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Continuous Analogue of Newton Method in Beam Dynamics Problems

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An algorithm of the continuous analogue of Newton method (CANM) is proposed for the solving of the boundary value problems of beam transport.
The efficiency of CANM has been practically shown on a number of the problems of beam dynamics leading to the solving of ordinary differential and integral equations.
The solving of the problem of determining the optimal (in sense of some criterions of quality) parameters P_i for charged particles transportation systems taking into account different nonlinear effects, is given. The results of the calculation of the consistent "invisible" straight section (insertion) of the accelerator obtained with the help of CANM are shown.

Key words and phrases: Newton method, beam dynamics.

1. Introduction

The creating of new accelerators and reconstruction of existing ones requires to solve complicated nonlinear problems. Mathematical modeling of these problems leads to interesting mathematical problems. Most of them are such complicated nonlinear problems (including inverse ones) that the only way for their investigation is to develop and implement a numerical method with the use of computers. In particular, among these problems, there is a problem of determination of the optimum (in the sense of some criterions of quality) parameters P_i for charged particles transportation systems taking into account some nonlinear effects; a problem of a design of matched "invisible" long–straight sections of an accelerator. All these physical problems can be formulated as mathematical problems of solution of boundary values problems for the systems of two second order nonlinear ordinary differential equations.

A number of physical problems [1,2] reduced to the solving of ordinary differential or integral equations, have been solved effectively with a continuous analogue of Newton method (CANM). That is why the authors of this paper decided to apply CANM to the solution of the problems mentioned above.

CANM it was actively used in works of mathematical modeling of problems of dynamics of particles [2–4]. The ideologist and scientific the professor E.P. Zhidkov was the head of these works in LIT JINR. Use of this method has allowed to solve a number of the physical problems connected with with enough split-hair accuracy

Problems of reconstruction of synchrophasetron JINR and problem of creation of the new superconducting accelerator on 1.5 Gev.

The purpose of the present work is to give detailed enough description of algorithm of numerical modeling of problems of dynamics of the charged particles in which basis the continuous analog of a method of Newton is put.

This method allows to fit parameters, make placements and estimate errors of placements for the elements of a system.

2. Physical Statement of the Problem

The scheme of the model of a charged particle transportation system with a chosen system of coordinates is shown in Fig. 1. One of the possible variants of the structure of "invisible" section for superconducting synchrotron is shown in Fig. 2. Physical statement of the problem of charged particle transportation is made in following way.

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The given charged particles beam's direction and spatial position at the entrance of a transportation system — $(\alpha_0,\ \alpha_{0\perp},\ s_0,\ x_0,\ y_0)$.

$$\left.\begin{matrix} s_0 \\ x_0 \\ y_0 \end{matrix}\right\} \text{ – coordinates of a trajectory}'\text{s initial point in the rectangular system of coordinates of an installation.}$$

α_0 — the angle contained by a tangent to the projection of the trajectory on the plane "SX" at point $(s_0,\ x_0,\ y_0)$ and S-axis (in radians); $\alpha_{0\perp}$ — the angle contained by a tangent to the projection of the trajectory on the plane "SY"and S-axis (in radians) at point $(s_0,\ x_0,\ y_0)$.

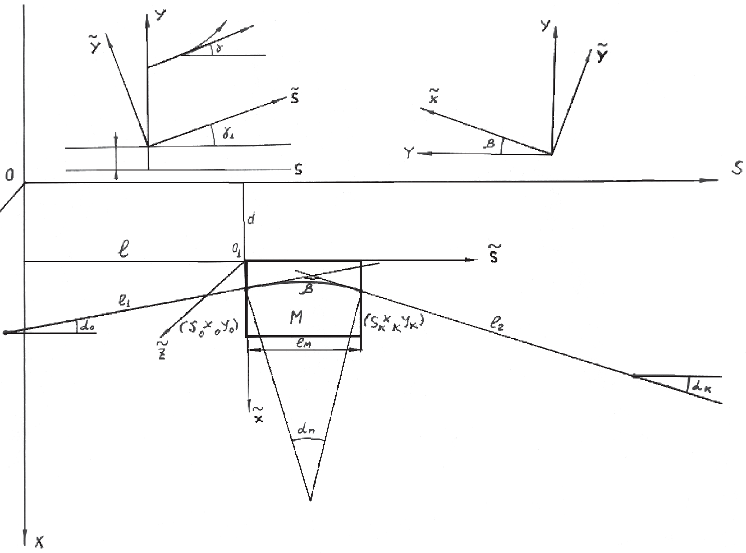

Figure 1. **The system of coordinates utilized for calculations: SX — basic rectangular system of coordinates; S_1X_1 and S_2X_2-systems of coordinates of the 1-st and 2-nd magnets**

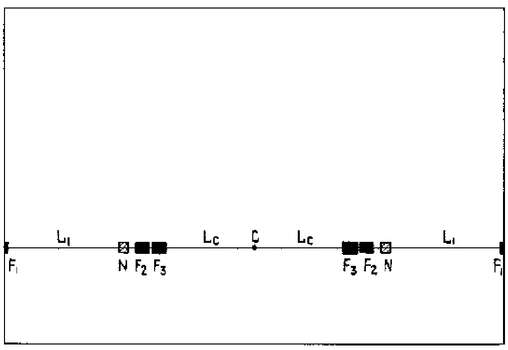

Figure 2. **Structure of the "invisible" section**

It is necessary to determine such parameters ($P_1, P_2, P_3, \ldots$; for example, p-momentum of a particle) so that for the given initial position and direction of the particle one can obtain its final position and direction (also given) — $(\alpha_k,\ \alpha_{k\perp},\ s_k,\ x_k,\ y_k)$.

$$\left.\begin{matrix} s_k \\ x_k \\ y_k \end{matrix}\right\} - \text{coordinates of a trajectory's final point in the rectangular system of coordinates of an installation.}$$

α_k and $\alpha_{k\perp}$ are of the same meaning as α_0 and $\alpha_{0\perp}$ but at point $(s_k,\ x_k,\ y_k)$.

3. Mathematical Statement of the Problem

Movement of a particle through a magnetic field is described by complete equations in the rectangular system of coordinates of an installation [5]:

$$\begin{cases} \dfrac{\mathrm{d}^2 x}{\mathrm{d}s^2} = \dfrac{A}{B_0 R_0}\varphi(s, x, y, x'_s, y'_s, B_s, B_x, B_y, P_i), \\ \dfrac{\mathrm{d}^2 y}{\mathrm{d}s^2} = \dfrac{A}{B_0 R_0}\psi(s, x, y, x'_s, y'_s, B_s, B_x, B_y, P_i), \end{cases} \tag{1}$$

where B_0R_0 — a magnetic rigidity of the particle and $\dfrac{1}{B_0R_0} = \dfrac{e}{p}$, p — momentum of the particle, $A = \sqrt{1 + (x'_s)^2 + (y'_s)^2}$, P_i — some parameters, their mathematical and physical meanings are defined in any particular case.

The components of the field $B(B_s, B_x, B_y)$ are defined either analytically or numerically (if the field is given as a table) in any particular physical case. B_s, B_x, B_y components are nonlinear functions of (s, x, y) and may depend on parameters P_i.

The boundary conditions are:

$$\begin{cases} x(s_0, P_i) = x_0, \\ y(s_0, P_i) = y_0, \\ x(s_k, P_i) = x_k = a, \\ y(s_k, P_i) = y_k = b, \end{cases} \qquad \begin{cases} x'_s(s_0, P_i) = \operatorname{tg}\alpha_0, \\ y'_s(s_0, P_i) = \operatorname{tg}\alpha_{0\perp}, \\ x'_s(s_k, P_i) = \operatorname{tg}\alpha_k = c, \\ y'_s(s_k, P_i) = \operatorname{tg}\alpha_{k\perp} = d. \end{cases} \tag{2}$$

The system of equations (1) may be transformed into a system of the first — order equations by substitution : $x'_s = x_1$ and $y'_s = y_1$

$$\begin{cases} (x_1)'_s = \dfrac{A}{B_0R_0}\varphi(s, x, y, x_1, y_1, B_y, P_i), \\ (y_1)'_s = \dfrac{A}{B_0R_0}\psi(s, x, y, x_1, y_1, B_y, P_i), \\ x'_s = x_1, \\ y'_s = y_1. \end{cases} \tag{3}$$

Then the boundary condition (2) may be rewritten as

$$\begin{cases} x(s_0, P_i) = x_0, \\ y(s_0, P_i) = y_0, \\ x_1(s_0, P_i) = \operatorname{tg}\alpha_0, \\ y_1(s_0, P_i) = \operatorname{tg}\alpha_{0\perp}, \end{cases} \qquad \begin{cases} x(s_k, P_i) = a, \\ y(s_k, P_i) = b, \\ x_1(s_k, P_i) = c, \\ y_1(s_k, P_i) = d. \end{cases} \tag{4}$$

Mathematical statement of the boundary value problem is as follows:

– It is necessary to determine such parameters P_i that the trajectory of a charged particle will satisfy equations (3) and following boundary conditions:

$$\begin{cases} x(s_k, P_i) = x_k(P_i) = a, \\ y(s_k, P_i) = y_k(P_i) = b, \\ x_1(s_k, P_i) = x_{1k}(P_i) = c, \\ y_1(s_k, P_i) = y_{1k}(P_i) = d. \end{cases} \tag{5}$$

– Let us rewrite boundary conditions (5) in a new form :

$$\begin{cases} f_1(s_k, P_i) = x_k(s_k, P_i) - a = 0, \\ f_2(s_k, P_i) = y_k(s_k, P_i) - b = 0, \\ f_3(s_k, P_i) = x_{1k}(s_k, P_i) - c = 0, \\ f_4(s_k, P_i) = y_{1k}(s_k, P_i) - d = 0. \end{cases} \tag{6}$$

The obtained system of four equations makes it possible to determine four parameters P_i, $i = 1 \div 4$. Satisfaction of the boundary conditions (5) achieved by these parameters fitting. The system of Eqs. (6) may be solved by the method of introduction t [6] if considering parameters P_i as functions of t, i.e. $P_i = P_i(t)$.

From this method we have:

$$\begin{cases} \frac{\partial}{\partial t} f_1(s_k, P_i) = -f_1(s_k, P_i), \\ \frac{\partial}{\partial t} f_2(s_k, P_i) = -f_2(s_k, P_i), \\ \frac{\partial}{\partial t} f_3(s_k, P_i) = -f_3(s_k, P_i), \\ \frac{\partial}{\partial t} f_4(s_k, P_i) = -f_4(s_k, P_i) \end{cases} \tag{7}$$

or, in detail,

$$\begin{cases} \sum\limits_{i=1}^{4} (x_k)'_{P_i} P'_{it} = -(x_k - a), \\ \sum\limits_{i=1}^{4} (y_k)'_{P_i} P'_{it} = -(y_k - b), \\ \sum\limits_{i=1}^{4} (x'_k)'_{P_i} P'_{it} = -(x'_k - c), \\ \sum\limits_{i=1}^{4} (y'_k)'_{P_i} P'_{it} = -(y'_k - d). \end{cases} \tag{8}$$

The values of ${P_i}'$, $i = 1 \div 4$ can be determined from the Eqs. (8).

Then, using the next formula

$$P'_{it} = \frac{P_i(t + \Delta t) - P_i(t)}{\Delta t}, \tag{9}$$

where Δt — a step for parameter t, we get the values of P_i for the next step on t:

$$P_i(t + \Delta t) = P_i(t) + P'_{it} \Delta t. \tag{10}$$

The values of $P_i(t_0)$ are given (the initial approximation for $t_0 = 0$).

In order to solve the system of equations (8), it is necessary to determine its coefficients $(x_k)'_{P_i}$, $(x'_k)'_{P_i}$, $(y_k)'_{P_i}$, $(y'_k)'_{P_i}$.

For this purpose we should solve following Cauchy problems for equations (3), (4):

1) Cauchy problem for the set of parameters P_1, P_2, P_3, P_4. Having solved this problem, we get:

$$\begin{cases} x_k(P_1, P_2, P_3, P_4), \\ y_k(P_1, P_2, P_3, P_4), \\ x'_k(P_1, P_2, P_3, P_4), \\ y'_k(P_1, P_2, P_3, P_4). \end{cases} \tag{11}$$

2) Cauchy problem for the set of parameters $P_1 + \Delta P_1, P_2, P_3, P_4$, where ΔP_1 — a step of the increment for corresponding parameter.
We obtain

$$\begin{cases} x_k(P_1 + \Delta P_1, P_2, P_3, P_4), \\ y_k(P_1 + \Delta P_1, P_2, P_3, P_4), \\ x'_k(P_1 + \Delta P_1, P_2, P_3, P_4), \\ y'_k(P_1 + \Delta P_1, P_2, P_3, P_4). \end{cases} \tag{12}$$

From (11) and (12) one can obtain

$$\begin{cases} (x_k)'_{P_1} = \dfrac{x_k(P_1 + \Delta P_1, P_2, P_3, P_4) - x_k(P_1, P_2, P_3, P_4)}{\Delta P_1}, \\ (x'_k)'_{P_1} = \dfrac{x'_k(P_1 + \Delta P_1, P_2, P_3, P_4) - x'_k(P_1, P_2, P_3, P_4)}{\Delta P_1}. \end{cases} \tag{13}$$

In the same way we get $(y_k)'_{P_1}$, $(y'_k)'_{P_1}$. The procedure described above can be used for calculating the derivatives with respect to P_2, P_3, P_4. It enables us to solve the system (8) and to obtain $P_i(t + \Delta t)$, $i = 1, 4$. So the first step of integration over t variable is complete. The next step is implemented by repeating the above procedure with new initial parameters obtained by formula (10).

This iterative process will be carried out until the given accuracy is attained.

4. Application of the Described Method to Some Physical Problems

The following physical problems have been solved by using the described method:

– calculation of the bend — focusing beam transport system for fast extraction of the beam from the Synchrophasetron at LHE, JINR [3];
– optimization of the long — straight matched sections of the 1.5 Gev proton superconducting synchrotron JINR and analysis of nonlinear aberrations in these sections [4].

Problem 1. A transportation system for beam fast extraction at the Synchrophasetron contains two turning magnets (see Fig. 1). As such magnets have a field with large gradient ($n = \dfrac{dB}{dR}\dfrac{R}{B} = -140$), the direction of the beam at the exit of the system depends essentially not only on the values of magnetic induction B_1 and B_2 but also on the magnet's position in space (turns around axes and slopes). The field into the magnet's iron is given as two – dimensional tables by of the results of the measurement. The values of the component $B_y(s, x, y)$ were measured in magnet's median planes in relative units. Using of tables is necessary because of complexity of the field distribution law at the edges of the magnets. In order to get a value of magnetic field in Gauss units one has to multiply a value from the tables by the value of magnetic induction B_1 or B_2, correspondingly, for the first and second magnets.

The values of components of the magnetic field can be obtained by formulae:

$$\begin{cases} B_y(s,x,y) = B_y(s,x,0) - \dfrac{y^2}{2}\left(\dfrac{\partial^2 B_y(s,x,0)}{\partial s^2} + \dfrac{\partial^2 B_y(s,x,0)}{\partial x^2}\right), \\ B_x(s,x,y) = -\dfrac{\partial B_y(s,x,0)}{\partial x} y, \\ B_s(s,x,y) = \dfrac{\partial B_y(s,x,0)}{\partial x} y. \end{cases} \tag{14}$$

The formulae (14) are obtained by Taylor expansion of $B_y(s,x,y)$ about the point $(s,x,0)$ up to second-order taking into account the following conditions:

$$\frac{\mathrm{d}^2 B_y}{\mathrm{d}s^2} + \frac{\mathrm{d}^2 B_y}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 B_y}{\mathrm{d}y^2} = 0 \quad \text{and} \quad \mathrm{rot}\,\vec{B} = 0. \tag{15}$$

The position of the magnets in space is determined by parameters:

$\left.\begin{matrix} d_1 \\ d_2 \end{matrix}\right\}$ – distances from the centres of the magnets to the "S" axis;

$\left.\begin{matrix} l_1 \\ l_2 \end{matrix}\right\}$ – distances from the centres of the magnets to the "X" axis;

$\left.\begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix}\right\}$ – horizontal angles of turn of the magnets around the "Y" axis;

$\left.\begin{matrix} \gamma_{1\perp} \\ \gamma_{2\perp} \end{matrix}\right\}$ – vertical angles of turn of the magnets around the "X" axis;

$\left.\begin{matrix} \beta_1 \\ \beta_2 \end{matrix}\right\}$ –vertical angles of turn of the magnets around the "S" axis.

Different combinations of the parameters by four have been taken as varied parameters. Each combination contained the values of magnetic induction B_1 and B_2 because these parameters exert a substantial influence on the trajectory of a particle. The table 1 contains coordinates and angles of beam at the exit of the system depending on the values of parameters $P_1 = B_1$, $P_2 = B_2$, $P_3 = d_1$, $P_4 = d_2$. The complete description of these results can be found in [5].

Table 1

Coordinates and angles of beam at the exit of the system depending on the values of parameters $P_1 = B_1$, $P_2 = B_2$, $P_3 = d_1$, $P_4 = d_2$

d_1	d_2	B_1	B_2	y	$\mathrm{tg}\,\alpha$
55	77	-100	-99	$143,4899$	$0,2573$
55	77	-99	-99	$143,3140$	$0,2569$
55	77	-99	-100	$143,3640$	$0,2573$
55	77	-99	-99	$143,3730$	$0,2610$
55	77	-99	-99	$143,6530$	$0,2600$

Problem 2. The organization of a beam's injection–extraction system, placement of an accelerating station etc. require long and free sections (see Fig. 2) in the modern rigidly focusing synchrotrons. The second physical problem is related to investigations of nonlinear aberrations in quadrupole lenses of the above mentioned section for the 1.5 Gev proton superconducting synchrotron (LHE of JINR) and its matching with

nonlinearities taken into consideration. It is worth mentioning that most papers devoted to design of a matched section are based on linear approximation. In this paper all calculations for the free section were implemented taking into account the nonlinear effects in lenses according to the method described above. Varied parameters are gradients of lenses (G_1, G_2, G_3), their lengths (l_1, l_2, l_3) and drift distances (L_1, L_2, L_3). The magnetic field of a quadrupole lens of great length almost does not depend on s, i.e. $B_s = 0$ and expansions of components B_x, B_y are:

$$\begin{cases} B_x = Gy[1 + d_6(5x^4 - 10x^2y^2 + y^4)+ \\ \qquad\qquad + d_{10}(9x^8 - 84x^6y^2 + 126x^4y^4 - 36x^2y^6 + y^8) + \cdots], \\ B_y = Gx[1 + d_6(5y^4 - 10x^2y^2 + x^4)+ \\ \qquad\qquad + d_{10}(9y^8 - 84y^6x^2 + 126x^4y^4 - 36y^2x^6 + x^8) + \cdots], \end{cases} \tag{16}$$

where G — a quadrupole gradient of a corresponding lens, $d_n = R^{(n-2)}c_n$, R — a radius is equal to half-aperture of a corresponding lens, c_n — a relative value of a field's nonlinearity in a lens. Numerical experiments showed that the best matching can be obtained by an insertion of corrective elements into the structure of the section (see Figs. 3, 4).

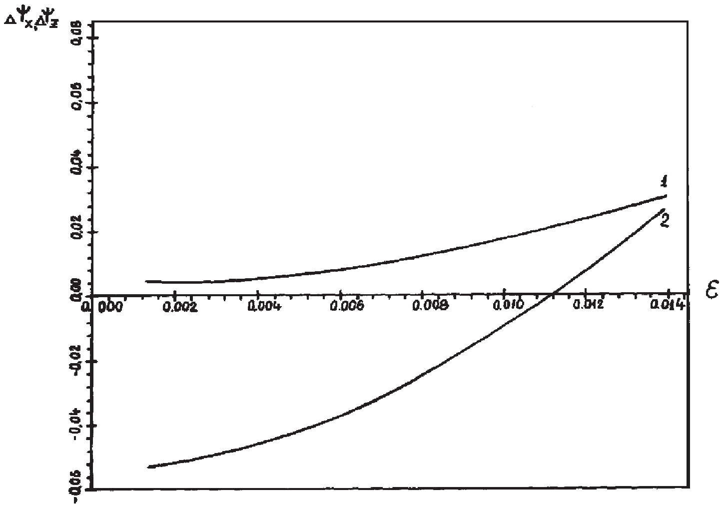

Figure 3. **Dependens of the phase shift $\nabla\psi_x$ (1) and $\nabla\psi_z$ (2) of magnitude emittensa beam for a linear coordination (without additional item)**

Computed optimum parameters of such elements for free section for the 1.5 Gev proton superconducting synchrotron are:

$$G_N = -464.296 \text{ Gs/cm}, \quad l_N = 13.2 \text{ cm}, \quad L_N = 10 \text{ cm}.$$

The complete description of the numerical modeling is given in [4].

5. Conclusions

The general approach to the solution of the problems of charged particles transport and matching of long-straight sections of accelerators is proposed. It is based on a continuous analogue of Newton method (CANM).

The developed method proved to be effective for solution of the described above boundary value problem of charged particles transport and allows to fit parameters

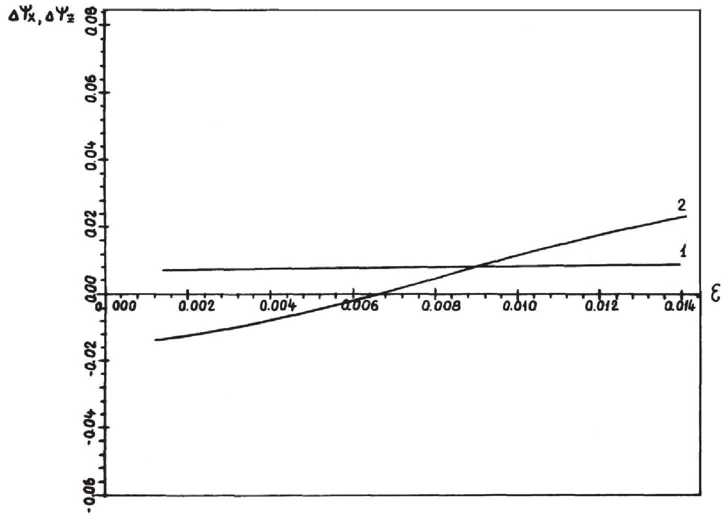

Figure 4. **Dependens of the phase shift $\bigtriangledown\psi_x$ (1) and $\bigtriangledown\psi_z$ (2) of magnitude emittensa beam for a unlinear coordination (with additional item)**

of the system and carry out the placement of system's elements according to initial and final positions and directions of a beam. The errors of element's placement can be estimated also.

1. The following physical problems have been solved:
   - parameters of proton's transport system for fast extraction of a beam at the Synchrophasetron of LHE, JINR have been computed and errors of turning magnets placement in the system have been obtained.
   - computation for a long matched section of the 1.5 Gev superconducting synchrotron (SPIN) have been carried out. Nonlinear aberrations have been taken into account. The results showed that nonlinear aberrations causes substantial dismatching in the matched sections and using of corrective nonlinear elements is the best way to eliminate this effect.
2. Further development of the proposed numerical method in connection with the transport systems is to try to obtain optimum parameters taking into consideration not only beam's direction but also extent of a beam's "spot".

References

1. *Zhidkov E. P., Puzynin I. V.* // Magazine of Calculus Mathematics and the Mathematical Physics (MCMMPH). — 1967. — Vol. 7, No 5. — P. 1086.
2. *Zhidkov E. P., Ryltseva T. V., Feoktistov B. V.* // Magazine of calculus mathematics and the mathematical physics (MCMMPH). — 1970. — Vol. 10, No 5. — P. 1199.
3. *Zhidkov E. P., Kulakova E. M., Polyakova R. V. et al.* — Preprint JINR, P9-6430, Dubna. — 1972.
4. *Zhidkov E. P., Polyakova R. V., Shelaev I. A., Yudin I. P.* — Preprint JINR, P11-11867, Dubna. — 1978.
5. *Steffen K.* High Energy Beam Optics. — Moscow: Mir, 1969.
6. *Zhidkov E. P., Puzynin I. V.* // Doklady Akademii Nauk SSSR. — 1968. — Vol. 180, No 1. — P. 18.

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Непрерывный аналог метода Ньютона для решения задачи динамики пучка

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На ряде физических задач, приводящих к решению обыкновенных дифференциальных и интегральных уравнений, практически показана эффективность непрерывного аналога метода Ньютона (НАМН), поэтому авторам данной работы показалось естественным развитие этого метода для решения задач при численном моделировании, связанном с проблемами создания новых ускорителей и реконструкции старых. НАМН, предлагаемый в данной работе для решения нелинейных задач транспортировки заряженных частиц, позволяет оптимальным образом подобрать параметры элементов системы транспортировки и произвести их расстановку, а также сделать оценку допусков на эти параметры.

Ключевые слова: метод Ньютона, динамика пучка.