mathrm{d}s}\right) = 0. \quad (15)$$

Equation (15) can be rewritten as

$$\Phi(\varphi)\frac{\mathrm{d}^2x^n}{\mathrm{d}s^2} + \Phi'(\varphi)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}x^n}{\mathrm{d}s} - \frac{\partial\varphi}{\partial x_n}\right) = 0. \quad (16)$$

These four equations describe the dynamics of particles of nuclear matter under the action of nuclear forces.

It should be noted that the four equations (16) are not independent, since the first equation ($n = 0$) is a consequence of the other three equations.

Actually, multiplying the left-hand side of equation (16) by $\mathrm{d}x_n/\mathrm{d}s$ and summing over n, we get zero, since

$$\frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\mathrm{d}x^n}{\mathrm{d}s} \equiv 1, \quad \frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\partial\varphi}{\partial x^n} \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}s}, \quad \frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\mathrm{d}^2x^n}{\mathrm{d}s^2} \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\mathrm{d}x^n}{\mathrm{d}s}\right) \equiv 0. \quad (17)$$

Consider equations (16) in the nonrelativistic case. Then from them we find

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2} + \Psi(\varphi)\frac{\partial\varphi}{\partial x^k} = 0, \quad k = 1, 2, 3, \quad (18)$$

where

$$\Psi(\varphi) = \frac{c^2\Phi'(\varphi)}{\Phi(\varphi)}, \quad t = \frac{x^0}{c}, \quad \left|\frac{\mathrm{d}x^k}{\mathrm{d}t}\right| \ll c. \quad (19)$$

Let us determine the function $\Phi(\varphi)$. For this purpose we will state a principle concerning the physical sense of the potential φ of nuclear forces. As noted above, in the Yakawa's theory the value $m_p\varphi$ presents the proton's potential energy in the nuclear field. We will not change this physical sense of the potential φ in the nonlinear generalization under consideration of the Yukawa's theory. This can be justified by the idea that the considered scalar theory of the nuclear field should be related to the scalar theory of the electrostatic field. Namely, the only difference of the first of these fields from the second should be the non-zero mass of its carriers. Then in the nuclear field, as well as in the electrostatic one, the potential energy of a probe particle should be proportional to the scalar field potential. Thus, we come to the following principle.

Principle. In the nonlinear scalar theory of the nuclear field, as well as in the Yukawa's theory, the value $m_p\varphi$ presents the proton's potential energy.

Let us turn to the dynamic equations (18) in the nonrelativistic case. Taking into account (19), from them we find

$$\frac{\mathrm{d}^2x^k}{\mathrm{d}t^2}+c^2\frac{\partial\ln|\Phi(\varphi)|}{\partial x^k}=0,\quad k=1,2,3. \tag{20}$$

Consider a stationary nuclear field. Then multiplying (20) by $m_p\mathrm{d}x^k/\mathrm{d}t$ and simming it over k, we get

$$\frac{m_pv^2}{2}+m_pc^2\ln|\Phi(\varphi)|=\mathrm{const},\quad v=\sum_{k=1}^{3}\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}\right)^2. \tag{21}$$

Equation (21) presents the nonrelativistic law of energy conservation for a proton moving in a stationary nuclear field. In it the first summand is the proton's kinetic energy and the second summand is its potential energy.

On the other hand, as follows from the above principle, this law of energy conservation should have the form

$$\frac{m_pv^2}{2}+m_p\varphi=\mathrm{const}. \tag{22}$$

Comparing equations (21) and (22) and taking into account (13), we come to the following formula for the function $\Phi(\varphi)$:

$$\Phi(\varphi)=c^2\mathrm{e}^{\varphi/c^2}. \tag{23}$$

Let us use formula (23) and set the constant k in (9) equal to zero in order to have the Yukawa's equation for free pions. Then the field equation (9) and the dynamic equations (16) acquire the form

$$\partial^n\partial_n\varphi+\left(\frac{m_\pi c}{\hbar}\right)^2\varphi=-4\pi\left(\frac{G}{m_p}\right)^2\rho_0(\mathbf{r},t)\mathrm{e}^{\varphi/c^2}, \tag{24}$$

$$c^2\frac{\mathrm{d}^2x^n}{\mathrm{d}s^2}+\frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}x^n}{\mathrm{d}s}-\frac{\partial\varphi}{\partial x_n}=0. \tag{25}$$

It should be noted that the exponential multiplier in the right-hand side of equation (24) allowed us in our papers [7,8] to describe the effect of nuclear saturation and the experimental values of binding energies and radii of medium and heavy nuclei. The approach used in the present paper to describe nuclear fields, which is based on action (11) and the above principle, gives a theoretical justification for investigations in [7,8].

Let us generalize the dynamic equations (25) in the case of simultaneous action of nuclear and electromagnetic fields. For this purpose we add the following summand

$S_{\rm em}$ to action (11) corresponding to an electromagnetic field with potentials A_n and strengths F_{mn} [6]:

$$S_{\rm em} = -c^{-1}\int {\rm d}q \int A_n {\rm d}x^n - (16\pi c)^{-1}\int F_{mn}F^{mn}\,{\rm d}^4x, \tag{26}$$

where

$${\rm d}q = \theta_0 {\rm d}V_0, \quad x^n = x^n(\tau). \tag{27}$$

Here θ_0 and ${\rm d}V_0$ are the charge density and the three-dimensional volume of a small part of the field source, respectively, in its comoving local inertial frame, ${\rm d}q$ is the charge of this small part of the source and τ is an arbitrary timelike parameter.

Let us apply the principle of least action to the sum of actions (11) and (26) with respect to the trajectory $x^n(\tau)$ of a small part of the field source with rest mass ${\rm d}m_0$ and charge ${\rm d}q$. As a result, instead of (16) we obtain the Euler-Lagrange equations

$${\rm d}m_0\left[\Phi(\varphi)\frac{{\rm d}^2x^n}{{\rm d}s^2} + \Phi'(\varphi)\left(\frac{{\rm d}\varphi}{{\rm d}s}\frac{{\rm d}x^n}{{\rm d}s} - \frac{\partial\varphi}{\partial x_n}\right)\right] - {\rm d}q\; F^n_{.m}\frac{{\rm d}x^m}{{\rm d}s} = 0. \tag{28}$$

Substituting expression (23) for $\Phi(\varphi)$ into (28) and using (12) and (27), we come to the following dynamic equations:

$$\rho_0 {\rm e}^{\varphi/c^2}\left[c^2\frac{{\rm d}^2x^n}{{\rm d}s^2} + \frac{{\rm d}\varphi}{{\rm d}s}\frac{{\rm d}x^n}{{\rm d}s} - \frac{\partial\varphi}{\partial x_n}\right] - \theta_0 F^n_{.m}\frac{{\rm d}x^m}{{\rm d}s} = 0. \tag{29}$$

When $\varphi = 0$ these equations coincide with the classical dynamic equations for charged matter in the electromagnetic field [6] in which the field strengths $F^n_{.m}$ satisfy the Maxwell equations

$$\partial_n F^n_{.m} = 4\pi\theta_0\frac{{\rm d}x_m}{{\rm d}s}. \tag{30}$$

Equations (29) can be represented in the form

$${\rm e}^{\varphi/c^2}\partial^n\varphi + \gamma F^n_{.m}\frac{{\rm d}x^m}{{\rm d}s} = W^n, \tag{31}$$

where

$$W^n = {\rm e}^{\varphi/c^2}\left(c^2\frac{{\rm d}^2x^n}{{\rm d}s^2} + \frac{{\rm d}\varphi}{{\rm d}s}\frac{{\rm d}x^n}{{\rm d}s}\right), \tag{32}$$

$$\gamma = \frac{\theta_0}{\rho_0}. \tag{33}$$

From equations (31) we get

$$\partial_n W^n = {\rm e}^{\varphi/c^2}\partial^n\partial_n\varphi + \partial_n F^n_{.m}\gamma\frac{{\rm d}x^m}{{\rm d}s} + U, \tag{34}$$

$$U = \frac{1}{c^2}{\rm e}^{\varphi/c^2}\partial^n\varphi\;\partial_n\varphi + F^n_{.m}\partial_n\left(\gamma\frac{{\rm d}x^m}{{\rm d}s}\right). \tag{35}$$

Using equations (24) and (30), from (34) we find

$$4\pi\left(\frac{G}{m_p}\right)^2\rho_0{\rm e}^{2\varphi/c^2} - 4\pi\theta_0\gamma = H, \tag{36}$$

$$H = U - \partial_n W^n - \left(\frac{m_\pi c}{\hbar}\right)^2{\rm e}^{\varphi/c^2}\varphi. \tag{37}$$

As follows from (33), formula (36) can be represented as

$$4\pi\rho_0\left[\left(\frac{G}{m_p}\right)^2 e^{2\varphi/c^2}-\gamma^2\right]=H, \tag{38}$$

where H is determined by expressions (32), (35) and (37) and can take on every values.

From formula (38) we find an extreme value of the nuclear potential $\varphi=\overline{\varphi}$ such that the density ρ_0 becomes infinite:

$$\frac{G}{m_p}e^{\overline{\varphi}/c^2}=|\gamma|\,. \tag{39}$$

Consider a moving proton. Then from (33) we get the following value of γ:

$$\gamma=\frac{e_p}{m_p}, \tag{40}$$

where e_p and m_p are the charge and rest mass of the proton, respectively.

From (39) and (40) we find the extreme value of the nuclear potential $\overline{\varphi}=\overline{\varphi}_p$ inside the proton:

$$\overline{\varphi}_p=-c^2\ln(G/e_p). \tag{41}$$

Consider a proton after its interaction with another particle. As follows from equation (24), it coincides with the Yukawa's equation outside field sources. Therefore, from (24) we find that the nuclear field outside a proton generated it can be represented in the Yukawa's form

$$m_p\varphi=-\frac{g^2}{r}\exp(-m_\pi cr/\hbar), \quad r\geqslant r_p, \tag{42}$$

where r_p is the proton's radius, r is the distance counted from its center in its comoving inertial frame and $g^2/\hbar c$ is a dimensionless constant of strong interaction depending on interactions of the proton with other particles.

When these interactions are weak, we have $m_p\,|\varphi|\ll m_pc^2$ [2]. In this case equation (24) coincides with the Yukawa's equation (3) and from (4) we get $g=G$ and the dimensionless constant of strong interaction should have the value (5).

Consider now the case of a sufficiently large energy of interaction of a proton with other particles. In this case let us study the extreme value $\overline{g}$ of the constant g. As follows from (42), the extreme value $\overline{g}$ can be reached when the nuclear potential φ of a proton is equal to its extreme value $\overline{\varphi}_p$ at the proton's boundary. The distribution of the potential φ inside a proton depends on a particle interacting with it. That is why let us consider such an interaction when the extreme value $\overline{\varphi}_p$ of the proton's potential φ is reached at its boundary $r=r_p$. Then setting $\varphi=\overline{\varphi}_p$ and $r=r_p$ in formula (42), from (41) and (42) we get the following equation for the extreme value $\overline{g}$ of the constant g:

$$m_pc^2\ln(G/e_p)=(\overline{g}^2/r_p)\exp(-m_\pi cr_p/\hbar). \tag{43}$$

From this equation we find the extreme value of the dimensionless constant of strong interaction:

$$\frac{\overline{g}^2}{\hbar c}=\left(\frac{m_pcr_p}{\hbar}\right)\ln(G/e_p)\exp(m_\pi cr_p/\hbar). \tag{44}$$

Substituting the value $r_p=1.17$ fm [2, p. 127] and the value (5) of G into (44), we obtain

$$\frac{\overline{g}^2}{\hbar c}=14.9 \tag{45}$$

which is very close to the experimental value of this constant approximately equal to 15 [5].

This result gives a good experimental verification of the suggested nonlinear equations (24) and (29) for nuclear matter moving in nuclear and electromagnetic fields.

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Об одном нелинейном обобщении мезонной теории Юкавы ядерных сил

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Мы рассматриваем нелинейное обобщение мезонной теории Юкавы ядерных сил, принадлежащее к классу теорий с лагранжианами, предложенными Л. Шиффом. Основываясь на этом базисе, мы изучаем нелинейные уравнения для ядерного потенциала и динамические уравнения для ядерных частиц, содержащие одну неизвестную функцию данного потенциала. Чтобы найти её, мы предлагаем принцип, означающий пропорциональность потенциала ядерного поля потенциальной энергии нуклонов в этом поле. Найденные уравнения применяются для изучения динамики ядерных частиц, движущихся при совместном действии ядерных и электромагнитных сил. Показывается, что в случае достаточно интенсивного взаимодействия нуклонов предложенные уравнения дают значение $14,9$ для безразмерной константы сильного взаимодействия, являющееся близким к экспериментальной величине.