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# Complete Foliations with Transverse Rigid Geometries and Their Basic Automorphisms

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The notion of rigid geometry is introduced. Rigid geometries include Cartan geometries as well as rigid geometric structures in the sense of Gromov. Foliations $(M, F)$ with transverse rigid geometries are investigated. An invariant $\mathfrak{g}_0$ of a foliation $(M, F)$ with transverse rigid geometry, being a Lie algebra, is introduced. We prove that if, for some foliation $(M, F)$ with transverse rigid geometry, $\mathfrak{g}_0$ is zero, then there exists a unique Lie group structure on its full basic automorphism group. Some estimates of the dimensions of this group depending on the transverse geometry are obtained. Examples, illustrating the main results, are constructed.

**Key words and phrases:** rigid geometry, foliation, basic automorphism, holonomy group.

## 1. Introduction

One of the basic objects associated with a geometric structure on a smooth manifold is its automorphism group. Among the central problems, there is the question whether the automorphism group can be endowed with a (finite-dimensional) Lie group structure [1].

In the theory of foliations with transverse geometries, automorphisms are understood as diffeomorphisms mapping leaves onto leaves and preserving transverse geometries. The group of all automorphisms of a foliation $(M, F)$ with transverse geometry is denoted by $\mathcal{A}(M, F)$. Let $\mathcal{A}_L(M, F)$ be the normal subgroup of $\mathcal{A}(M, F)$ formed by automorphisms mapping each leaf onto itself. The quotient group $\mathcal{A}(M, F)/\mathcal{A}_L(M, F)$ is called the full basic automorphism group and is denoted by $\mathcal{A}_B(M, F)$.

In the investigation of foliations $(M, F)$ with transverse geometry it is naturally to ask the above problem about the existence of a Lie group structure for the full group $\mathcal{A}_B(M, F)$ of basic automorphisms of $(M, F)$.

Leslie [2] was first who solved a similar problem for smooth foliations on compact manifolds. For foliations with complete transversal projectable affine connection this problem was studied by Belko [3].

The leaf space $M/F$ of the foliation is a diffeological space, and the group $\mathcal{A}_B(M, F)$ can be considered as a subgroup of the diffeological Lie group $\mathrm{Diff}(M/F)$. For Lie foliations with dense leaves on a compact manifold, the diffeological Lie groups $\mathrm{Diff}(M/F)$ are computed by Hector and Macias-Virgos [4].

In this work we introduce a notion of a rigid structure. Cartan geometries [1] and rigid geometric structures in the sense of Gromov [5, 6] are rigid structures in our sense. At the same time almost complex and symplectic structures don't belong to rigid structures. A manifold equipped with a rigid structure is called a rigid geometry.

We investigate foliations admitting rigid geometries as transverse structures and call them by foliations with transverse rigid geometries (TRG). Cartan foliations [7, 8] and $G$-foliations, where $G$ is a Lie group of finite type, are foliations with TRG. In particular, Riemannian, pseudo-Riemannian, Lorenz, projective and conformal foliations belong to the class of foliations under investigation. The category of foliations with TRG is denoted by $\mathfrak{F}_{TRG}$. The group $\mathcal{A}_B(M, F)$ is an invariant of $(M, F)$ in the category $\mathfrak{F}_{TRG}$.

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We assume that all the foliations under consideration are modelled on effective rigid geometries. We construct the foliated bundle for a foliation $(M, F)$ with TRG and reduce problems on the automorphism groups and the basic automorphism groups of $(M, F)$ to the analogous problems for $e$-foliations (Theorems 3 and Proposition 9). Emphasize that these statements are proved without assumption of completeness of $(M, F)$.

For any complete foliation $(M, F)$ with TRG we define the structure Lie algebra $\mathfrak{g}_0(M, F)$ and show that $\mathfrak{g}_0(M, F)$ is an invariant of this foliation in the category $\mathfrak{F}_{TRG}$ (Proposition 5). One of the main results of this work is the proof of the theorem asserting that if $\mathfrak{g}_0(M, F)$ is zero, then there exists a unique Lie group structure on $\mathcal{A}_B(M, F)$. We also obtain some estimates of the dimensions of these Lie groups depending on the transverse geometry (Theorem 5).

We give different interpretations of holonomy groups of complete foliations with TRG (Theorem 4) and find some other sufficient conditions for the existence of a Lie group structure on $\mathcal{A}_B(M, F)$ (Theorem 6). In particular, it is shown that the structure Lie algebra of any complete proper foliation with TRG is zero, and $\mathcal{A}_B(M, F)$ is a Lie group (Corollary 2).

We demonstrate that, for a foliation with TRG covered by a fibration, the condition $\mathfrak{g}_0(M, F) = 0$ is equivalent to the discreteness of its global holonomy group (Theorem 7).

Examples of computations of the full basic automorphism group of a foliation with TRG are constructed. Examples 1 and 2 also show that the group $\mathcal{A}_B(M, F)$ depends on the transverse rigid geometry of the foliation $(M, F)$.

# 2. Rigid geometries

**Parallelizable manifolds.** Recall that a manifold admitted an $e$-structure is called *parallelizable.* In other words, a parallelizable manifold is a pair $(P, \omega)$, where $P$ is a smooth manifold and $\omega$ is a smooth non-degenerate $\mathbb{R}^m$-valued 1-form $\omega$ on $P$, i. e., $\omega_u \colon T_uP \to \mathbb{R}^m$ is an isomorphism of the vector spaces for each $u \in P$. Here $m = \dim P$.

**Rigid structures.** We will use notations from [9]. Denote by $P(N, H)$ a principal $H$-bundle with the projection $p \colon P \to N$. Suppose that the action of $H$ on $P$ is a right action and $R_a$ is the diffeomorphism of $P$, corresponding to an element $a \in H$.

Two principal bundles $P(N, H)$ and $\tilde{P}(\tilde{N}, \tilde{H})$ are called *isomorphic* if $H = \tilde{H}$ and there exists a diffeomorphism $\Gamma \colon P \to \tilde{P}$ such that $\Gamma \circ R_a = \tilde{R}_a \circ \Gamma$, $\forall a \in H$, where $\tilde{R}_a$ is the transformation of $\tilde{P}$, corresponding to an element $a$.

**Def 1.** Let $P(N, H)$ be a principal $H$-bundle and $(P, \omega)$ be a parallelizable manifold satisfying the following condition:
(S) there is an inclusion $\mathfrak{h} \subset \mathbb{R}^m$ of the vector space of the Lie algebra $\mathfrak{h}$ of the Lie group $H$ into vector space $\mathbb{R}^m$ such that $\omega(A^*) = A$, $\forall A \in \mathfrak{h}$, where $A^*$ is the fundamental vector field on $P$ corresponding to $A$.

Then $\xi = (P(N, H), \omega)$ is called a *rigid structure* on the manifold $N$. A pair $(N, \xi)$ is called a *rigid geometry.*

**Def 2.** Let $\xi = (P(N, H), \omega)$ and $\tilde{\xi} = (\tilde{P}(\tilde{N}, \tilde{H}), \tilde{\omega})$ be two rigid structures. An isomorphism $\Gamma \colon P \to \tilde{P}$ of the principal bundles $P(N, H)$ and $\tilde{P}(\tilde{N}, \tilde{H})$ satisfying the equality $\Gamma^*\tilde{\omega} = \omega$ is called an *isomorphism* of the rigid structures $\xi$ and $\tilde{\xi}$.

Any isomorphism $\Gamma$ of rigid structures $\xi$ and $\tilde{\xi}$ defines a map $\gamma \colon N \to \tilde{N}$ such that $p \circ \Gamma = \gamma \circ p$, and $\gamma$ is a diffeomorphism from $N$ to $\tilde{N}$. The projection $\gamma$ is called an *isomorphism* of the rigid geometries $(N, \xi)$ and $(\tilde{N}, \tilde{\xi})$.

**Induced rigid geometries.** Let $\xi = (P(N, H), \omega)$ be a rigid structure on a manifold $N$ with the projection $p \colon P \to N$. Let $V$ be an arbitrary open subset of the manifold $N$, let $P_V := p^{-1}(V)$ and $\omega_V := \omega|_{P_V}$. Then $\xi_V := (P_V(V, H), \omega_V)$ is also a rigid structure.

**Def 3.** The pair $(V, \xi_V)$ defined above is called an *induced rigid geometry on the open subset* $V$ of $N$.

**Gauge transformations.** Let $\mathcal{A}(\xi)$ be the group of all automorphisms of a rigid structure $\xi = (P(N,H), \omega)$. It is a Lie group as a closed subgroup of the group $\mathcal{A}(P,\omega)$ of all automorphism of a parallelizable manifold $(P,\omega)$. Denote by $\mathcal{A}(N,\xi)$ the group of all automorphisms of the geometry $(N,\xi)$, i. e., $\mathcal{A}(N,\xi) := \{\gamma \in \mathrm{Diff}(N) \mid \exists\, \Gamma \in \mathcal{A}(\xi) : p\circ\Gamma = \gamma\circ p\}$. Consider the natural group epimorphism $\chi\colon \mathcal{A}(\xi) \to \mathcal{A}(N,\xi)\colon \Gamma \mapsto \gamma$, where $\gamma$ is the projection of $\Gamma$ with respect to $p\colon P \to N$.

**Def 4.** Let $\xi = (P(N,H), \omega)$ be a rigid structure on a manifold $N$ with the projection $p\colon P \to N$. The group $\mathrm{Gauge}(\xi) := \{\Gamma \in \mathcal{A}(\xi) \mid p\circ\Gamma = p\}$ is called a *group of gauge transformations of the rigid structure* $\xi$.

Remark that $\mathrm{Gauge}(\xi)$ is a closed normal Lie subgroup of the group $\mathcal{A}(\xi)$, because it is the kernel of the natural group epimorphism $\chi\colon \mathcal{A}(\xi) \to \mathcal{A}(N,\xi)$.

**Effectiveness of rigid geometries.**

**Def 5.** A rigid structure $\xi = (P(N,H), \omega)$ is called *effective* if for an arbitrary open subset $V$ in $N$ the induced rigid structure $\xi_V = (P_V(V,H), \omega_V)$ has the trivial group of gauge transformations, i. e., $\mathrm{Gauge}(\xi_V) = \{\mathrm{id}_{P_V}\}$. A rigid geometry $(N,\xi)$ is said *to be effective* if $\xi$ is an effective structure.

**Pseudogroup of local automorphisms.** Let $(N,\xi)$ be a rigid geometry. For arbitrary open subsets $V$, $V' \subset N$ an isomorphism $V \to V'$ of the induced rigid geometries $(V,\xi_V)$ and $(V',\xi_{V'})$ is called a *local automorphism* of $(N,\xi)$. The family $\mathcal{H}$ of all local automorphisms of a rigid geometry $(N,\xi)$ forms a pseudogroup of local automorphisms. Denote it by $\mathcal{H} = \mathcal{H}(N,\xi)$. Recall that a pseudogroup $\mathcal{H}$ of local diffeomorphisms of manifold $N$ is called *quasi-analytic* if the existence of an open subset $V \subset N$ and an element $\gamma \in \mathcal{H}$ such that $\gamma|_V = \mathrm{id}_V$ implies that $\gamma|_{D(\gamma)} = \mathrm{id}_{D(\gamma)}$ in the entire (connected) domain $D(\gamma)$ on which $\gamma$ is defined.

**Proposition 1.** *The pseudogroup $\mathcal{H} = \mathcal{H}(N,\xi)$ of all local automorphisms of an effective rigid geometry $(N,\xi)$ is quasi-analytic.*

**Proof.** Let $\gamma$ be an element of $\mathcal{H} = \mathcal{H}(N,\xi)$ such that $\gamma|_V = \mathrm{id}_V$ for some open subset $V$ in $N$. The effectiveness of the rigid geometry $(N,\xi)$ implies $\Gamma = \mathrm{id}_{P_V}$, where $\Gamma$ is a local automorphism of $\xi$ having the projection $\gamma|_V$ with respect to $p\colon P \to N$. Let the domain $D = D(\gamma)$ of $\gamma$ be an open connected subset of $N$ such that $D\setminus V \neq \emptyset$. Consider an automorphism $\tilde{\Gamma}$ of the induced rigid structure $\xi_D$ with the projection $\gamma$. Since $\tilde{\Gamma}^*\omega_D = \omega_D$, $\tilde{\Gamma}$ is an isomorphism of the parallelizable manifold $(P_D, \omega_D)$. It is known that two automorphisms of a connected parallelizable manifold, which coincide at one point, coincide at any point. Therefore it follows from the equality $\tilde{\Gamma}|_{P_V} = \Gamma = \mathrm{id}_{P_V}$ that $\tilde{\Gamma}|_{CP_D} = \mathrm{id}_{CP_D}$ for each connected component $CP_D$ of $P_D$. Thus, $\tilde{\Gamma} = \mathrm{id}_{P_D}$, hence $\gamma = \mathrm{id}_D$. □

# 3. Foliations with transverse rigid geometries. Foliated bundles

**Foliations with transverse rigid geometries (TRG).** A foliation $(M,F)$ of codimension $q$ on an $n$-manifold $M$ has a transverse rigid geometry $(N,\xi)$, where $N$ is a $q$-manifold, if $(M,F)$ is defined by a cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ modeled on $(N,\xi)$, i. e.,

1) $\{U_i\}$ is an open covering of $M$;
2) $f_i\colon U_i \to N$ are submersions with connected fibres;
3) $\gamma_{ij}\circ f_j = f_i$ on $U_i \cap U_j$,

with $\gamma_{ij}$ is a local automorphism of $(N, \xi)$. The topological space $N$ is not assumed to be connected. Without loss of generality, we will suppose that $N = \cup_{i\in J} f_i(U_i)$ and the family $\{(U_i, f_i)\}$ is maximal as it is generally used in manifold theory.

Let $\Sigma$ be the set of fibres of the submersions $f_i$ belonging to the cocycle $\eta$. One can easily check that $\Sigma$ is a base of a certain topology $\tau$ in $M$. The connected components of the topological space $(M, \tau)$ form a partition $F = \{L_\alpha |\ \alpha \in A\}$.

**Def 6.** We call $(M, F)$, where $F$ is the partition mentioned above, a *foliation with transverse rigid geometry* $(N, \xi)$, and $L_\alpha$ are called its *leaves*. The cocycle $\eta$ modelled on $(N, \xi)$ is said to be an $(N, \xi)$*-cocycle.*

Let $(M, F)$ be a foliation defined by an $(N, \xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$, where $(N, \xi)$ is an effective rigid geometry. Effectiveness of $\xi$ guarantees the existence of a unique isomorphism $\Gamma_{ij}$ of the induced rigid structures $\xi_{f_j(U_i\cap U_j)}$ and $\xi_{f_i(U_i\cap U_j)}$, whose projection coincides with $\gamma_{ij}$. Hence, in the case $U_i \cap U_j \cap U_k \neq \emptyset$, the equality $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ implies the equality
$(\Gamma_1)$ $\Gamma_{ij} \circ \Gamma_{jk} = \Gamma_{ik}$.

The following two equalities are direct corollaries of the effectiveness of $\eta$ and $(\Gamma_1)$ :
$(\Gamma_2)$ $\Gamma_{ii} = \mathrm{id}_{P_i}$ and $(\Gamma_3)$ $\Gamma_{ij} = (\Gamma_{ji})^{-1}$.

**Assumptions.** *In this work we will assume that each rigid geometry is effective and all the foliations under consideration are modeled on effective rigid geometries.*

**Notations.** We denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on a manifold $N$. If $Q$ is a smooth distribution on $M$, then $\mathfrak{X}_Q(M) := \{X \in \mathfrak{X}(M) \mid X_u \in Q_u, \forall u \in M\}$. If $Q$ is an integrable distribution and defines a foliation $F$, where $Q = TF$, we also use notation $\mathfrak{X}_F(M)$ for $\mathfrak{X}_Q(M)$.

**Foliated bundles.** Now we construct the foliated bundle for a foliation with TRG.

**Theorem 1.** *Let $(M, F)$ be a foliation with a transverse rigid geometry $(N, \xi)$, where $\xi = (P(N, H), \omega)$. Then there exist a principal $H$-bundle $\pi\colon \mathcal{R} \to M$, an $H$-invariant foliation $(\mathcal{R}, \mathcal{F})$ whose leaves are projected by $\pi$ onto the leaves of $(M, F)$ and an $\mathbb{R}^m$-valued 1-form $\tilde{\omega}$ on $\mathcal{R}$, where $m = \dim P$, that satisfy the following conditions:*
*(i) the map $\tilde{\omega}_u\colon T_u(\mathcal{R}) \to \mathbb{R}^m$, $\forall u \in \mathcal{R}$, is surjective; moreover, $\ker \tilde{\omega}_u = T_u\mathcal{F}$;*
*(ii) there is an inclusion $\mathfrak{h} \subset \mathbb{R}^m$ of the vector space of the Lie algebra $\mathfrak{h}$ of the Lie group $H$ into $\mathbb{R}^m$ such that $\tilde{\omega}(A^*) = A$, $\forall A \in \mathfrak{h}$, where $A^*$ is the fundamental vector field on $\mathcal{R}$ corresponding to $A$;*
*(iii) the foliation $(\mathcal{R}, \mathcal{F})$ is an e-foliation;*
*(iv) the restriction $\pi_{\mathcal{L}}$ on an arbitrary leaf $\mathcal{L}$ of the foliation $(\mathcal{R}, \mathcal{F})$ is a regular covering map onto a leaf of $(M, F)$, and the subgroup $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of the Lie group $H$ is the group of deck transformations.*

**Proof.** Supppose that the foliation $(M, F)$ with transverse rigid geometry is defined by a $(N, \xi)$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}$, where $\xi = (P(N, H), \omega)$, and let $p\colon P \to N$ be the projection of the principal $H$-bundle $P(N, H)$. Denote $V_i := f_i(U_i)$, $P_i := p^{-1}(V_i)$ and $p_i := p|_{P_i}$. Without loss of generality, we can assume that $U_i$ and $V_i$ are contractible open sets. Let $\mathcal{R}_i := f_i^* P_i := \{(x, z) \in U_i \times P_i \mid f_i(x) = p_i(z)\}$, $\bar{f}_i\colon \mathcal{R}_i \to P_i\colon (x, z) \mapsto z$ and $\pi_i\colon \mathcal{R}_i \to U_i\colon (x, z) \mapsto x$, $\forall (x, z) \in \mathcal{R}_i$. We have $p_i \circ \bar{f}_i = f_i \circ \pi_i$. The formula $(x, z) \cdot a := (x, z \cdot a)$, $\forall (x, z) \in \mathcal{R}_i$, $\forall a \in H$, defines a right action of the group $H$ on $\mathcal{R}_i$. Thus we have the principal $H$-bundle $\pi_i\colon \mathcal{R}_i \to U_i$ with the simple $H$-invariant foliation $\mathcal{F}_i = \{\bar{f}_i^{-1}(z) \mid z \in P_i\}$. Let $\omega_i := \omega|_{P_i}$. We have the $\mathbb{R}^m$-valued 1-form $\bar{\omega}_i := \bar{f}_i^* \omega_i$ defined on $\mathcal{R}_i$. Moreover, $\bar{\omega}_i(X) = 0$ for $X \in \mathfrak{X}(\mathcal{R}_i)$ if and only if $X \in \mathfrak{X}_{\mathcal{F}_i}(\mathcal{R}_i)$.

Let $Y := \bigsqcup_{i\in J} \mathcal{R}_i$ be the disjunct union of the manifolds $\mathcal{R}_i$. Let us introduce an equivalence relation $\rho$ in $Y$. Denote a point $u \in Y \cap \mathcal{R}_i$ by the pair $(i, u)$. Two points $(i, u)$ and $(j, w)$ in $Y$ are said to be $\rho$-equivalent if
(1) $\pi_i(u) = \pi_j(w) \in U_i \cap U_j$;
(2) $\bar{f}_i(u) = (\Gamma_{ij} \circ \bar{f}_j)(w)$,

where $\Gamma_{ij}$ is the isomorphism of the rigid structures $\xi_{f_j(U_i\cap U_j)}$ and $\xi_{f_i(U_i\cap U_j)}$ whose projection $\gamma_{ij}$ belongs to the $(N,\xi)$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}$.

The above equality $(\Gamma_2)$ implies that $\rho$ is reflexive. The relation $(\Gamma_3)$ guarantees the symmetry of $\rho$, while the relation $(\Gamma_1)$ implies the transitivity of $\rho$. Thus $\rho$ is indeed an equivalence relation in $Y$. Hence we have the quotient space $\mathcal{R} := Y/\rho$, the quotient mapping $\varphi\colon Y \to \mathcal{R}$ and the surjective projection $\pi\colon \mathcal{R} \to M$, where $\pi$ maps the equivalence class $[(i,u)] \in \mathcal{R}$ of a point $(i,u) \in Y$ to the point $\pi_i(u) \in M$. The restriction $\varphi_i := \varphi|_{\mathcal{R}_i}\colon \mathcal{R}_i \to \mathcal{R}$ is injective. Therefore $\varphi_i$ is a bijective map onto image $\tilde{U}_i := \varphi(\mathcal{R}_i)$; it will be denoted by $\varphi_i\colon \mathcal{R}_i \to \tilde{U}_i$. A smooth structure in $\mathcal{R}$ is well defined by assuming that each bijection $\varphi_i$ is a diffeomorphism of $\mathcal{R}_i$ and $\tilde{U}_i$.

Let $x$ be any point in $\mathcal{R}$ and $\tilde{U}_i \ni x$. Set $x\cdot a := \varphi_i^{-1}(x)\cdot a$, $\forall a \in H$. This definition does not depend on the choice of $\tilde{U}_i$ containing $x$ because all $\Gamma_{ij}$ are isomorphisms of the corresponding principal $H$-bundles. Thus $\mathcal{R}$ becomes the total space of the principal $H$-bundle. The quotient manifold $\mathcal{R}/H$ can be identified with the manifold $M$, while the projection onto the quotient can be identified with $\pi\colon \mathcal{R} \to M$.

Define an 1-form $\tilde{\omega}$ on $\mathcal{R}$ by the formula $\tilde{\omega}|_{\tilde{U}_i} := (\varphi_i^{-1})^*\bar{\omega}_i$. If $U_i \cap U_j \neq \emptyset$, then $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$ and $\Gamma_{ij}^*\omega_i = \omega_j$ because $\Gamma_{ij}$ is an isomorphism of the respective rigid structures, which lies over the local automorphism $\gamma_{ij}$ of $(N,\xi)$. Since $\bar{\omega}_i = \bar{f}_i^*\omega_i$, we have the equality $(\varphi_i^{-1})^*\bar{\omega}_i = (\varphi_j^{-1})^*\bar{\omega}_j$ on $\tilde{U}_i \cap \tilde{U}_j$. Thus the 1-form $\tilde{\omega}$ is well defined.

The foliations $\mathcal{F}_i$ on $\mathcal{R}_i$ and hence the foliations $(\varphi_i)_*\mathcal{F}_i$ on $\tilde{U}_i$ are glued together by $\rho$ into a foliation $\mathcal{F}$ on the manifold $\mathcal{R}$ such that $\mathcal{F}|_{\tilde{U}_i} = (\varphi_i)_*\mathcal{F}_i$. It follows from the definition of $\tilde{\omega}$ that $\tilde{\omega}(X) = 0$ for $X \in \mathfrak{X}(\mathcal{R})$ if and only if $X \in \mathfrak{X}_{\mathcal{F}}(\mathcal{R})$.

The invariance of the foliations $\mathcal{F}_i$, $i \in J$, with respect to the action of the group $H$ implies the $H$-invariance of the foliation $\mathcal{F}$ on $\mathcal{R}$.

The equality $\tilde{\omega}(A^*) = A$, $\forall A \in \mathfrak{h}$, is a consequence of the equality $(S)$ for $\omega$ and the definitions of the principal $H$-bundle $\pi\colon \mathcal{R} \to M$ with the 1-form $\tilde{\omega}$.

We emphasize that the $(P,\omega)$-cocycle $\{\tilde{U}_i, \tilde{f}_i, \{\Gamma_{ij}\}\}$ defines the foliation $(\mathcal{R},\mathcal{F})$. Thus $(\mathcal{R},\mathcal{F})$ is an $e$-foliation.

From the construction of the foliation $(\mathcal{R},\mathcal{F})$ it follows that the restriction $\pi|_{\mathcal{L}}$ onto an arbitrary leaf $\mathcal{L}$ of $(\mathcal{R},\mathcal{F})$ is a covering mapping onto some leaf $L$ of the foliation $(M,F)$. Fix a point $x \in L$ and a point $u \in \mathcal{L} \cap \pi^{-1}(x)$. For any point $u' \in \mathcal{L} \cap \pi^{-1}(x)$ there exists a unique element $b \in H$ such that $u' = u\cdot b$. Invariance of the lifted foliation $(\mathcal{R},\mathcal{F})$ with respect to the action of the Lie group $H$ implies that $R_b(\mathcal{L}) = \mathcal{L}$, hence $b \in H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$. Thus the subgroup $H(\mathcal{L})$ of the group $H$ acts transitively on the set $\mathcal{L} \cap \pi^{-1}(x)$, with $L = \mathcal{L}/H(\mathcal{L})$. Therefore the covering mapping $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L$ is regular, and $H(\mathcal{L})$ is its deck transformation group. □

**Def 7.** The principal $H$-bundle $\mathcal{R}(M,H)$ with the $H$-invariant foliation $(\mathcal{R},\mathcal{F})$ constructed in the proof of Theorem 1 is called *the foliated bundle for the foliation* $(M,F)$ *with transverse rigid geometry* $(N,\xi)$ and $(\mathcal{R},\mathcal{F})$ is called *the lifted foliation.*

**Remark 1.** The lifted $e$-foliation $(\mathcal{R},\mathcal{F})$ is defined by $(P,\omega)$-cocycle $\{\tilde{U}_i, \tilde{f}_i, \{\Gamma_{ij}\}\}$.

**Remark 2.** If $H$ is disconnected, $\mathcal{R}$ may be also disconnected. In this case all the connected components of $\mathcal{R}$ are mutually diffeomorphic, and we will consider one of them. Thus, we assume that the space of the foliated bundle $\mathcal{R}$ is connected.

# 4. Completeness and a structure Lie algebra of a foliation with TRG

**Completeness of foliations with TRG.** Let $(M,F)$ be an arbitrary smooth foliation on a manifold $M$ and $TF$ be the distribution on $M$ formed by the vector spaces tangent to the leaves of the foliation $F$. The vector quotient bundle $TM/TF$ is called the transverse vector bundle of the foliation $(M,F)$. Let us identify $TM/TF$

with an arbitrary smooth distribution $\mathfrak{M}$ on $M$ that is transverse to the foliation $(M, F)$, i. e., $TM = TF \oplus \mathfrak{M}$.

Let $(M, F)$ be a foliation with TRG and $(\mathcal{R}, \mathcal{F})$ be the lifted foliation. It is natural to identify the transverse vector bundle $T\mathcal{R}/T\mathcal{F}$ with a distribution $\overline{\mathfrak{M}} := \pi^*\mathfrak{M}$ on $\mathcal{R}$, i. e., with a distribution defined by the equality $\overline{\mathfrak{M}}_u := \{X_u \in T_u\mathcal{R} \mid \pi_* X_u \in \mathfrak{M}_x\}$, where $x = \pi(u)$ and $u \in \mathcal{R}$.

**Def 8.** A foliation $(M, F)$ with transverse rigid geometry is said to be $\mathfrak{M}$-*complete* if any transverse vector field $X \in \mathfrak{X}_{\overline{\mathfrak{M}}}(\mathcal{R})$ such that $\tilde{\omega}(X) = \text{const}$ is complete. A foliation $(M, F)$ with TRG of arbitrary codimension $q$ is said to be *complete* if there exists a smooth $q$-dimensional transverse distribution $\mathfrak{M}$ on $M$ such that $(M, F)$ is $\mathfrak{M}$-complete.

**Remark 3.** In other words, $(M, F)$ is an $\mathfrak{M}$-complete foliation iff the lifted $e$-foliation $(\mathcal{R}, \mathcal{F})$ is complete with respect to the distribution $\overline{\mathfrak{M}}$ in the sense of Conlon [10]. Remark that a complete $e$-foliation in the sense of Conlon is also complete in the sense of Molino [11].

**Ehresmann connections for foliations.** Let $(M, F)$ be a foliation of codimension $q$ and $\mathfrak{M}$ be a smooth $q$-dimensional distribution on $M$ that is transverse to the foliation $F$. The piecewise smooth integral curves of the distribution $\mathfrak{M}$ are said to be *horizontal,* and the piecewise smooth curves in the leaves are said to be vertical. A piecewise smooth mapping $H$ of the square $I_1 \times I_2$ to $M$ is called a *vertical-horizontal homotopy* if the curve $H|_{s\times I_2}$ is vertical for any $s \in I_1$ and the curve $H|_{I_1\times t}$ is horizontal for any $t \in I_2$. In this case, the pair of paths $(H|_{I_1\times\{0\}}, H|_{\{0\}\times I_2})$ is called the *base* of $H$. It is well known that there exists at most one vertical-horizontal homotopy with a given base. A distribution $\mathfrak{M}$ is called an *Ehresmann connection for a foliation* $(M, F)$ (in the sense of Blumenthal and Hebda [12]) if, for any pair of paths $(\sigma, h)$ in $M$ with a common starting point $\sigma(0) = h(0)$, where $\sigma$ is a horizontal curve and $h$ is a vertical curve, there exists a vertical-horizontal homotopy $H$ with the base $(\sigma, h)$. If the distribution $\mathfrak{M}$ is integrable, then the connection is said to be *integrable.* For a simple foliation $F$, i. e., such that it is formed by the fibers of a submersion $r\colon M \to B$, a distribution $\mathfrak{M}$ is an Ehresmann connection for $F$ if and only if $\mathfrak{M}$ is an Ehresmann connection for the submersion $r$, i. e., if and only if any smooth curve in $B$ possesses horizontal lifts.

**Proposition 2.** *If $(M, F)$ is an $\mathfrak{M}$-complete foliation with TRG, then $\mathfrak{M}$ is an Ehresmann connection for this foliation.*

**Proof.** The distribution $\overline{\mathfrak{M}} := \pi^*\mathfrak{M}$ is an Ehresmann connection for the lifted foliation $(\mathcal{R}, \mathcal{F})$, because $(M, F)$ is $\mathfrak{M}$-complete. So in view of $F = \pi_*\mathcal{F}$ and $\mathfrak{M} = \pi_*\overline{\mathfrak{M}}$, we see that $\mathfrak{M}$ is an Ehresmann connection for $(M, F)$. □

**Structure Lie algebra.** Applying of the relevant results of Molino [11] on complete $e$-foliations, we obtain the following theorem.

**Theorem 2.** *Let $(M, F)$ be a complete foliation with TRG and $(\mathcal{R}, \mathcal{F})$ be its lifted e-foliation. Then:*
*(i) the closure of the leaves of the foliation $\mathcal{F}$ are fibers of a certain locally trivial fibration $\pi_b\colon \mathcal{R} \to W$;*
*(ii) the foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ induced on the closure $\overline{\mathcal{L}}$ is a Lie foliation with dense leaves with the structure Lie algebra $\mathfrak{g}_0$, that is the same for any $\mathcal{L} \in \mathcal{F}$.*

**Def 9.** The structure Lie algebra $\mathfrak{g}_0$ of the Lie foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ is called *the structure Lie algebra* of the complete foliation $(M, F)$ and is denoted by $\mathfrak{g}_0 = \mathfrak{g}_0(M, F)$.

**Remark 4.** If $(M, F)$ is a Riemannian foliation on a compact manifold, this notion coincides with the notion of a structure Lie algebra in the sense of Molino [11].

**Def 10.** The fibration $\pi_b\colon \mathcal{R} \to W$ satisfying Theorem 2 is called a *basic fibration* for $(M, F)$.

# 5. Category of foliations with TRG

**Category of foliations.** Denote by $\mathfrak{Fol}$ the category of foliations, objects of which are foliations, morphisms of two arbitrary foliations $(M, F)$ and $(M', F')$ are smooth maps $M \to M'$ mapping leaves of the foliation $(M, F)$ into leaves of the foliation $(M', F')$; a composition of morphisms coincides with the composition of maps.

**Category of foliations with TRG.** Let $(M, F)$ and $(M', F')$ are foliations with transverse rigid geometries $(N, \xi)$ and $(N', \xi')$ defined by an $(N, \xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ and an $(N', \xi')$-cocycle $\eta' = \{U'_r, f'_r, \{\gamma'_{rs}\}\}$, respectively. Let $f\colon M \to M'$ be a morphism which is a local isomorphism in the category $\mathfrak{Fol}$.

Hence for any $x \in M$ and $y := f(x)$ there exist neighborhoods $U_k \ni x$ and $U'_k \ni y$ from $\eta$ and $\eta'$ respectively and a diffeomorphism $\lambda\colon V_k \to V'_s$, where $V_k := f_k(U_k)$ and $V'_s := f'_s(U'_s)$, satisfying the relations $f(U_k) = U'_s$ and $\lambda \circ f_k = f'_s \circ f|_{U_k}$. We will say that $f$ preserves *transverse rigid structure* if the diffeomorphism $\lambda\colon V_k \to V'_s$ is an isomorphism of the induced rigid geometries $(V_k, \xi_{V_k})$ and $(V'_s, \xi'_{V'_s})$.

This notion is well defined, i. e., it does not depend of the choice of neighborhoods $U_k$ and $U'_k$ from the cocycles $\eta$ and $\eta'$.

By a *TRG-morphism of two foliations* $(M, F)$ *and* $(M', F')$ *with transverse rigid geometries* we mean a morphism $f\colon M \to M'$ in the category $\mathfrak{Fol}$ which preserves transverse rigid structure. The category $\mathfrak{F}_{TRG}$ objects of which are foliations with TRG, morphisms are TRG-morphisms, is called *the category of foliations with transverse rigid geometries.*

**Isomorphisms in the category $\mathfrak{F}_{TRG}$.** Remark that for any $e$-foliation $(\mathcal{R}, \mathcal{F})$ the lifted foliation coincides with $(\mathcal{R}, \mathcal{F})$. Using this we easily get the following lemma.

**Lemma 1.** *Let $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ be two e-foliations with transverse rigid geometries $(P, \omega)$ and $(P', \omega')$ respectively, where $(P, \omega)$ and $(P', \omega')$ are parallelizable manifolds. Let $\tilde{\omega}$ and $\tilde{\omega}'$ be the basic 1-forms on $\mathcal{R}$ and $\mathcal{R}'$ defined according to Theorem 1. Then a diffeomorphism $f\colon \mathcal{R} \to \mathcal{R}'$ is an isomorphism in the category $\mathfrak{F}_{TRG}$ if and only if $f$ is an isomorphism in the category $\mathfrak{Fol}$ and $f^*\tilde{\omega}' = \tilde{\omega}$.*

**Proposition 3.** *Let $(M, F)$ and $(M', F')$ be two foliations with TRG, let $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ be the corresponding lifted foliations. Then a diffeomorphism $f\colon M \to M'$ is an isomorphism in the category $\mathfrak{F}_{TRG}$ if and only if there exists an isomorphism $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$ of the lifted foliations in the category $\mathfrak{F}_{TRG}$ such that $R'_a \circ \hat{f} = \hat{f} \circ R_a$, $\forall a \in H$, where $R_a$, $R'_a$ are the right translations by an element $a \in H$ on $\mathcal{R}$ and $\mathcal{R}'$ accordingly.*

**Proof.** We will use the notations introduced in the proof of Theorem 1. Let the foliation with TRG $(M, F)$ is defined by an $(N, \xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$. Recall that the lifted foliation $(\mathcal{R}, \mathcal{F})$ is defined by a $(P, \omega)$-cocycle $\tilde{\eta} = \{\tilde{U}_i, \tilde{f}_i, \{\Gamma_{ij}\}\}$, where $\tilde{U}_i := \pi^{-1}(U_i)$, and the local isomorphisms $\Gamma_{ij}$ of the rigid structure $\xi$ lie over the local isomorphisms $\gamma_{ij}$ of the rigid geometry $(N, \xi)$. For the objects, concerning to the foliation $(M', F')$, we will use primes.

At first, suppose that $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$ is an isomorphism of the $e$-foliations $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ satisfying the condition $R'_a \circ \hat{f} = \hat{f} \circ R_a$, $\forall a \in H$. Then the projection $f\colon M \to M'$ of $\hat{f}$ is well defined by the equality $\pi' \circ \hat{f} = f \circ \pi$, where $\pi\colon \mathcal{R} \to M$ and $\pi'\colon \mathcal{R}' \to M'$ are the projections of the foliated bundles.

Consider an arbitrary point $x \in M$ and $y := f(x) \in M'$. There are neighborhoods $U_k \ni x$ and $U'_s \ni y$ from the $(N, \xi)$-cocycle and the $(N', \xi')$-cocycle defining the foliations $(M, F)$ and $(M', F')$ respectively, with $f(U_k) = U'_s$. Then $\hat{f}(\tilde{U}_k) = \tilde{U}'_s$. The lifted $e$-foliations $(\tilde{U}_k, \mathcal{F}|_{\tilde{U}_k})$ and $(\tilde{U}'_s, \mathcal{F}'|_{\tilde{U}'_s})$ are defined by the submersions $\tilde{f}_k\colon \tilde{U}_k \to P_k$ and $\tilde{f}'_s\colon \tilde{U}'_s \to P'_s$ accordingly. Besides, $R'_a \circ \hat{f} = \hat{f} \circ R_a$, $\forall a \in H$. Hence, according to Lemma 1, a diffeomorphism $\Gamma\colon P_k \to P'_s$ defined by the relation $\Gamma \circ \tilde{f}_k = \tilde{f}'_s \circ \hat{f}|_{\tilde{U}'_s}$ is a local isomorphism of the rigid structures $\xi$ and $\xi'$. Put $V_k = f_k(U_k)$ and $V'_s = f'_s(U'_s)$.

Let $\gamma\colon V_k \to V'_s$ be the projection of $\Gamma$, then $\gamma$ is an isomorphism of the rigid geometries induced on $V_k$ and $V'_s$. Thus, $f$ is an isomorphism of the foliations $(M, F)$ and $(M', F')$ in the category $\mathfrak{F}_{TRG}$.

Converse, suppose that $f\colon M \to M'$ is an isomorphism of the foliations $(M, F)$ and $(M', F')$ in the category $\mathfrak{F}_{TRG}$. Construct $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$ in the following way. Let $x$ be any point in $M$ and $y := f(x) \in M'$. Let $U_k \ni x$ and $U'_s \ni y$ be neighborhoods from the cocycles $\eta$ and $\eta'$ respectively, with $f(U_k) = U'_s$. Consider $\mathcal{R}_k := f_k^*(P_k)$, where $P_k = \tilde{f}_k(\tilde{U}_k)$. Then $\mathcal{R}_k = \{(x, z) \in U_k \times P_k \mid f_k(x) = p_k(z)\}$, $\mathcal{R}'_s$ is defined similarly. Since $f$ is an isomorphism in the category $\mathfrak{F}_{TRG}$, by definition, there exists a diffeomorphism $\gamma\colon V_k \to V'_s$ which is an isomorphism of the induced rigid geometries $(V_k, \xi_{V_k})$ and $(V'_s, \xi'_{V'_s})$, and $\gamma \circ f_k = f'_s \circ f|_{U_k}$.

Since the rigid geometries $(N, \xi)$ and $(N', \xi')$ are effective, there is a unique isomorphism $\Gamma\colon P_k \to P'_s$ of the induced rigid structures $\xi_{V_k}$ and $\xi'_{V'_s}$ with the projection $\gamma$. Then $\Gamma^*\omega' = \omega$ and $\Gamma \circ R_a = R'_a \circ \Gamma$, $\forall a \in H$. Define a map $h\colon \mathcal{R}_k \to \mathcal{R}'_s$ by the equality

$$h(x, z) := (f(x), \Gamma(z)), \quad \forall (x, z) \in \mathcal{R}_k.$$

According to the definition of the foliated bundle for $(M, F)$, the bijections $\varphi_i\colon \mathcal{R}_i \to \tilde{U}_i$ are isomorphisms of the simple foliations with TRG defined by the submersions $\bar{f}_i\colon \mathcal{R}_i \to P_i$ and $\tilde{f}_i\colon \tilde{U}_i \to P_i$ respectively. An analogous assertion holds for the foliation $(M', F')$. Hence $h\colon \mathcal{R}_k \to \mathcal{R}'_s$ is an isomorphism of the foliations mentioned above in the category $\mathfrak{F}_{TRG}$.

Put, by definition, $\hat{f}|_{\tilde{U}_k} := \varphi'_s \circ h \circ \varphi_k^{-1}$ for any neighborhood $\tilde{U}_k$ from the cocycle $\eta$. It is not difficult to check that this equality defines the map $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$, where $\hat{f}$ satisfies the following conditions: $(i)$ $\hat{f}^*\tilde{\omega}' = \tilde{\omega}$ and $(ii)$ $R'_a \circ \hat{f} = \hat{f} \circ R_a$, $\forall a \in H$. Therefore, by Lemma 1, $\hat{f}$ is an isomorphism of the lifted $e$-foliations satisfying (ii). □

**Proposition 4.** *Let $(M, F)$ and $(M', F')$ be two foliations with transverse rigid geometries $(N, \xi)$ and $(N'\xi')$ accordingly. Let $\hat{f}_1$ and $\hat{f}_2\colon \mathcal{R} \to \mathcal{R}'$ be two isomorphisms of $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ satisfying the equalities $R'_a \circ \hat{f}_i = \hat{f}_i \circ R_a$, $i = 1, 2$, $\forall a \in H$. If their projections $h_i\colon M \to M'$ coincide: $h_1 = h_2$, then $\hat{f}_1 = \hat{f}_2$.*

**Proof.** The map $\hat{f} := \hat{f}_2^{-1} \circ \hat{f}_1\colon \mathcal{R} \to \mathcal{R}$ is an isomorphism of $(\mathcal{R}, \mathcal{F})$ satisfying the relation $R'_a \circ \hat{f} = \hat{f} \circ R_a$, $\forall a \in H$, where the projection of $\hat{f}$ is $f = h_2^{-1} \circ h_1 = \mathrm{id}_M$. For any $x \in M$ we have $y = f(x) = x$. Therefore, we can take $U_s = U_k \ni x$ in the definition of morphisms of the category $\mathfrak{F}_{TRG}$. Then we have $\tilde{U}_k = \tilde{U}_s$. As above, let $f_k\colon U_k \to V_k$ be a submersion from the cocycle defining $(M, F)$ and $p_k := p|_{P_k}$. Then $\gamma = \mathrm{id}_{V_k}$ and $\Gamma \circ p_k = p_k$, where $\Gamma\colon P_k \to P_k$ is an automorphism of the induced rigid structure $\xi_{V_k} = (P_k(H, V_k), \omega_k)$. Therefore $\Gamma \in \mathrm{Gauge}(\xi_{V_k})$. Due to the effectiveness of the transverse rigid geometry $(N, \xi)$, we necessarily have $\Gamma = \mathrm{id}_{P_k}$. The equality $\Gamma \circ \tilde{f}_k = \tilde{f}_k \circ \hat{f}|_{\tilde{U}_k}$ implies $\tilde{f}_k = \tilde{f}_k \circ \hat{f}|_{\tilde{U}_k}$, i. e., $\hat{f}|_{\tilde{U}_k} \in \mathcal{A}_L^H(\tilde{U}_k, \mathcal{F}|_{\tilde{U}_k})$. For any $x \in M$, the neighborhoods $\{U_k \mid x \in U_k\}$ from the $(N, \xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ form a base of the topology of the manifold $M$ at $x$. Hence $\hat{f}(u) = u$, $\forall u \in \pi^{-1}(x)$. Since $x$ is an arbitrary point in $M$, we have $\hat{f} = \mathrm{id}_{\mathcal{R}}$. Thus, $\hat{f}_1 = \hat{f}_2$. □

**A foliated natural functor.** By analogy to Proposition 3 and 4 it is not difficult to prove that for any morphism $f\colon M \to M'$ of foliations $(M, F)$ and $(M, F')$ in the category $\mathfrak{F}_{TRG}$ there exists a unique morphism $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$ of the lifted foliations $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ satisfying the equality $R'_a \circ \hat{f} = \hat{f} \circ R_a$ $\forall a \in H$. Set $\Phi(M, F) := (\mathcal{R}, \mathcal{F})$ and $\Phi(f) := \hat{f}$, then we get a covariant functor $\Phi$ from the category $\mathfrak{F}_{TRG}$ to the category of foliated bundles. This functor is a foliated natural bundle in sense of Wolak [13], [14, Chapter II].

**Automorphism groups of foliations with TRG.** Let $(M, F)$ be a foliation with a fixed transverse rigid structure $(N, \xi)$. Denote by $\mathcal{A}(M, F)$ the group of all automorphisms of $(M, F)$ in the category $\mathfrak{F}_{TRG}$. We say also that $\mathcal{A}(M, F)$ is the *full group of automorphisms.*

**Theorem 3.** *Let $(M, F)$ be a foliation with TRG. Let $(\mathcal{R}, \mathcal{F})$ be the lifted foliation and $\mathcal{A}^H(\mathcal{R}, \mathcal{F}) = \{f \in \mathcal{A}(\mathcal{R}, \mathcal{F}) \mid f \circ R_a = R_a \circ f, \quad \forall a \in H\}$. Then the map $\mu\colon \mathcal{A}^H(\mathcal{R}, \mathcal{F}) \to \mathcal{A}(M, F)\colon \hat{f} \mapsto f$, where $f$ is the projection of $\hat{f} \in \mathcal{A}^H(\mathcal{R}, \mathcal{F})$ with respect to $\pi\colon \mathcal{R} \to M$, is a natural group isomorphism.*

**Proof.** By Proposition 3, the map $\mu$ is well defined and surjective. It is clear that $\mu$ is a group homomorphism. According to Proposition 4, $\mu$ is injective. Thus, $\mu$ is a group isomorphism. □

**Remark 5.** Due to Theorem 3, problems concerning to automorphism groups of foliations with TRG are reduced to the analogous problems for automorphism groups of the lifted $e$-foliations.

**Invariance of the structure Lie algebra.** The following statement shows that the structure Lie algebra $\mathfrak{g}_0(M, F)$ of a foliation $(M, F)$ with TRG is an invariant in the category $\mathfrak{F}_{TRG}$.

**Proposition 5.** *Let $(M, F)$ and $(M', F')$ be two foliations with TRG isomorphic in the category $\mathfrak{F}_{TRG}$. Then their structure Lie algebras $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_0(M', F')$ are isomorphic.*

**Proof.** Let $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$ be the lifted foliations for $(M, F)$ and $(M', F')$ respectively. Suppose that there exists an isomorphism $f\colon M \to M'$ of the foliations $(M, F)$ and $(M', F')$ in $\mathfrak{F}_{TRG}$. Then by Proposition 3 there exists a map $\hat{f}\colon \mathcal{R} \to \mathcal{R}'$ which is an isomorphism of $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F}')$. Let $\mathcal{L}$ be an arbitrary leaf of $(\mathcal{R}, \mathcal{F})$, then $\mathcal{L}' = \hat{f}(\mathcal{L})$ is a leaf of $(\mathcal{R}', \mathcal{F}')$. Since $\hat{f}$ is a homeomorphism, $\hat{f}$ maps the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ onto the closure $\overline{\mathcal{L}'}$ of $\mathcal{L}'$, i. e., $\hat{f}(\overline{\mathcal{L}}) = \overline{\mathcal{L}'}$. Thus, $\hat{f}|_{\overline{\mathcal{L}}}\colon \overline{\mathcal{L}} \to \overline{\mathcal{L}'}$ is an isomorphism of the induced Lie foliations $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ and $(\overline{\mathcal{L}'}, \mathcal{F}'|_{\overline{\mathcal{L}'}})$ with dense leaves. It is known [11] that the structure Lie algebra of a Lie foliation with dense leaves is an invariant in the category of foliations $\mathfrak{Fol}$. Therefore the Lie algebras $\mathfrak{g}_0(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ and $\mathfrak{g}_0(\overline{\mathcal{L}'}, \mathcal{F}'|_{\overline{\mathcal{L}'}})$ are isomorphic. By definition $\mathfrak{g}_0(M, F) = \mathfrak{g}_0(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ and $\mathfrak{g}_0(M', F') = \mathfrak{g}_0(\overline{\mathcal{L}'}, \mathcal{F}'|_{\overline{\mathcal{L}'}})$, hence the Lie algebras $\mathfrak{g}_0(M, F)$ and $\mathfrak{g}_0(M', F')$ are isomorphic. □

# 6. Different interpretations of holonomy groups

**Holonomy groups of foliations with Ehresmann connections.** Let $(M, F)$ be a foliation with an Ehresmann connection $\mathfrak{M}$ (see Section 3). Let $\Omega_x$ be the set of horizontal curves with an initial point $x$. It is not difficult to prove that the map $\Phi_x\colon \Omega_x \times \pi_1(L, x) \to \Omega_x\colon (\sigma, [h]) \mapsto \tilde{\sigma}$, where $[h] \in \pi_1(L, x)$, $H$ is a vertical-horizontal homotopy with the base $(\sigma, h)$, and $\tilde{\sigma}(s) := H(s, 1)$, $s \in I_1$, defines a right action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set $\Omega_x$.

**Def 11.** Since $K_{\mathfrak{M}}(L, x) := \{[h] \in \pi_1(l, x) \mid \Phi_x(\sigma, [h]) = \sigma, \forall \sigma \in \Omega_x\} = \ker \Phi_x$ is a normal subgroup in $\pi_1(L, x)$, the quotient group $H_{\mathfrak{M}}(L, x) := \pi_1(L, x)/\ker \Phi_x$ is well defined [12]. The group $H_{\mathfrak{M}}(L, x)$ is called the *$\mathfrak{M}$-holonomy group of the leaf $L$ of the foliation $(M, F)$ with the Ehresmann connection $\mathfrak{M}$.*

It is known that there is a natural group epimorphism $\delta\colon H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ onto the germ holonomy group $\Gamma(L, x)$ of the leaf $L = L(x)$ such that

$$\beta = \delta \circ \alpha, \tag{$*$}$$

where $\alpha\colon \pi_1(L, x) \to H_{\mathfrak{M}}(L, x)$ and $\beta\colon \pi_1(L, x) \to \Gamma(L, x)$ are the natural projections onto the corresponding quotient groups.

The following assertion is a consequence of Theorem 7 proved by the author in [15].

**Proposition 6.** *Let $(M, F)$ be a foliation with an Ehresmann connection $\mathfrak{M}$. The natural group epimorphism $\delta\colon H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the relation $(*)$ is an isomorphism if and only if the holonomy pseudogroup of the foliation $(M, F)$ is quasi-analytic.*

**Equivalent approaches to the notion of holonomy groups.**

**Theorem 4.** *Let $(M, F)$ be an $\mathfrak{M}$-complete foliation with TRG defined by an $(N, \xi)$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}$. Let $L = L(x)$, $x \in M$, be an arbitrary leaf of this foliation and $\mathcal{L} = \mathcal{L}(u)$, $u \in \pi^{-1}(x)$, be the corresponding leaf of the lifted foliation $(\mathcal{R}, \mathcal{F})$. Then the germ holonomy group $\Gamma(L, x)$ of the leaf $L$ is isomorphic to each of the following five groups:*
*(i) the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$;*
*(ii) the group $\mathcal{H}_v$ formed by germs of local diffeomorphisms belonging to the isotropy subpseudogroup of the holonomy pseudogroup $\mathcal{H}$ of local automorphisms of the transverse rigid geometry $(N, \xi)$ at point $v = f_i(x)$, where $x \in U_i$;*
*(iii) the group of deck transformations of the regular covering map $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L$;*
*(iv) the subgroup $H(\mathcal{L}) = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$ of the Lie group $H$;*
*(v) the holonomy group $\Phi(u)$ of the integrable connection $T(\mathcal{F}|_{\pi^{-1}(L)})$ in the principal $H$-bundle with the projection $\pi|_{\pi^{-1}(L)}\colon \pi^{-1}(L) \to L$.*

**Proof.** According to Proposition 2, an $\mathfrak{M}$-complete foliation $(M, F)$ with TRG has an Ehresmann connection $\mathfrak{M}$. Recall that the holonomy pseudogroup $\mathcal{H}$ is a subpseudogroup of the pseudogroup $\mathcal{H}(N, \xi)$ of all local automorphisms of the transverse rigid geometry $(N, \xi)$. According to Proposition 1 $\mathcal{H}$ is a quasi-analytic pseudogroup. Therefore applying Proposition 6 we see that $\nu\colon H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ is a natural group isomorphism.

Recall that according to Theorem 1 the restriction $\pi|_{\mathcal{L}}\colon \mathcal{L} \to L$ is a regular covering with the deck transformations group $H(\mathcal{L})$. Then there is a normal subgroup $\rho_*(u)$ of the fundamental group $\pi_1(L, x)$ and a group isomorphism $\mu_u\colon \pi_1(L, x)/\rho_*(u) \to H(\mathcal{L})$.

Denote by $\alpha\colon \pi_1(L, x) \to \Gamma(L, x)$ and $\beta\colon \pi_1(L, x) \to \Phi(u)$ the natural group epimorphisms. It is enough to show that $\ker\alpha = \ker\beta = \rho_*(u)$. Let $[h] \in \ker\alpha$, then $h$ is a loop at $x$. Consider a chain $U_1, \ldots, U_k$, $U_i \cap U_{i+1} \neq \emptyset$, $\forall i \in \{1, \ldots, k-1\}$, of neighborhoods from the $(N, \xi)$-cocycle $\eta$ that covers the set $h([0, 1])$. Let $f_i\colon U_i \to V_i$ be submersions and $\gamma_{js}$ be the corresponding local automorphisms of the rigid geometry $(N, \xi)$ from $\eta$. According to Proposition 4 for each $\gamma_{js}$ there is a unique local automorphism $\Gamma_{js}$ of $\xi$ lying over $\gamma_{js}$.

The composition of projections $\gamma := \gamma_{1k} \circ \gamma_{kk-1} \circ \ldots \circ \gamma_{21}$ is defined in a neighborhood of the point $v := f_1(x)$ of the manifold $N$. The triviality of the germ of $\gamma$ at $v$ is a consequence of the choice of $[h] \in \ker\alpha$. Therefore there exists a neighborhood $V \ni v$ in $N$ such that $\gamma|_V = \mathrm{id}_V$. Due to the effectiveness of $\xi$, the automorphism $\Gamma := \Gamma_{1k} \circ \Gamma_{kk-1} \circ \ldots \circ \Gamma_{21}$ satisfies the equality $\Gamma|_{P_V} = \mathrm{id}_{P_V}$.

Denote by $\tilde{h}$ the path in the leaf $\mathcal{L}$ with the origin $u = \tilde{h}(0)$ covering the loop $h$. In the sequel, we will use notations of the proof of Theorem 1. From the definition of the lifted foliation $(\mathcal{R}, \mathcal{F})$ it is follows that the chain $\tilde{U}_1, \ldots, \tilde{U}_k$, where $\tilde{U}_i = \pi^{-1}(U_i)$, covers the set $\tilde{h}([0, 1])$. As $\tilde{f}_1(u) \in P_V$, the equality $\Gamma|_{P_V} = \mathrm{id}_{P_V}$ implies $\Gamma(\tilde{f}_1(u)) = \tilde{f}_1(u)$. Hence $\tilde{f}_1(\tilde{h}(1)) = \tilde{f}_1(\tilde{h}(0))$. Therefore $\tilde{h}(1) = \tilde{h}(0) = u$ and $h \in \ker\beta$. Thus, $\ker\alpha \subset \ker\beta$.

The equality $\ker\beta = \rho_*(u)$ follows directly from the definition of $\Phi(u)$ [7]. To complete the proof, we have to show the implication $\rho_*(u) \subset \ker\alpha$. Take any $[h] \in \rho_*(u)$. Let $\tilde{h}$ be the loop in $\mathcal{L}$ covering $h$ with the origin at $u = \tilde{h}(0)$. Then $\tilde{h}(1) = \tilde{h}(0) = u$. Consider an arbitrary chain $U_1, \ldots, U_r$, $U_i \cap U_{i+1} \neq \emptyset$, $\forall i \in \{1, \ldots, r-1\}$, of neighborhoods belonging to the $(N, \xi)$-cocycle $\eta$ that covers the set $h([0, 1])$. Let $\gamma_{js}$ be the corresponding local automorphisms of $(N, \xi)$ from $\eta$. Let $\Gamma_{js}$ be the unique local automorphism of $\xi$ with the projection $\gamma_{js}$. It is well known that any $e$-foliation has no holonomy. Then the holonomy diffeomorphism $\Gamma := \Gamma_{1r} \circ \Gamma_{rr-1} \circ \ldots \circ \Gamma_{21}$ has the trivial germ at the point $\tilde{f}_1(u)$. Therefore its projection $\gamma := \gamma_{1r} \circ \gamma_{rr-1} \circ \ldots \circ \gamma_{21}$ has the trivial germ at point $v = f_1(x)$. Since $\gamma$ is a local holonomy diffeomorphism along the loop $h$, we have $[h] \in \ker \alpha$. □

# 7. Foliations with the zero structure Lie algebra

**Proposition 7.** *Let $(M, F)$ be a complete foliation with TRG. Suppose that $\mathfrak{g}_0(M, F) = 0$. Let $\pi_b \colon \mathcal{R} \to W$ be the basic fibration. Then:*
*(i) the formula*

$$\Phi^W \colon W \times H \to W \colon (w, a) \mapsto \pi_{\mathrm{b}}(R_a(u)) \ \ \forall (w, a) \in W \times H, \ \ \forall u \in \pi_{\mathrm{b}}^{-1}(w)$$

*defines a smooth locally free action of the Lie group $H$ on the basic manifold $W$;*
*(ii) there is a homeomorphism $s \colon M/F \to W/H$ between the leaf space $M/F$ and the orbit space $W/H$ satisfying the equality $k \circ \pi_b = s \circ q \circ \pi$, where $k \colon W \to W/H$ is the quotient map onto $W/H$, $q \colon M \to M/F$ is the quotient map onto $M/F$;*
*(iii) the equality $\pi_b^* \bar{\omega} = \tilde{\omega}$ defines an $\mathbb{R}^m$-valued non-degenerate 1-form $\bar{\omega}$ on $W$ such that $\bar{\omega}(A_W^*) = A$, where $A_W^*$ is the fundamental vector field on $W$ defined by an element $A \in \mathfrak{h} \subset \mathbb{R}^m$.*

**Proof.** (i) Since $\mathfrak{g}_0(M, F) = 0$, by Theorem 2 the lifted foliation $(\mathcal{R}, \mathcal{F})$ is formed by the fibres of the basic fibration $\pi_b \colon \mathcal{R} \to W$. The action $\Phi^W$ is well defined, because the lifted foliation $(\mathcal{R}, \mathcal{F})$ is $H$-invariant. Smoothness of the action of $H$ on $\mathcal{R}$ and smoothness of $\pi_b$ imply smoothness of $\Phi^W$. Take any point $w \in W$ and $u \in \pi_b^{-1}(w)$. Let $\mathcal{L} = \mathcal{L}(u)$ and $L := \pi(\mathcal{L})$, then $x = \pi(u) \in L$. Recall that $H(\mathcal{L}) = \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L}\}$. Let $H_w$ be the isotropy subgroup of $H$ at $w$. From the definition of the action $\Phi^W$ of the Lie group $H$ on $W$ it follows that $H(\mathcal{L}) = H_w$. The condition $\mathfrak{g}_0(M, F) = 0$ implies that the lifted foliation $(\mathcal{R}, \mathcal{F})$ is proper, hence the induced foliation $(\pi^{-1}(L), \mathcal{F}|_{\pi^{-1}(L)})$ is also proper. Therefore the orbit $u \cdot H(\mathcal{L}) = \mathcal{L} \cap \pi^{-1}(x)$ is a discrete subset of the orbit $u \cdot H$. So $H(\mathcal{L})$ is a discrete subgroup of the Lie group $H$.

Thus, each isotropy group of the action $\Phi^W$ is discrete, i. e., $\Phi^W$ is a locally free action.

(ii) Consider an arbitrary point $x \in M$, $u \in \pi^{-1}(x)$ and $w = \pi_b(u)$. From (i) it is follows that $\pi_b(\pi^{-1}(L(x)) = w \cdot H$. Hence the map $s \colon M/F \to W/H \colon [L] \mapsto [w \cdot H]$, where $[L]$ is the leaf $L$ considered as a point of $M/F$ and $[w \cdot H]$ is the orbit of $H$ considered as a point of $W/H$, is well defined and satisfies the equality $k \circ \pi_b = s \circ q \circ \pi$ stated in (ii). Since $k$ and $q$ are open and continuous maps, this relation implies that the bijection $s$ is a homeomorphism.

(iii) This statement is a consequence of the assertion (ii) of Theorem 1 and of the definition of the 1-form $\bar{\omega}$. □

**Corollary 1.** *If $\mathfrak{g}_0(M, F) = 0$, then the holonomy group $\Gamma(L, x)$ is isomorphic to the isotropy group $H_w$, where $w \in \pi_b(\pi^{-1}(x))$, of the induced action $\Phi^W$ of the Lie group $H$ on the basic manifold $W$.*

**Proof.** As shown in the proof of Proposition 7, $H(\mathcal{L}) = H_w$. Therefore, according to Theorem 4, the holonomy group $\Gamma(L, x)$ of a leaf $L$ of this foliation is isomorphic to the isotropy group $H_w$. □

# 8. The groups of basic automorphisms of foliations with TRG

Let $\mathcal{A}(M, F)$ be the full automorphism group of a foliation $(M, F)$ with TRG. We denote by $\mu \colon \mathcal{A}^H(\mathcal{R}, \mathcal{F}) \to \mathcal{A}(M, F)$ the group isomorphism defined in Theorem 3.

**Leaf automorphisms.** The group

$$\mathcal{A}_L(M,F) := \{f \in \mathcal{A}(M,F) \mid f(L_\alpha) = L_\alpha,\ \forall L_\alpha \in F\}$$

is a normal subgroup of $\mathcal{A}(M,F)$ which is called the *leaf automorphism group* of $(M,F)$.

**Proposition 8.** *Consider the subgroup of leaf automorphisms* $\mathcal{A}^H_L(\mathcal{R},\mathcal{F}) := \{f \in \mathcal{A}^H(\mathcal{R},\mathcal{F}) \mid \hat{f}(\mathcal{L}_\alpha) = \mathcal{L}_\alpha,\ \forall \mathcal{L}_\alpha \in \mathcal{F}\}$ *of the group* $\mathcal{A}^H(\mathcal{R},\mathcal{F})$*. Then the restriction*

$$\mu_L := \mu|_{\mathcal{A}^H_L(\mathcal{R},\mathcal{F})} \colon \mathcal{A}^H_L(\mathcal{R},\mathcal{F}) \to \mathcal{A}_L(M,F)$$

*is a group isomorphism.*

**Proof.** Let $\hat{f} \in \mathcal{A}^H_L(\mathcal{R},\mathcal{F})$ and $f := \mu(\hat{f})$. Consider an arbitrary leaf $L \in F$. There exists a leaf $\mathcal{L} \in \mathcal{F}$ such that $\pi|_{\mathcal{L}} \colon \mathcal{L} \to L$ is a covering map. As $\hat{f}(\mathcal{L}) = \mathcal{L}$ and $\pi \circ \hat{f} = f \circ \pi$ then $f(L) = L$, hence $f \in \mathcal{A}_L(M,F)$. Thus we have an inclusion $\mu(\mathcal{A}^H_L(\mathcal{R},\mathcal{F})) \subset \mathcal{A}_L(M,F)$.

Let us show that the map $\mu_L$ is surjective. Take an arbitrary element $g \in \mathcal{A}_L(M,F)$. According to Theorem 3 there is a unique element $\hat{g} \in \mathcal{A}^H(\mathcal{R},\mathcal{F})$ lying over $g$. Let $u$ be an arbitrary point in $\mathcal{R}$, $u' := \hat{g}(u)$, $x = \pi(u)$, $L = L(x)$. There are neighborhoods $U_j \ni x$ and $U_i \ni y = g(x)$ from the $(N,\xi)$-cocycle $\eta = \{U_i, f_i, \{\gamma_{ij}\}\}$ defining the foliation $(M,F)$. Remark that the points $v := f_j(x)$ and $v' := f_i(y)$ belong to the same orbit of the holonomy pseudogroup $\mathcal{H}(M,F)$ of the foliation $(M,F)$. Recall that each element of $\mathcal{H}(M,F)$ is a local automorphism of the transverse rigid geometry $(N,\xi)$. Therefore there exists a local automorphism $\gamma_{ij} \in \mathcal{H}(M,F)$ such that $\gamma_{ij}(v) = v'$. Effectiveness of the transverse rigid geometry $(N,\xi)$ implies the existence of a unique local automorphism $\Gamma_{ij}$ of the rigid structure $\xi$ from the holonomy pseudogroup $\mathcal{H}(\mathcal{R},\mathcal{F})$ with the projection $\gamma_{ij}$. In the notations of the proof of Theorem 1 $\tilde{\eta} = \{\tilde{U}_i, \tilde{f}_i, \{\Gamma_{ij}\}\}$ is a $(P,\omega)$-cocycle defining the $e$-foliation $(\mathcal{R},\mathcal{F})$. Let $w = \tilde{f}_j(u)$, $w' = \tilde{f}_i(u')$ then $w' = \Gamma_{ij}(w)$, i. e., the points $w$ and $w'$ belong to the same orbit of the holonomy pseudogroup $\mathcal{H}(\mathcal{R},\mathcal{F})$. Therefore the points $u$ and $u'$ belong to the same leaf $\mathcal{L}$ of $(\mathcal{R},\mathcal{F})$, i. e., $\hat{g}(\mathcal{L}) = \mathcal{L}$. Hence $\hat{g} \in \mathcal{A}^H_L(\mathcal{R},\mathcal{F})$.

Thus $\mu_L$ is an isomorphism of the groups $\mathcal{A}^H_L(\mathcal{R},\mathcal{F})$ and $\mathcal{A}_L(M,F)$. □

**Basic automorphisms of foliations with TRG.** Remark that the quotient group $\mathcal{A}^H_B(\mathcal{R},\mathcal{F}) := \mathcal{A}^H(\mathcal{R},\mathcal{F})/\mathcal{A}^H_L(\mathcal{R},\mathcal{F})$ is well defined.

**Def 12.** The quotient group

$$\mathcal{A}_B(M,F) := \mathcal{A}(M,F)/\mathcal{A}_L(M,F)$$

is called the *basic automorphism group* of the foliation $(M,F)$ with TRG.

Let $(M,F)$ be a foliation with TRG. Let $M/F$ be the leaf space of $(M,F)$, and $q\colon M \to M/F$ be the natural projection onto the leaf space which maps any $x \in M$ to the leaf $L(x)$ considered as a point $[L(x)]$ in $M/F$. Each $f \in \mathcal{A}(M,F)$ maps an arbitrary leaf $L$ of $F$ onto some leaf of this foliation. Hence the equality $\tilde{f}([L]) = [f(L)]$ defines a mapping $\tilde{f}$ of the leaf space $M/F$ onto itself such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\ q\ } & M/F \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ M & \xrightarrow{\ q\ } & M/F. \end{array} \qquad (1)$$

is commutative. Since $q$ is an open and continuous mapping, (1) implies that $\tilde{f}$ is a homeomorphism of the leaf space $M/F$. Denote by $\mathcal{A}(M/F)$ the set of all such homeomorphisms of $M/F$. Then

$$\tilde{q}\colon \mathcal{A}(M,F) \to \mathcal{A}(M/F)\colon f \mapsto \tilde{f}$$

is a group epimorphism with the kernel $\ker \tilde{q} = \mathcal{A}_L(M,F)$. Therefore the basic automorphism group $\mathcal{A}_B(M,F)$ is canonically isomorphic to the group $\mathcal{A}(M/F)$. Thus the basic automorphism group $\mathcal{A}_B(M,F)$ can be considered as a group, $\mathcal{A}(M/F)$, of homeomorphisms of the leaf space $M/F$ of this foliation.

Let us emphasize that the basic automorphism group $\mathcal{A}_B(M,F)$ of a foliation $(M,F)$ with TRG is an invariant of this foliation in the category $\mathfrak{F}_{TRG}$.

**Proposition 9.** *Let $(M,F)$ be a foliation with TRG and $(\mathcal{R},\mathcal{F})$ be the lifted foliation. Denote by $\mathcal{A}_B^H(\mathcal{R},\mathcal{F})$ the quotient group $\mathcal{A}^H(\mathcal{R},\mathcal{F})/\mathcal{A}_L^H(\mathcal{R},\mathcal{F})$. There exists a natural group isomorphism $\chi\colon \mathcal{A}_B^H(\mathcal{R},\mathcal{F}) \to \mathcal{A}_B(M,F)$ satisfying the commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^H(\mathcal{R},\mathcal{F}) & \xrightarrow{\ \mu\ } & \mathcal{A}(M,F) \\ {\scriptstyle r}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ \mathcal{A}_B^H(\mathcal{R},\mathcal{F}) & \xrightarrow{\ \chi\ } & \mathcal{A}_B(M,F), \end{array} \qquad (2)$$

*where $r$ and $s$ are the associated group epimorphisms onto the quotient groups.*

**Proof.** By Theorem 3, the map $\chi\colon \mathcal{A}_B^H(\mathcal{R},\mathcal{F}) \to \mathcal{A}_B(M,F)\colon \hat{h}\cdot\mathcal{A}_L^H(\mathcal{R},\mathcal{F}) \mapsto h\cdot\mathcal{A}_L(M,F)$, where $h$ is the projection of $\hat{h} \in \mathcal{A}^H(\mathcal{R},\mathcal{F})$ with respect to $\pi\colon \mathcal{R} \to M$, is well defined. According to Proposition 8, $\mu(\ker r) = \ker s$, where $\ker r$ and $\ker s$ are the kernels of the epimorphisms $r$ and $s$, respectively. Hence there exists an isomorphisms of the quotient groups $\mathcal{A}_B^H(\mathcal{R},\mathcal{F})$ and $\mathcal{A}_B(M,F)$ satisfying the diagram (2). □

## 9. Conditions guarantee that $\mathcal{A}_B(M,F)$ is a Lie group

**Uniqueness of a Lie group structure.** The next proposition follows from Proposition 1 proved by Bagaev and the author in [16].

**Proposition 10.** *Let $\mathcal{A}(P,\omega)$ be the Lie group of all automorphisms of a parallelizable manifold $(P,\omega)$. If a group $G$ is realized as a closed subgroup of $\mathcal{A}(P,\omega)$, then $G$ admits a unique topology and a unique smooth structure that make it into a Lie group. This topology coincides with the compact-open topology.*

**The case $\mathfrak{g}_0(M,F) = 0$.** A leaf $L$ of a foliation $(M,F)$ is called *closed* if $L$ is a closed subset in the topology of the manifold $M$. Further we use the term "a closed leaf" only in this sense.

Let $(M,F)$ be a complete foliation with TRG and $\pi_b\colon M \to W$ be the basic fibration. Suppose that $\mathfrak{g}_0(M,F) = 0$, then according to Theorem 2 the leaves of the lifted foliation $(\mathcal{R},\mathcal{F})$ coincide with the fibres of the basic fibration $\pi_b\colon \mathcal{R} \to W$. Hence the basic manifold $W$ can be identified with the leaf space $\mathcal{R}/\mathcal{F}$ of the foliation $(\mathcal{R},\mathcal{F})$, and $\pi_b$ can be identified with the projection $\tilde{q}\colon \mathcal{R} \to \mathcal{R}/\mathcal{F}$.

Applying the commutative diagram (1) to the foliation $(\mathcal{R},\mathcal{F})$ we see that each automorphism $h \in \mathcal{A}^H(\mathcal{R},\mathcal{F})$ induces a diffeomorphism $\tilde{h}$ of the manifold $W$ such that $\pi_b \circ h = \tilde{h} \circ \pi_b$. Since $h^*\tilde{\omega} = \tilde{\omega}$, then, from the definition of the non-degenerate $\mathbb{R}^m$-valued 1-form $\bar{\omega}$ on $W$ satisfying Proposition 7, it is follows that $\tilde{h}^*\bar{\omega} = \bar{\omega}$. From the definition of the action $H$ on $W$ it is follows that $h \circ R_a = R_a \circ h$, $\forall a \in H$.

As above, we denote by $\mathcal{A}(W,\bar{\omega})$ the group of all automorphisms of the parallelizable manifold $(W,\bar{\omega})$, i. e., $\mathcal{A}(W,\bar{\omega}) := \{f \in \mathrm{Diff}W \mid f^*\bar{\omega} = \bar{\omega}\}$. It is well known that $\mathcal{A}(W,\bar{\omega})$ admits a unique Lie group structure. There is a natural bijection between the identity component $\mathcal{A}_e(W,\bar{\omega})$ of $\mathcal{A}(W,\bar{\omega})$ and the orbit $\mathcal{A}_e(W,\bar{\omega})\cdot v$ of a point $v\in W$, being a closed submanifold of $W$. This bijection induces a smooth structure on $\mathcal{A}_e(W,\bar{\omega})$ [9]. According to Proposition 10, the topology of $\mathcal{A}(W,\bar{\omega})$ is the compact-open topology.

Let $\mathcal{A}^H(W) := \{f\in\mathcal{A}(W,\bar{\omega}) \mid f\circ R_a = R_a\circ f\}$ and let $\mathcal{A}^H_e(W)$ be the identity component of $\mathcal{A}^H(W)$. Then $\mathcal{A}^H(W)$ and $\mathcal{A}^H_e(W)$ are closed Lie subgroups of $\mathcal{A}(W,\bar{\omega})$.

**Proposition 11.** *Let $(M,F)$ be a complete foliation with TRG and $\mathfrak{g}_0(M,F)=0$. Then the map*

$$\nu\colon \mathcal{A}^H_B(\mathcal{R},\mathcal{F})\to\mathcal{A}^H(W)\colon h\cdot\mathcal{A}^H_L(\mathcal{R},\mathcal{F})\mapsto\tilde{h}$$

*where $h\in\mathcal{A}^H(\mathcal{R},\mathcal{F})$ and $\tilde{h}$ is the projection of $h$ with respect to $\pi_b\colon\mathcal{R}\to W$ is a group isomorphism onto an open-closed Lie subgroup of the Lie group $\mathcal{A}^H(W)$.*

**Proof.** At first, consider the map $\alpha\colon\mathcal{A}^H(\mathcal{R},\mathcal{F})\to\mathcal{A}^H(W)\colon h\mapsto\tilde{h}$, where $\tilde{h}$ is the projection of $h$ with respect to $\pi_b\colon M\to W$. As shown above, $\tilde{h}\in\mathcal{A}^H(W)$. It is clear that $\alpha$ is a group homomorphism with the kernel $\ker\alpha$, being equal to the normal subgroup $\mathcal{A}^H_L(\mathcal{R},\mathcal{F})$. Therefore, there exists a group isomorphism $\nu\colon\mathcal{A}^H_B(\mathcal{R},\mathcal{F})\to\mathcal{A}^H(W)$, satisfying the equality $\alpha=\nu\circ r$, where $r\colon\mathcal{A}^H(\mathcal{R},\mathcal{F})\to\mathcal{A}^H_B(\mathcal{R},\mathcal{F})$ is the natural projection onto the quotient group $\mathcal{A}^H_B(\mathcal{R},\mathcal{F})=\mathcal{A}^H(\mathcal{R},\mathcal{F})/\mathcal{A}^H_L(\mathcal{R},\mathcal{F})$.

It is enough to prove that $\mathrm{im}\alpha$ is an open-closed subgroup in $\mathcal{A}^H(W)$.

If $\mathcal{A}^H(W)$ is a discrete Lie group, then $\mathrm{im}\nu=\mathrm{im}\alpha$ is also a discrete Lie group.

Now suppose that $\dim\mathcal{A}^H(W)\geqslant 1$. Let $\mathfrak{a}$ be the Lie algebra of the Lie group $\mathcal{A}^H(W)$ and $B$ be any element of $\mathfrak{a}$. Denote by $B^*$ the fundamental vector field defined by $B$. Then $X:=B^*$ is a complete vector field and it defines a 1-parameter group $\varphi^X_t$, $t\in(-\infty,\infty)$, of diffeomorphisms of $W$. The condition $\varphi^X_t\in\mathcal{A}^H(W)$, $\forall t\in(-\infty,\infty)$ is equivalent to the following relations: 1) $L_X A^*_W=0$, $\forall A\in\mathfrak{h}$; 2) $L_X\bar{\omega}=0$.

Since $\pi_b\colon\mathcal{R}\to W$ is a submersion with an Ehresmann connection $\overline{\mathfrak{M}}$, there exists a unique vector field $Y\in\mathfrak{X}_{\overline{\mathfrak{M}}}(\mathcal{R})$ such that $\pi_{b*}Y=X$. Remark that completeness of the vector field $X$ implies completeness of the vector field $Y$. Hence $Y$ defines a 1-parameter group $\psi^Y_t$, $t\in(-\infty,\infty)$, of diffeomorphisms of the manifold $\mathcal{R}$. Let us show that $\psi^Y_t\in\mathcal{A}^H(\mathcal{R},\mathcal{F})$, $\forall t\in(-\infty,\infty)$, i. e., we have to check the validity of the following facts: 1) the map $\psi^Y_t$, $t\in(-\infty,\infty)$, is an isomorphism of $(\mathcal{R},\mathcal{F})$ in the category $\mathfrak{F}ol$; 2) $L_Y\tilde{\omega}=0$; 3) $L_Y A^*=0$, $\forall A\in\mathfrak{h}$.

1) The equality $\pi_{b*}Y=X$ implies the relation $\pi_b\circ\psi^Y_t=\phi^X_t\circ\pi_b$ for any fixed $t\in(-\infty,\infty)$, hence $\psi^Y_t(\pi_b^{-1}(v))=\pi_b^{-1}(\phi^X_t(v))$, $\forall v\in W$, and $\psi^Y_t$ is an isomorphism of the lifted foliation $(\mathcal{R},\mathcal{F})$ in the category $\mathfrak{Fol}$.

2) Take arbitrary $u\in\mathcal{R}$ and $Z_0\in\overline{\mathfrak{M}}_u$. There is a unique vector field $Z\in\mathfrak{X}_{\overline{\mathfrak{M}}}(\mathcal{R})$ such that $Z|_u=Z_0$ and $\tilde{\omega}(Z)=\tilde{\omega}(Z_0)=\mathrm{const}$. Put $Z_W:=\pi_{b*}Z$ and apply the following formula [9]:

$$(L_X\bar{\omega})(Z_W)=X(\bar{\omega}(Z_W))-\bar{\omega}([X,Z_W]). \tag{3}$$

The relation $\tilde{\omega}=\bar{\omega}\circ\pi_{b*}$ implies that $\bar{\omega}(Z_W)=\tilde{\omega}(Z_0)=\mathrm{const}$, so $X(\bar{\omega}(Z_W))=0$. By the choice of $X$ we have $L_X\bar{\omega}=0$. Hence the equality (3) gives

$$\bar{\omega}([X,Z_W])=0. \tag{4}$$

In the formula

$$(L_Y\tilde{\omega})(Z)=Y(\tilde{\omega}(Z))-\tilde{\omega}([Y,Z]) \tag{5}$$

the first term $Y(\tilde{\omega}(Z))=0$, because $\tilde{\omega}(Z)=\mathrm{const}$. The relations $\tilde{\omega}=\bar{\omega}\circ\pi_{b*}$ and (4) imply the following chain of equalities:

$$\tilde{\omega}([Y,Z])=\bar{\omega}(\pi_{b*}[Y,Z])=\bar{\omega}([\pi_{b*}Y,\pi_{b*}Z])=\bar{\omega}([X,Z_W])=0.$$

Therefore (5) implies that $(L_Y\tilde{\omega})(Z)=0$ and $(L_Y\tilde{\omega})(Z_0)=0$. Thus, $L_Y\tilde{\omega}=0$.

3) Denote by $(W,F^H)$ the foliation formed by the connected components of orbits of the action $\Phi^W$ of $H$ on $W$. Let $(\mathcal{R},\mathcal{F}^H)$ be the foliation formed by the connected components of orbits of the Lie group $H$ on $\mathcal{R}$.

At any point $u\in\mathcal{R}$ there is an neighborhood $\mathcal{W}$ foliated with respect to both foliations $(\mathcal{R},\mathcal{F})$ and $(\mathcal{R},\mathcal{F}^H)$ which meets each leaf of these foliations in at most one connected subset. We can suppose that the basic fibration $\pi_b\colon\mathcal{R}\to W$ is trivial in the neighborhood $\pi_b^{-1}(\mathcal{V})$, where $\mathcal{V}:=\pi_b(\mathcal{W})$. Put $U=\pi(\mathcal{W})$. Let $r\colon U\to U/(F|_U)$ and $s\colon\mathcal{V}\to\mathcal{V}/(\mathcal{F}^H|_{\mathcal{V}})$ be the quotient maps. We can identify $U/(F|_U)$ and $\mathcal{V}/(\mathcal{F}^H|_{\mathcal{V}})$ with the manifold $V$ such that the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\pi_b} & \mathcal{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle s} \\ U & \xrightarrow{r} & V, \end{array} \tag{6}$$

where the restrictions of $\pi$ and $\pi_b$ onto $\mathcal{W}$ are denoted by the same letters, is commutative. Without loss of generality, we can assume that $\mathfrak{M}|_U$ is an Ehresmann connection for the submersion $r$ and $\overline{\mathfrak{M}}|_{\mathcal{W}}$ is an Ehresmann connection for the submersion $\pi_b$.

By the choice of $X$, for any $A\in\mathfrak{h}$ we have the equality $L_XA^*_W=0$, i. e., $[A^*_W,X]=0$. Since the fundamental vector fields span the tangent spaces to the leaves of the foliation $(W,F^H)$, it is not difficult to check that $X$ is a foliated vector field for this foliation. Hence the vector field $X_V:=s_*X|_{\mathcal{V}}$ is well defined. There is a unique vector field $Y_U\in\mathfrak{X}_{\mathfrak{M}}(U)$ such that $r_*Y_U=X_V$. In other words, $Y_U$ is the $\mathfrak{M}$-horizontal lift of $X_V$. The commutative diagram (6) implies the relation $\pi_{b*}Y_{\mathcal{W}}=Y_U$, hence $Y$ is a foliated vector field with respect to the foliation $(\mathcal{R},\mathcal{F}^H)$. Therefore,

$$[A^*,Y]\in\mathfrak{X}_{\mathcal{F}^H}(\mathcal{R}). \tag{7}$$

According to Theorem 1, we have the equalities $\tilde{\omega}(A^*)=\bar{\omega}(A^*_W)=A$, hence $A^*$ is a vector field foliated with respect to $(\mathcal{R},\mathcal{F})$. So we have the following chain of equalities

$$\pi_{b*}[A^*,Y]=[\pi_{b*}A^*,\pi_{b*}Y]=[A^*_W,X]=0,$$

hence,

$$[A^*,Y]\in\mathfrak{X}_{\mathcal{F}}(\mathcal{R}). \tag{8}$$

The relations (7) and (8) imply the equality $[A^*,Y]=0$, $\forall A\in\mathfrak{h}$.

Thus, we proved the inclusion $\mathcal{A}^H_e(W)\subset\mathrm{im}\alpha=\mathrm{im}\nu$. Therefore $\mathrm{im}\nu$ is an open-closed Lie subgroup of the Lie group $\mathcal{A}^H(W)$. □

**Theorem 5.** *Let $(M,F)$ be a complete foliation with a transverse rigid geometry $(N,\xi)$, where $\xi=(P(N,H),\omega)$. Suppose that the structure Lie algebra $\mathfrak{g}_0(M,F)$ is zero. Then:*
*(i) the full basic automorphism group $\mathcal{A}_B(M,F)$ is realized as an open-closed subgroup of the Lie group $\mathcal{A}^H(W)$ and admits a Lie group structure with the following estimate of its dimension:*

$$\dim\mathcal{A}_B(M,F)\leqslant\dim P; \tag{9}$$

*(ii) if either there exists an isolated closed leaf $L$ or the set of closed leaves of the foliation $(M,F)$ is countable, then*

$$\dim\mathcal{A}_B(M,F)\leqslant\dim H; \tag{10}$$

*(iii) there exists a unique topology and a unique smooth structure on the full group $\mathcal{A}_B(M,F)$ of basic automorphisms of the foliation $(M,F)$, making $\mathcal{A}_B(M,F)$ into a Lie group. This topology coincides with the compact-open topology, when $\mathcal{A}_B(M,F)$ is realized as a subgroup of the group $\mathcal{A}^H(W)$.*

**Proof.** (i) Applying Propositions 9 and 11, we get that the map $\beta := \nu \circ \chi^{-1} \colon \mathcal{A}_B(M,F) \to \mathcal{A}^H(W)$ is a group isomorphism of the full group of basic automorphisms $\mathcal{A}_B(M,F)$ onto an open-closed subgroup im$\beta$ of the Lie group $\mathcal{A}^H(W)$. We identity $\mathcal{A}_B(M,F)$ with im$\beta$ and consider $\mathcal{A}_B(M,F)$ as an open-closed subgroup of the Lie group $\mathcal{A}^H(W)$. Hence $\mathcal{A}_B(M,F)$ admits a Lie group structure, and the following estimates of dimensions hold:

$$\dim \mathcal{A}_B(M,F) \leqslant \dim \mathcal{A}^H(W) \leqslant \dim \mathcal{A}(W,\bar{\omega}) \leqslant \dim W = \dim P = m.$$

(ii) Suppose that there exists a closed leaf $L$ of the foliation $(M,F)$. Then $\pi_b(\pi^{-1}(L))$ is a closed orbit of the action $\Phi^W$ of the Lie group $H$ on $W$. Let us to fix an arbitrary point $v$ in this orbit. Let $L^W = L^W(v)$ be a leaf of the foliation $(W,F^W)$. Then the leaf $L^W$ is a closed subset in $W$. It is known [1] that the smooth structure of the Lie group $\mathcal{A}^H_e(W)$ coincides with the smooth structure induced by the bijection of the identity component $\mathcal{A}^H_e(W)$ of the Lie group $\mathcal{A}^H(W)$ onto the closed submanifold $\mathcal{A}^H_e(W)\cdot v$ of $W$, where $\mathcal{A}^H_e(W)\cdot v$ is the orbit of the point $v$. Any $g \in \mathcal{A}^H_e(W)$ maps each closed orbit of the Lie group $H$ onto some closed orbit of $H$. Since $\mathcal{A}^H_e(W)\cdot v$ is connected and either the orbit $v\cdot H$ is isolated or if the set of closed orbits of $H$ is countable, so $\mathcal{A}^H_e(W)\cdot v \subset L^W$. Therefore $\dim \mathcal{A}_B(M,F) \leqslant \dim \mathcal{A}^H_e(W)\cdot v \leqslant \dim F^H = \dim H$.

(iii) Applying the statement (i) proved above and Proposition 10 to the group $\mathcal{A}_B(M,F)$ we get the statement (iii). □

**Remark 6.** Theorem 5 does not exclude the triviality of the full group $\mathcal{A}_B(M,F)$.

**Remark 7.** The main result of the work [3] by Belko is the theorem asserting that if there exists a closed leaf of a foliation $(M,F)$ with complete transversally projectable affine connection, then the group $\mathcal{A}_B(M,F)$ is a Lie group. This statement is not correct. It's proof essentially uses the fact that existence of a closed leaf of this foliation implies that the lifted foliation is simple. It is not true, in general. Let us consider a foliation $(M,F)$ from Example 3 (in Section 10), when $r = 1/\pi$, as affine foliation. It has a compact leaf, but $\mathfrak{g}_0(M,F) = \mathbb{R}^1 \neq 0$, hence the lifted foliation is not simple. Thus the foliation $(M,F)$ is a Lie foliation with non-zero structure Lie algebra $\mathfrak{g}_0(M,F)$, and the group $\mathcal{A}_B(M,F)$ is not a Lie group.

**Discrete holonomy groups of leaves.** Let $(M,F)$ be a complete foliation with TRG. Let $\pi\colon \mathcal{R} \to M$ be the projection of the foliated bundle over $(M,F)$.

**Def 13.** We say that the holonomy group of a leaf $L \ni x$ of the foliation $(M,F)$ is *discrete* if there exists a point $u \in \pi^{-1}(x)$ such that the group $H(\mathcal{L}) := \{a \in H \mid R_a(\mathcal{L}) = \mathcal{L},\ \mathcal{L} = \mathcal{L}(u) \in \mathcal{F}\}$ is a discrete subgroup of the Lie group $H$.

Let $u' \in \pi^{-1}(x)$ and $u' \neq\in \mathcal{L} = \mathcal{L}(u)$. In this case the subgroup $H(\mathcal{L}')$ is conjugate to the subgroup $H(\mathcal{L})$ in the Lie group $H$. Hence $H(\mathcal{L})$ is a discrete subgroup of $H$ iff $H(\mathcal{L}')$ is a discrete subgroup of $H$. Thus, by Theorem 4 the notion of discrete holonomy group of leaf $L$ is well defined.

Recall that a leaf $L$ of a foliation $(M,F)$ is said to be *proper* if $L$ is an embedded submanifold in $M$. A foliation $(M,F)$ is called *proper* if each its leaf is proper.

**Proposition 12.** *Let $(M,F)$ be a complete foliation with TRG. If there exists proper leaf $L$ with discrete holonomy group then the structure Lie algebra $\mathfrak{g}_0(M,F)$ is zero.*

**Proof.** Let $L$ be a proper leaf with discrete holonomy group. Let $x \in L$, $u \in \pi^{-1}(x)$, and $\mathcal{L} = \mathcal{L}(u)$. Since $L$ is proper, there exists a foliated neighborhood $U$ of the point $x$ such that $L$ meets $U$ in a connected subset $L \cap U$. Then there is a neighborhood $\mathcal{U}$ of $u$ foliated with respect to $(\mathcal{R},\mathcal{F})$ such that an embedded submanifold $\pi^{-1}(L)$ of $\mathcal{R}$ meets $\mathcal{U}$ in a connected subset. Since the subgroup $H(\mathcal{L})$ of $H$ is discrete, $\mathcal{L}\cap\pi^{-1}(x) = u\cdot H(\mathcal{L})$ is a discrete subset of the fiber $\pi^{-1}(x)$. Therefore, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of $u$ foliated with respect to the foliation $(\mathcal{R},\mathcal{F})$ such that $\mathcal{L}\cap\mathcal{V}$

is connected. By [17, Theorem 4.11], it follows that the leaf $\mathcal{L}$ is proper. Thus the complete $e$-foliation $(\mathcal{R}, \mathcal{F})$ has a proper leaf $\mathcal{L}$. It is known [10] that such a foliation is formed by the fibers of a locally trivial fibration. Hence all leaves of the lifted foliation $(\mathcal{R}, \mathcal{F})$ are closed and the structure Lie algebra $\mathfrak{g}_0(M, F)$ is zero. □

**Theorem 6.** *Let $(M, F)$ be a complete foliation with transverse rigid geometry $(N, \xi)$, where $\xi = (P(N, H), \omega)$. If at least one of the following conditions holds:*
*(i) there exists a proper leaf $L$ with discrete holonomy group;*
*(ii) there is a closed leaf $L$ with discrete holonomy group;*
*(iii) there exists a proper leaf $L$ with finite holonomy group;*
*(iv) there is a closed leaf $L$ with finite holonomy group,*
*then the basic automorphism group $\mathcal{A}_B(M, F)$ admits a Lie group structure of dimension at most $\dim P$, and this structure is unique.*

**Proof.** Remark that any closed leaf of a foliation is proper and each finite holonomy group is a discrete one. Hence we have implications $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$. According to Proposition 12 the existence of a proper leaf $L$ with discrete holonomy group guarantees the equality $\mathfrak{g}_0(M, F) = 0$. Thus, applying Theorem 5 we get the required assertion. □

It is well known that any foliation has leaves without holonomy. Therefore, the following statement is a consequence of the assertion (iii) of Theorem 6.

**Corollary 2.** *For any proper complete foliation $(M, F)$ with TRG the basic automorphism group $\mathcal{A}_B(M, F)$ admits a unique Lie group structure.*

## 10. Foliations covered by fibrations

**$(G, X)$-foliations.** It is said that a group of diffeomorphisms of a manifold $X$ *acts quasi-analytically,* if existence of an element $g \in G$ and an open subset $U$ in $X$ such that $g|_U = \mathrm{id}_U$ implies $g = \mathrm{id}_X$.

Let $G$ be a Lie group of diffeomorphisms of a manifold $X$, which acts quasi-analytically on $X$. Recall that $(M, F)$ is a $(G, X)$-foliation if $(M, F)$ is defined by an $\eta$-cocycle $\{U_i, f_i, \{\gamma_{ij}\}\}$, where $f_i \colon U_i \to V_i$ is a submersion onto an open subset of $X$, for each $\gamma_{ij}$ there is $g \in G$ such that $\gamma_{ij} = g|_{f_j(U_i \cap U_j)}$ and $\{V_i\}$ is a covering of $X$. Uniqueness of such a $g \in G$ is a consequence of quasi-analyticity of the action of $G$.

**Def 14.** Let $(X, \xi)$ be a rigid geometry, where $\xi = (P(H, X), \omega)$, let $G$ be an automorphism group of $(X, \xi)$. A $(G, X)$-foliation $(M, F)$ is called a $(G, X)$-*foliation with transverse rigid structure.*

**Foliations covered by fibrations.** Let $f \colon \tilde{M} \to M$ be the universal covering map.

**Def 15.** We say that a foliation $(M, F)$ *is covered by a fibration* if the induced foliation $\tilde{F} := f^*F$ on $\tilde{M}$ is formed by the leaves of a submersion $r \colon \tilde{M} \to B$ onto a $q$-dimensional manifold $B$, where $q$ is codimension of the foliation $(M, F)$.

**Proposition 13.** *Let $(M, F)$ be a foliation with TRG, which admits an Ehresmann connection $\mathfrak{M}$ and is covered by a fibration $r \colon \tilde{M} \to B$, where $f \colon \tilde{M} \to M$ is the universal covering map. Then:*
*(i) $B$ is simply connected, the submersion $r \colon \tilde{M} \to B$ is a locally trivial fibration;*
*(ii) the manifold $B$ admits a rigid geometry $\zeta$ locally isomorphic to $(N, \xi)$;*
*(iii) there is a group epimorphism $\alpha \colon \pi_1(M) \to \Psi$ onto some subgroup $\Psi$ of automorphisms of the rigid geometry $(B, \zeta)$: the group $\Psi$ is called the global holonomy group of the foliation covered by a fibration; the foliation $(M, F)$ is a $(\Psi, B)$-foliation with TRG;*
*(iv) for any $x \in M$, the holonomy group $\Gamma(L, x)$ of the leaf $L = L(x)$ is isomorphic to the isotropy subgroup $\Psi_y$, where $y \in r(f^{-1}(x))$.*

**Proof.** (i) By Proposition 2, the distribution $\mathfrak{M}$ is an Ehresmann connection for foliation $(M, F)$. It is not difficult to check that the distribution $\tilde{\mathfrak{M}} = f^*\mathfrak{M}$ is an Ehresmann connection for the foliation $(\tilde{M}, \tilde{F})$, hence $\tilde{\mathfrak{M}}$ is an Ehresmann connection for the submersion $r\colon \tilde{M} \to B$. It is well known that a submersion admitting an Ehresmann connection is a locally trivial fibration. Thus, the foliation $(\tilde{M}, \tilde{F})$ is formed by the fibres of the locally trivial fibration $r\colon \tilde{M} \to B$. Hence $r\colon \tilde{M} \to B$ is a fibration with the covering homotopy property. Applying the exact homotopy sequence and the fact that the leaves of $r$ are arcwise connected and the manifold $\tilde{M}$ is simply connected, we see that the basic manifold $B$ is also simply connected.

(ii) Let $f_i\colon U_i \to V_i$ be a submersion from the $(N, \xi)$-cocycle $\eta$, defining $(M, F)$. Without loss of generality, we can assume that $U_i$ is a regularly covered neighborhood, i. e., $f^{-1}(U_i) = \sqcup \mathcal{W}_\alpha$ is a disjunct sum of neighborhoods $\mathcal{W}_\alpha$ such that $f|_{\mathcal{W}_\alpha}\colon \mathcal{W}_\alpha \to U_i$ is a diffeomorphism. Therefore there exists a diffeomorphism $\gamma_\alpha\colon \mathcal{V}_\alpha = r(\mathcal{W}_\alpha) \to V_i$ satisfying the equality $\gamma_\alpha \circ r = f_i \circ f$ on $\mathcal{W}_\alpha$. The diffeomorphism $\gamma_\alpha$ induces a rigid geometry $\zeta_\alpha = (\tilde{P}_\alpha(\mathcal{V}_\alpha, H), \omega_\alpha)$, where $\tilde{P}_\alpha := \gamma_\alpha^* P_\alpha$ on $\mathcal{V}_\alpha$, such that $\gamma_\alpha\colon \mathcal{V}_\alpha \to V_i$ is an isomorphism of $(\mathcal{V}_\alpha, \zeta_\alpha)$ and $(V_i, \xi_{V_i})$. By a straightforward verification, one can show that there exists a unique rigid structure $\zeta = (\tilde{P}(H, B), \beta)$ on $B$ such that $\zeta|_{\mathcal{V}_\alpha} = \zeta_\alpha$.

(iii) Let us consider the fundamental group $\pi_1(M, x)$, $x \in M$, as the group $G$ of deck transformations of the universal covering map $f\colon \tilde{M} \to M$. Since each $g \in G$ is an isomorphism of the induced foliation $(\tilde{M}, \tilde{F})$ and the basic manifold $B$ of the fibration $r\colon \tilde{M} \to B$ can be considered as the leaf space $\tilde{M}/\tilde{F}$, $g$ defines a map $\psi\colon B \to B$ satisfying the relation $r \circ g = \psi \circ r$. Hence $\psi$ is a diffeomorphism of $B$. Moreover, from the definition of the rigid geometry $\zeta$ on $B$ it follows that $\psi \in \mathcal{A}(B, \zeta)$. Denote by $\Psi$ the group of all such $\psi$. Then there is a group epimorphism $\chi\colon \pi_1(M, x) \to \Psi\colon g \mapsto \psi$, where $r \circ g = \psi \circ r$.

(iv) As $f\colon \tilde{M} \to M$ is the covering map, we can consider $(M, F)$ as a $(\Psi, B)$-foliation. Then the holonomy pseudogroup $\mathcal{H}$ of $(M, F)$ is determined by the group $\Psi$. Since $\Psi$ acts quasi-analytically on $B$, for each $y \in B$ the group $\mathcal{H}_y$, which consists of germs at $y$ of transformations from the isotropy subpseudogroup of the holonomy pseudogroup $\mathcal{H}$, is isomorphic to the isotropy subgroup $\Psi_y$. According to Theorem 4, the holonomy group $\Gamma(L, x)$ is isomorphic to $\mathcal{H}_y$, where $y \in r(f^{-1}(x))$, and hence the group $\Gamma(L, x)$ is isomorphic to $\Psi_y$. □

According to Proposition 2 a complete foliation $(M, F)$ with TRG admits an Ehresmann connection, hence the following assertion is true.

**Corollary 3.** *If $(M, F)$ is a complete foliation with TRG, then the statements of Proposition 13 are valid for $(M, F)$.*

**Basic automorphism groups of foliations with TRG covered by fibrations.** In the following theorem we give and apply another interpretation of the structure Lie algebra of a foliation $(M, F)$ with TRG covered by a fibration.

**Theorem 7.** *Let $(M, F)$ be a complete foliation with TRG covered by a fibration $r\colon \tilde{M} \to B$, where $f\colon \tilde{M} \to M$ is the universal covering map. Let $\Psi$ be the global holonomy group of $(M, F)$ considered as a subgroup of the Lie group $\mathcal{A}(B, \zeta)$ of all automorphisms of the rigid geometry $(B, \zeta)$, which was introduced in Proposition 13. Then:*
*(i) the structure Lie algebra $\mathfrak{g}_0(M, F)$ is isomorphic to the Lie algebra of the Lie group $\overline{\Psi}$, where $\overline{\Psi}$ is the closure of $\Psi$ in the full Lie group of automorphisms $\mathcal{A}(B, \zeta)$ which is a Lie group;*
*(ii) the equality $\mathfrak{g}_0(M, F) = 0$ is equivalent to the condition that $\Psi$ is a discrete subgroup of the Lie group $\mathcal{A}(B, \zeta)$;*
*(iii) if $\Psi$ is a discrete subgroup of the Lie group $\mathcal{A}(B, \zeta)$, then the full group of basic automorphisms $\mathcal{A}_B(M, F)$ admits a Lie group structure, and this structure is unique.*

**Proof.** As above, let $\pi\colon \mathcal{R} \to M$ be the projection of the foliated bundle over $(M, F)$ and $f\colon \tilde{M} \to M$ be the universal covering map. Put $\tilde{\mathcal{R}} := f^*\mathcal{R} = \{(y, u) \in$

$\tilde{M} \times \mathcal{R} \mid f(y) = \pi(u)\}$, $\tilde{\pi} \colon \tilde{\mathcal{R}} \to \tilde{M} \colon (y, u) \mapsto y$, $\varphi \colon \tilde{\mathcal{R}} \to \mathcal{R} \colon (y, u) \mapsto u$. A right action of $H$ on $\tilde{\mathcal{R}}$ is defined by the equality $(y, u) \cdot a = (y, u \cdot a)$, $\forall a \in H$. It is easy to see that a principal $H$-bundle $\tilde{\pi} \colon \tilde{\mathcal{R}} \to \tilde{M}$ equipped with a foliation $(\tilde{\mathcal{R}}, \tilde{\mathcal{F}})$, where $\tilde{\mathcal{F}} = \varphi^* \tilde{\mathcal{F}}$, is the foliated bundle for the foliation $(\tilde{M}, \tilde{F})$. Since $(\tilde{M}, \tilde{F})$ is a simple foliation defined by submersion $r \colon \tilde{M} \to B$, so $(\tilde{\mathcal{R}}, \tilde{\mathcal{F}})$ is also simple foliation defined by the projection of the basic fibration $\tilde{\pi}_b \colon \tilde{\mathcal{R}} \to \tilde{W}$. In general, when the Lie group $H$ is not connected, the manifold $\tilde{\mathcal{R}}$ is not simply connected. Remark that the lifted $e$-foliation $(\mathcal{R}, \mathcal{F})$ is covered by fibration $\tilde{\pi}_b \colon \tilde{\mathcal{R}} \to \tilde{W}$. The fundamental group $G = \pi_1(M, x)$ acts on $\tilde{\mathcal{R}}$ by the formula $g(y, u) := (g(y), u), \forall (y, u) \in \tilde{\mathcal{R}}, \forall g \in G$. Hence $g \circ R_a = R_a \circ g, \forall a \in H, \forall g \in G$. Moreover, each $g$ is an automorphism of the foliation $(\tilde{\mathcal{R}}, \tilde{\mathcal{F}})$ in the category $\mathfrak{Fol}$. Therefore $G$ induces a group $\tilde{\Psi} \subset Diff(\tilde{W})$. Let $s \colon \tilde{W} \to B$ be a map defined by the equality $s \circ \tilde{\pi}_b = \tilde{\pi} \circ r$. Analogously to proof of Proposition 13, a rigid structure $\zeta = (\tilde{W}(H, B), \theta)$ with the projection $s \colon \tilde{W} \to B$ is defined, and $(B, \zeta)$ is a rigid geometry, $\tilde{\Psi} \subset \mathcal{A}(\zeta)$. Furthermore the group isomorphism $\mathcal{A}(\zeta) \to \mathcal{A}(\mathcal{B}, \zeta)$ maps $\tilde{\Psi}$ onto $\Psi$.

Consider any leaf $\mathcal{L} = \mathcal{L}(u)$, $u \in \mathcal{R}$, of $(\mathcal{R}, \mathcal{F})$. Let $z \in \varphi^{-1}(u)$ and $d = \tilde{\pi}_b(z)$. Since $\tilde{\Psi} \subset \mathcal{A}(\zeta) \subset \mathcal{A}(\tilde{\mathcal{W}}, \theta)$, so $\overline{\tilde{\Psi} \cdot d} = (cl\tilde{\Psi}) \cdot d$, where $\overline{\tilde{\Psi} \cdot d}$ is the closure of the orbit $\tilde{\Psi} \cdot d$ in $\tilde{W}$, and $cl\tilde{\Psi}$ is a closure of $\tilde{\Psi}$ in the Lie group $\mathcal{A}(\zeta)$. Hence the closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ in $\mathcal{R}$ satisfies the equality $\overline{\mathcal{L}} = \varphi(\tilde{\pi}_b^{-1}((cl\tilde{\Psi}) \cdot d))$. Denote by $(cl\tilde{\Psi})_e$ the identity component of the Lie group $cl\tilde{\Psi}$, then $(cl\tilde{\Psi})_e \cdot d$ and $\mathbb{L} := \tilde{\pi}_b^{-1}((cl\tilde{\Psi})_e \cdot d)$ are connected smooth manifolds, with $\varphi|_{\mathbb{L}} \colon \mathbb{L} \to \overline{\mathcal{L}}$ is a regular covering map. The induced foliation $(\varphi|_{\mathbb{L}})^*(\mathcal{F}|_{\overline{\mathcal{L}}})$ is simple and is defined by a submersion $\tilde{\pi}_b|_{\mathbb{L}} \colon \mathbb{L} \to (cl\tilde{\Psi})_e \cdot d \cong (cl\tilde{\Psi})_e$. It is known [11] that this implies that the structure Lie algebra of the Lie foliation $(\overline{\mathcal{L}}, \mathcal{F}|_{\overline{\mathcal{L}}})$ with dense leaves is isomorphic to the Lie algebra of the Lie group $(cl\tilde{\Psi})_e$. Since $\Psi$ is the projection of $\tilde{\Psi}$ with respect to $s \colon \tilde{W} \to B$, so effectiveness of $\zeta$ implies that the Lie groups $\overline{\Psi}$ and $cl\tilde{\Psi}$ are isomorphic.

The statement (ii) is a direct consequence of the statement (i). Therefore the assertion (iii) follows from Theorem 5. □

**Basic automorphisms of foliations with integrable Ehresmann connections.**

**Proposition 14.** *Let $(M, F)$ be an $\mathfrak{M}$-complete foliation with TRG. Suppose that the distribution $\mathfrak{M}$ is integrable and, therefore, defines a foliation $(M, F^t)$, where $TF^t = \mathfrak{M}$. Then:*
*(i) the universal covering manifold $\tilde{M}$ can be identified with the product $L \times B$ of some manifolds $L$ and $B$, and $(M, F)$ is covered by the trivial fibration $r \colon L \times B \to B$, where $r$ is the canonical projection onto the second factor;*
*(ii) if the global holonomy group $\Psi$ is a discrete subgroup of the Lie group $\mathcal{A}(B, \zeta)$ of all automorphisms of the induced rigid geometry $(B, \zeta)$, then the full group of basic automorphisms $\mathcal{A}_B(M, F)$ admits a unique Lie group structure.*

**Proof.** By assumption, $M$ is endowed with two transverse foliations $(F, F^t)$ of dimensions $p$ and $q$, respectively, where $p + q = \dim M$. According to Proposition 2, the distribution $\mathfrak{M} = TF^t$ is an Ehresmann connection for the foliation $(M, F)$.

Let $f \colon \tilde{M} \to M$ be the universal covering map. Let $\tilde{F} := f^*F$, $\tilde{F}^t := f^*F^t$ be the induced foliations on $\tilde{M}$ and $\tilde{\mathfrak{M}} := T\tilde{F}^t$. Remark that $\tilde{\mathfrak{M}} = f^*\mathfrak{M}$ is an integrable Ehresmann connection for the foliation $(\tilde{M}, \tilde{F})$. In the terminology of Section 3, the simply connected manifold $\tilde{M}$ is endowed with two transverse foliations $(\tilde{F}, \tilde{F}^t)$ such that for any pair of curves $(\sigma, h)$ with a common initial point $\sigma(0) = h(0)$, where $\sigma$ is a horizontal curve and $h$ is a vertical curve, there exists a vertical-horizontal homotopy $H$ with the base $(\sigma, h)$. In other words, the conditions of the famous Kashiwabara's theorem about the decomposition of manifolds [18] (rediscovered by Blumenthal and

Hebda [12]) are satisfied. According to this theorem there exists a diffeomorphism $\Phi$ of $\tilde{M}$ onto a product of manifolds $L \times B$ which is an isomorphism in the category $\mathfrak{Fol}$ of two pairs of foliations: first, of $(\tilde{M}, \tilde{F})$ and $(L \times B, F_1)$, where $F_1 = \{L \times \{z\} \mid z \in B\}$, second, of $(\tilde{M}, \tilde{F}^t)$ and $(L \times B, F_2)$, where $F_2 = \{\{y\} \times B \mid y \in L\}$. We identify $\tilde{M}$ with $L \times B$ by means of $\Phi$, while the foliation $(\tilde{M}, \tilde{F})$ is identified with the trivial foliation $(L \times B, F_1)$. Therefore the foliation $(M, F)$ is covered by the trivial fibration $r\colon L \times B \to B$. Thus, $(M, F)$ satisfies Theorem 7. □

# 11. Examples

**Foliations obtained by suspension of a homomorphism.** Let $\rho\colon \pi_1(B, b_0) \to \mathrm{Diff}(T)$ be a homomorphism of the fundamental group of a manifold $B \ni b_0$ into the group of diffeomorphisms of a $q$-dimensional manifold $T$, and let $p\colon \hat{B} \to B$ be the universal covering mapping. Then we have a right action of the group $\Pi := \pi_1(B, b_0)$ on $\hat{B}$ by deck transformations. The equality

$$(x, t) \cdot g := (x \cdot g, \rho(g^{-1})(t)), \quad \forall (x, t) \in \hat{B} \times T, \quad \forall g \in \Pi,$$

defines a free right properly discontinuous smooth action of the group $\Pi$ on the product of manifolds $\hat{B} \times T$; therefore the quotient manifold $M := \hat{B} \times_{\Pi} T$ is defined. Let $\kappa\colon \hat{B} \times T \to M$ be the natural projection. Then $F := \{\kappa(\hat{B} \times \{t\}) \mid t \in T\}$ is a foliation of codimension $q$ on $M$; in this case, it is said that the *foliation* $(M, F)$ *is obtained by suspension of the homomorphism* $\rho$. For this foliation we will use the notation $(M, F) := \mathrm{Sus}(T, B, \rho)$ suggested in [19]. The image $\Psi := \mathrm{im}\rho$ is the global holonomy group of $(M, F)$.

**Transversally similar and transversally homothetic foliations.** Let $G$ be the similarity group of the Euclidean space $\mathbb{E}^q$, $q \geqslant 1$, and $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Then $G = CO(q) \ltimes \mathbb{R}^q$ is the semidirect product of the conformal group $CO(q) = \mathbb{R}^+ \cdot O(q)$ and the group $\mathbb{R}^q$. Let $H = CO(q)$ and $p\colon G \to G/H = \mathbb{E}^q$ be the canonical principal $H$-bundle. Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$, and $\omega$ be the Maurer-Cartan $\mathfrak{g}$-valued 1-form on $G$. Then $\xi = (G(\mathbb{E}^q, H), \omega)$ is an effective rigid geometry. Foliations with this transverse geometry $(\mathbb{E}^q, \xi)$ are called *transversally similarity foliations* [7].

Denote by $E$ the neutral element of the group $O(q)$. If $G = (\mathbb{R}^+ \cdot E) \ltimes \mathbb{R}^q$, $H = \mathbb{R}^+ \cdot E$, and $\omega$ is the Maurer-Cartan $\mathfrak{g}$-valued 1-form on the Lie group $G$, then foliations with the transverse effective rigid geometry $(\mathbb{E}^q, \xi)$, where $\xi = (G(\mathbb{E}^q, \mathbb{R}^+ \cdot E), \omega)$, are called *transversally homothetic foliations* [7].

**Example 1.** Let $B$ be a smooth $p$-dimensional manifold whose fundamental group $\pi_1(B, b)$ contains an element $\alpha$ of infinite order. For an arbitrary natural number $q \geqslant 1$, denote by $\mathbb{E}^q$ a $q$-dimensional Euclidean space. Define a homomorphism $\rho\colon \Pi := \pi_1(B, b) \to \mathrm{Diff}(\mathbb{E}^q)$ by setting $\rho(\alpha) = \psi$, where $\psi$ is the homothetic transformation of the Euclidean space $\mathbb{E}^q$ with the coefficient $\lambda \neq 1$, i. e. $\psi(x) = \lambda x$, $\forall x \in \mathbb{E}^q$, and $\rho(\beta) = \mathrm{id}_{\mathbb{E}^q}$ for any element $\beta \in \pi_1(B, b)$ such that $\beta \neq \alpha^k$ with some integer $k$. Then $(M, F) = \mathrm{Sus}(\mathbb{E}^q, B, \rho)$ is a proper transversally similar foliation with a unique closed leaf diffeomorphic to $B$.

According to Corollary 2, the full basic automorphism group $\mathcal{A}_B(M, F)$ of this foliation $(M, F)$ admits a Lie group structure. Let us compute the group $\mathcal{A}_B(M, F)$ and show that this fact is indeed true.

The group $\Pi_0 := \ker \rho$ acts on $\hat{B} \times \mathbb{E}^q$ properly discontinuously, hence the quotient manifold $\hat{B} \times_{\Pi_0} \mathbb{E}^q = B_0 \times \mathbb{E}^q$, where $B_0 := \hat{B}/\Pi_0$, is defined. The quotient group $\Psi_0 := \Pi/\Pi_0 \cong \mathbb{Z}$ acts from the right on the product of manifolds $B_0 \times \mathbb{E}^q$ such that $M = B_0 \times_{\Psi_0} \mathbb{E}^q$ and the quotient map $\kappa\colon M_0 := B_0 \times \mathbb{E}^q \to M$ is a regular covering map with the deck transformation group $\Psi_0$. The foliation $(M_0, F_0)$, where $F_0 := \kappa^* F$, is formed by the fibres of the projection $\mathrm{pr}_2\colon M_0 = B_0 \times \mathbb{E}^q \to \mathbb{E}^q$ onto the second factor.

The group $\mathcal{A}(\xi)$ is equal to the group of left translations of the Lie group $G = CO(q) \leftthreetimes \mathbb{R}^q$, hence we can identify $\mathcal{A}(\mathbb{E}^q, \xi) \cong \mathcal{A}(\xi)$ with $G$. For any $h \in G$ the transformation $h' = (\mathrm{id}_{B_0}, h)$ of $B_0 \times \mathbb{E}^q$ belongs to $\mathcal{A}(M_0, F_0)$. Therefore, the map $\alpha\colon \mathcal{A}(M_0, F_0) \to G\colon h' \mapsto h$, where $h \circ \mathrm{pr}_2 = \mathrm{pr}_2 \circ h'$, is a group epimorphism with $\ker \alpha = \mathcal{A}_L(M_0, F_0)$. Let us emphasize that $\hat{f} \in \mathcal{A}(M_0, F_0)$ lies over an automorphism $f \in \mathcal{A}(M, F)$ if and only if it satisfies the relation $\hat{f} \circ \Psi_0 = \Psi_0 \circ \hat{f}$. Remark that $\alpha(\Psi_0) = \Psi \subset \mathcal{A}(\mathbb{E}^q, \xi) = G$ is the global holonomy group of the foliation $(M, F)$. Let $N(\Psi)$ be the normalizer of $\Psi$ in the Lie group $G$. It is not difficult to check that the map

$$\beta\colon \mathcal{A}_B(M, F) \to N(\Psi)/\Psi\colon f \cdot \mathcal{A}_L(M, F) \mapsto \alpha(\hat{f}) \cdot \Psi,$$

where $\hat{f} \in \mathcal{A}(M_0, F_0)$ lies over $f$ with respect to the map $\kappa$, is a group isomorphism, hence $\mathcal{A}_B(M, F) \cong N(\Psi)/\Psi$.

In our case $\Psi = \langle \psi \rangle$ and $N(\Psi) = \mathbb{R}^+ \cdot O(q)$, therefore $\mathcal{A}_B(M, F) \cong U(1) \times O(q)$, where $U(1) \cong (\mathbb{R}^+ \cdot E)/\Psi$ is the compact 1-dimensional abelian group.

If $q = 1$, then $O(q) = \mathbb{Z}_2$ and $\mathcal{A}_B(M, F) \cong U(1) \times \mathbb{Z}_2$.

**Example 2.** Consider the foliation $(M, F)$ constructed in Example 1 as a transversally homothetic foliation, i. e., with a different transverse rigid geometry. In this case the Lie group $\mathcal{A}_B(M, F)$ is isomorphic to the quotient Lie group $N(\Psi)/\Psi$, where $N(\Psi)$ is the normalizer of $\Psi$ in the Lie group $(\mathbb{R}^+ \cdot E) \leftthreetimes \mathbb{R}^q$. Since $N(\Psi) = \mathbb{R}^+ \cdot E$, so $\mathcal{A}_B(M, F) \cong U(1)$.

**Remark 8.** In both examples 1 and 2 the foliation $(M, F)$ has a closed leaf and, in Theorem 3, the equality is achieved in the estimate (ii) of the dimension of $\mathcal{A}_B(M, F)$.

**Example 3.** Let $\psi$ be the rotation of the plane $\mathbb{E}^2$ about the point $0 \in \mathbb{E}^2$ through the angle $\delta = 2\pi r$. Consider an Euclidean metric $g$ on $\mathbb{E}^2$. Denote by $Iso(\mathbb{E}^2, g)$ the full isometry group of $(\mathbb{E}^2, g)$. Let $\rho\colon \pi_1(S^1, b) \cong \mathbb{Z} \to Iso(\mathbb{E}^2, g)$ be defined by the equality $\rho(1) := \psi$, $1 \in \mathbb{Z}$. Then we have a suspended Riemannian foliation $(M, F) := \mathrm{Sus}(\mathbb{E}^2, S^1, \rho)$. This foliation has a unique closed (compact) leaf.

There exists a group isomorphism between $\mathcal{A}_B(M, F)$ and the quotient group $N(\Psi)/\Psi$, where $\Psi = \langle \psi \rangle$ and $N(\Psi)$ is the normalizer of $\Psi$ in the Lie group $Iso(\mathbb{E}^2, g)$ identified with $O(2) \leftthreetimes \mathbb{R}^2$. Since $N(\Psi) = O(2)$, so $\mathcal{A}_B(M, F) = O(2)/\Psi$. Hence $\mathcal{A}_B(M, F)$ admits a Lie group structure if and only if $\Psi$ is a closed subgroup of $O(2)$ or, equivalent, when $\delta = 2\pi r$ for some rational number $r$.

If $\delta = 2\pi r$, where $r$ is a non-zero rational number, then $\mathcal{A}_B(M, F) \cong O(2)$.

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## Полные слоения с трансверсальными жесткими геометриями и их базовые автоморфизмы

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Введено понятие жестких геометрий. Жесткие геометрии включают картановы геометрии, а также жесткие геометрические структуры в смысле Громова. Исследуются слоения $(M, F)$ с трансверсальными жесткими геометриями. Найден инвариант $\mathfrak{g}_0(M, F)$ слоения $(M, F)$, представляющий собой алгебру Ли. Доказано, что при $\mathfrak{g}_0(M, F) = 0$ группа базовых автоморфизмов слоения $(M, F)$ допускает структуру группы Ли, причем эта структура единственна. Получены оценки размерностей этих групп в зависимости от трансверсальных геометрий. Построены примеры вычисления групп базовых автоморфизмов слоений.