UDC 517.917

Necessary Optimality Conditions for Stationary Nonlinear Hydrodynamic Disrupted Problems in a Bounded Domain

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In the paper we establish the optimal necessary conditions for guaranteeing uniquely the resolution of boundary hydrodynamic problems in a bounded domain so that they could accurately describe the studied hydrodynamic phenomenon.

Key words and phrases: necessary conditions of optimality, Command, optimal command, uniqueness, disruption, linearization, nonlinear.

1. Introduction

The focus of our research on such problems lies in the fact that for nonlinear systems of the type of Navier–Stokes in a three-dimensional space, we can not find a class of spaces where we could uniquely solve the problem at the border. This class is found by the linearization of systems of Navier–Stokes. However linearized systems often do not describe accurately the movement of liquid (or fluid). An intermediate case of investigation was proposed in [1], i.e., to the linearized system, nonlinear terms are added, which may allow us to more accurately describe the movement of liquid (or fluid) and at the same time allow the resolution in a unique way of the nonlinear problem relative to the boundaries, obtained by disrupting our initial system.

Using the Hadamard theorem for infinitely small Lipschitz constant, satisfying the conditions of these disturbances, the obtained disrupted problem at the borders has a unique solution $v = A(f, \alpha)$ where α is the value of the velocity v at the border (with $\alpha = 0$ for the studied problem), f is the second member of the perturbed obtained system, and A satisfies the Lipschitz conditions with respect to f, in the corresponding functional spaces.

For some smooth conditions on the Nemytsky–Hammerchtéin operator and using the theorem of Hadamard about strong derivation of inverse functions, the operator $A(f)$ is strongly differentiable in the sense of the corresponding f. This derivation is weaker than the Fréchet derivation. But it is quite sufficient to establish the necessary conditions of optimality of problems relative to those equations.

2. Statement of the Problem of Optimal Control

The physical processes that find their applications in technique, are generally controlled. It means that they can be achieved in many ways at the mercy of man. Therefore, we must find the best control according to particular criteria, in other words, the optimal control of the process.

The flow of an incompressible viscous fluid in a not empty and bounded domain Ω, is characterized by its velocity $v = v(x)$ and pressure $p = p(x)$.

Consider the associated system after disruption

$$\nu \vartriangle v(x) + M(x, v(x)) + \int\limits_{\Omega} k(x, y) g(y, v(y))\mathrm{d}y = \nabla P(x) + f(x), \tag{1}$$

$$\operatorname{div} v(x) = 0, \tag{2}$$

$$v\mid_{\partial\Omega}(x) = 0, \quad x \in \Omega \subset \mathbb{R}^3, \tag{3}$$

and functional with the following form

$$J_k(f) = \int\limits_{\Omega} P_k(x, v(x)), f(x))\mathrm{d}x, \quad k = \overline{0, 1, \ldots, s_1 + s_2}, \tag{4}$$

where P_k are Caratheodoric functions, that is they are measurable with regard to the triplet (x, v, f) and continue with regard to the couple (v, f) almost everywhere for all the elements x of Ω; ν is the kinematic coefficient of viscosity (or of tenacity) and it is considered to be constant. $\partial\Omega = S$ is the border of the domain Ω. In addition to that we have

$$|P_k(x, v, f)| \leqslant Q_k(x) + C_k\left(|v|^2 + |f|^2\right),$$

$$|\nabla_{(v,f)} P_k(x, v, f)| \leqslant D_k(x) + \widehat{C}_k\left(|v|^2 + |f|^2\right)$$

with P_k, $k = \overline{0, 1, \ldots, s_1 + s_2}$, derivable with respect to the pair (v, f), $Q_k(x) \in L_1(\Omega)$, $D_k(x) \in L_2(\Omega)$, C_k and $\widehat{C}_k$ are constants. More, P_k, P_{kv} and P_{kf} verify the Lipschiz condition from the pair (v, f); s_1 and s_2 are non negative integers.

According to [1] the following functions:

$$\begin{aligned} M: \quad \Omega \times \mathbb{R}^3 \times \mathbb{R}^9 &\to \mathbb{R}^3 \\ (x, \zeta, \eta) &\mapsto M(x, \zeta, \eta), \end{aligned}$$

$$\begin{aligned} g: \quad \Omega \times \mathbb{R}^3 \times \mathbb{R}^9 &\to \mathbb{R}^3 \\ (x, \zeta, \eta) &\mapsto g(x, \zeta, \eta), \end{aligned}$$

$$\begin{aligned} k: \quad \Omega \times \Omega &\to \mathbb{R}^9 \\ (x, y) &\mapsto k(x, y). \end{aligned}$$

are measurable and satisfying the following conditions:

$$\|M(x, \zeta, \eta)\| \leqslant c_0(\|\zeta\| + \|\eta\|) + d_1(x) \tag{5}$$

$$|g(x, \zeta, \eta)| \leqslant c_1(\|\zeta\| + \|\eta\|) + d_2(x) \tag{6}$$

where $d_i(x) \in L_2(\Omega)$, $i = 1, 2$.
Moreover M and g are continuously differentiable with respect to the correspondent (ζ, η) almost at each fixed point $x \in \Omega$, and

$$|M'_\zeta| + |M'_\eta| \leqslant c_2, \qquad \forall\zeta \in \mathbb{R}^3,\ \forall\eta \in \mathbb{R}^9 \tag{7}$$

$$|g'_\zeta| + |g'_\eta| \leqslant c_3, \qquad \forall\zeta \in \mathbb{R}^3,\ \forall\eta \in \mathbb{R}^9 \tag{8}$$

at almost every $x \in \Omega$, where c_i is a constant for $i = 0, 1, 2, 3$.

The function K defines a continuous integral operator $L_2(\Omega) \to L_2(\Omega)$, with the following form:

$$(K\varphi)(x) = \int\limits_{\Omega} k(x, y)\varphi(y)dy. \tag{9}$$

In the same way, the following operators have been defined in [1]:

$$\begin{array}{rrcl} N: & W_2^1(\Omega) & \to & L_2(\Omega) \\ & \vartheta & \mapsto & [N(\vartheta)](x) = N(\vartheta), \end{array} \tag{10}$$

and

$$\begin{array}{rrcl} G: & W_2^1(\Omega) & \to & L_2(\Omega) \\ & \vartheta & \mapsto & [G(\vartheta)](x) = G(\vartheta), \end{array} \tag{11}$$

by the formulae:

$$[N(v)](x) = M(x, v(x), \nabla v(x)) \tag{12}$$

$$[G(v)](x) = g(x, v(x), \nabla v(x)) \tag{13}$$

and the operators:

$$\begin{array}{rrcl} N'(\vartheta): & W_2^1(\Omega) & \to & L_2(\Omega) \\ & h(x) & \mapsto & [N'(\vartheta)](x) = N'(\vartheta)h, \end{array}$$

and

$$\begin{array}{rrcl} G': & W_2^1(\Omega) & \to & L_2(\Omega) \\ & h(x) & \mapsto & [G'(\vartheta)h](x) = G'(\vartheta)h, \end{array}$$

depending on the parameter $\vartheta \in W_2^1(\Omega)$ by the formulas:

$$[N'(\vartheta)]h(x) = M'_\zeta(x, \vartheta(x), \nabla\vartheta(x))h(x) + \sum_{i=1}^{3} M'_{\eta_i} \frac{\partial h(x)}{\partial x_i} \tag{14}$$

and

$$[G'(\vartheta)]h(x) = g'_\zeta h(x) + \sum_{i=1}^{3} g'_{\eta_i} \frac{\partial h(x)}{\partial x_i}, \tag{15}$$

where M'_{η_i}, g'_ζ, g'_{η_i} have for argument $(x, \vartheta(x), \nabla\vartheta(x))$ (the notations $N'(\vartheta)$ and $G'(\vartheta)$ are in [2]).
Let

$$U = \left\{ \vartheta \in W_2^1(\Omega):\ \exists! f \in \left(\overset{\circ}{J}{}_2^1\right)',\ \exists! \alpha \in B,\ M_1(\vartheta) = (f, \vartheta) \right\},$$

$$\|\vartheta\|_U = \|f\|_{\left(\overset{\circ}{J}{}_2^1\right)'} + \|\alpha\|_B.$$

Assuming that $[M_1(\vartheta)](x) = v\triangle\vartheta - \nabla P$,

$$[M_1(\vartheta)](x) \equiv M(x, \vartheta(x), \nabla\vartheta(x)) + \int_\Omega k(x, y) g(y, \vartheta(y), \nabla\vartheta(y)) dy.$$

It has been prooved in [1] that the operator

$$\begin{array}{rrcl} M_1: & (U, \|\,\|_U) & \to & \left(\overset{\circ}{J}{}_2^1\right)' \times B \\ & \vartheta & \mapsto & M_1(\vartheta) = (f, \vartheta) \end{array}$$

is an isomorphism. Where $\overset{\circ}{J}{}_2^1$ is the Hilbert space of vector functions, obtained by completing $\overset{\bullet}{J}(\Omega)$ according to the standard corresponding scalar product:

$$(u, \vartheta) = \int_\Omega (u\vartheta + u_x \vartheta_x) dx,$$

$\overset{\bullet}{J}(\Omega)$ is the set of infinitely differentiable vector functions and B has been defined by

$$\alpha \in B \equiv \{\alpha \in L_2(S) \,/\, \exists\, a \in W_2^1(\Omega),\, diva = 0,\ a|_S = \alpha\}$$

with $\|\alpha\|_B = Inf\{\|a\|_{W_2^1(\Omega)} \,:\, a \in W_2^1(\Omega),\, diva = 0,\ a|_S = \alpha\}$.

In [1], it was shown that by choosing $\omega = max(c_2, c_3)$ with the operators M and g, satisfying the conditions $(5) - (8)$ and if there is a number $\omega_0 > 0$ such that for any ω, we have $0 < \omega < \omega_0$ then the problem

$$\begin{cases} M(\vartheta) \equiv v\triangle\vartheta(x) + M(x, \vartheta(x), \nabla\vartheta(x)) + \int_\Omega k(x, y) g(y, \vartheta(y)\nabla\vartheta(y)) dy = \nabla p(x) + f(x), \\ \operatorname{div} \vartheta(x) = 0, \\ \vartheta|_S(x) = \alpha(x) \end{cases}$$

has a unique solution $\vartheta = A(f, \alpha)$ for all $f \in \overset{\circ}{J}{}_2^1$ and $\alpha \in B$ and more:

1) $A : \overset{\circ}{J}{}_2^1 \times B \to W_2^1(\Omega)$ is $s-$ continuous and $s-$ differentiable on $\overset{\circ}{J}{}_2^1 \times B$;
2) the operator A is strongly differentiable on $\overset{\circ}{J}{}_2^1 \times B$ as a mapping on the space $(W_2^1(\Omega), \sigma)$, where σ is a weak topology in $W_2^1(\Omega)$.

We also obtained in [1], that when the solution $\vartheta = A(f, \alpha)$ is $s-$ continuous and $s-$ differentiable as a mapping from $\overset{\circ}{J}{}_2^1 \times B$ to U, then A is $s-$continuous and $s-$ differentiable as a mapping from $L_2(\Omega) \times B$ in $W_2^1(\Omega)$. This is deduced from the continuity of A from U in $W_2^1(\Omega)$ and $L_2(\Omega)$ in $\big(\overset{\circ}{J}{}_2^1\big)'$, this is $\big(H(\Omega) \subset L_2(\Omega) \subset \big(\overset{\circ}{J}{}_2^1\big)'\big)$.

To obtain the result above stated, we had to show that the operators N and G are $s-$continuous and $s-$differentiable on $W_2^1(\Omega)$ and $G' = K * G$. Similary, it was shown that, since K is a continuous linear map and that the operators G and N satisfy the Lipschitz condition, then $K \circ G$ also satisfies the Lipschitz condition.

Therefore, what conditions the command applied to the system (for disruption) should be submitted to, so that the associated solution to the command coud be unique?

The problem is to choose a command $\overline{f}$ from U_0, where U_0 is a convex set in $L_2(\Omega)$, such that for the solution $\overline{v}(x)$ of the system (1)–(3), depending of that command $\overline{f}$, constraints persist, which are given in the form of inequalities

$$J_k(f) \leqslant 0, \quad k = \overline{1, s_1}, \tag{16}$$

given in the form of equalities

$$J_k(f) = 0, \quad k = \overline{s_1 + 1, s_1 + s_2}, \tag{17}$$

and that in addition to this, the functional $J_0(f)$ takes the smallest possible value

$$J_0(\overline{f}) = \inf_{U_0} J_0(f). \tag{18}$$

Such control is called optimal.

Definition 1. The function $v = A(f)$ is called generalized solution of system (1)–(3) in $W_2^1(\Omega)$, if it satisfies the integral identity $I = 0$, $\forall\varphi \equiv \varphi(x) \in \overset{\circ}{J}{}_2^1(\Omega)$, where

$$I = -\nu \int_\Omega v_x \varphi_x \mathrm{d}x \int_\Omega \left[M(x, v(x)) + \int_\Omega k(x, y) g(y, v(y)) \mathrm{d}y - f(x) \right] \varphi(x) \mathrm{d}x = 0. \quad (19)$$

Suppose that φ is sufficiently smooth. Then

$$-\nu \int_\Omega \sum_{j=1}^{3} \left(\frac{\partial v_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} \right) \mathrm{d}x = \nu \int_\Omega \sum_{j=1}^{3} \frac{v_i \partial^2 \varphi_i}{\partial x_j^2} \mathrm{d}x - \nu \int_{\partial\Omega} v_i \sum_{j=1}^{3} \frac{\partial \varphi_i}{\partial x_j} n_j \mathrm{d}x =$$

$$= \nu \int_\Omega \sum_{j=1}^{3} v_i \frac{\partial^2 \varphi_i}{\partial x_j^2} \mathrm{d}x - \nu \int_{\partial\Omega} v_i \sum_{j=1}^{3} \frac{\partial \varphi_i}{\partial n_j} \mathrm{d}s = \langle \nu \Big(\vartriangle \varphi_i - \frac{\partial \varphi_i}{\partial n} \delta_s \Big), v_i \rangle,$$

$\sum_{j=1}^{3} \langle \nu \vartriangle \varphi_i, v_i \rangle = \langle \nu \vartriangle \varphi, v \rangle$ by condition (3).

Thus we obtain

$$I = \langle \nu \vartriangle \varphi + (N'(v) + KG'(v))\varphi, v \rangle - \langle \varphi, f \rangle = 0. \quad (20)$$

Theorem 1. *Suppose that under the conditions of* (4) *(see [1])* $\overline{v}$ *is a solution of system (1)–(3), corresponding to the control* $\overline{f}(x) \in U_0$, *where*

$$f^\varepsilon(x) = \overline{f}(x) + \varepsilon \Big(f(x) - \overline{f}(x) \Big), \quad (0 \leqslant \varepsilon \leqslant 1),$$

and $v^\varepsilon(x)$ *is a solution relative to the control* $f^\varepsilon(x) \in U_0$. *Then*

$$\| v^\varepsilon(x) - \overline{v}(x) \|_{W_2^1(\Omega)} \leqslant C \left\| f^\varepsilon(x) - \overline{f}(x) \right\|_{L_2(\Omega)}. \quad (21)$$

Proof. Remark that $\delta v(x) = v^\varepsilon(x) - \overline{v}(x)$ satisfy the integral identity:

$$\nu \int_\Omega \varphi_x \delta v_x - \int_\Omega \varphi \left[M^\varepsilon - \overline{M} + \int_\Omega k(g^\varepsilon - \overline{g}) \mathrm{d}y \right] \mathrm{d}x + \int_\Omega \varphi \delta f \mathrm{d}x = 0, \quad \forall\varphi \in \overset{\circ}{J}{}_2^1(\Omega). \quad (22)$$

So, using condition (16) (see [1]) and the restriction (21) (see [1], for $\alpha = 0$), we obtain inequality (21). □

3. Derivation of the Functional

Consider the functional

$$J(f) = \int_\Omega P(x, v(x)), f(x)) \mathrm{d}x, \quad (23)$$

Let's prove that J is differentiable in $L_2(\Omega)$.

3.1. Formula for the Gradient of the Function

Consider the problem (1)–(3) with a disrupted control $f^\varepsilon \in L_2(\Omega)$, which is linked to the solution $v^\varepsilon(x)$ of the problem and the value of the functional $J(f^\varepsilon)$.

Denote the variations by: $\delta v = v^\varepsilon - \overline{v}, \delta f = f^\varepsilon - \overline{f}$. We have

$$\triangle\ J = \triangle\ J(\overline{f}) = J(f^\varepsilon) - J(\overline{f}) = \int\limits_\Omega \Big[P(x, v^\varepsilon, f^\varepsilon) - P(x, \overline{v}, \overline{f}) \Big] \mathrm{d}x,$$

$$\begin{aligned} P(x, v^\varepsilon, f^\varepsilon) - P(x, \overline{v}, \overline{f}) &= P(x, v^\varepsilon, f^\varepsilon) - P(x, \overline{v}, f^\varepsilon) + P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) = \\ &= \int\limits_0^1 P_v(x, \overline{v} + \vartheta\delta v, f^\varepsilon)\delta v \mathrm{d}\vartheta + P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) = \\ &= P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) + P_v(x, \overline{v}, \overline{f})\delta v + \int\limits_0^1 \Big[P_v(x, \widehat{v}, f^\varepsilon) - P_v(x, \overline{v}, \overline{f}) \Big] \delta v \mathrm{d}\vartheta = \\ &= P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) + P_v(x, \overline{v}, \overline{f})\delta v + \int\limits_0^1 \Big[P_v(x, \widehat{v}, f^\varepsilon) - P_v(x, \overline{v}, f^\varepsilon) \Big] \delta v \mathrm{d}\vartheta + \\ &\quad + \Big[P_v(x, \overline{v}, f^\varepsilon) - P_v(x, \overline{v}, \overline{f}) \Big] \delta v. \end{aligned}$$

Then

$$\triangle\ J = \int\limits_\Omega \Big[P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) \Big] \mathrm{d}x + \int\limits_\Omega P_v(x, \overline{v}, \overline{f})\delta v dx + \int\limits_\Omega (r_1 + r_2) \mathrm{d}x,$$

where

$$\begin{gathered} r_1 = \int\limits_0^1 \Big[P_v(x, \widehat{v}, f^\varepsilon) - P_v(x, \overline{v}, f^\varepsilon) \Big] \delta v \mathrm{d}\vartheta, \quad \widehat{v} = \overline{v} + \vartheta\delta v, \\ r_2 = \Big[P_v(x, \overline{v}, f^\varepsilon) - P_v(x, \overline{v}, \overline{f}) \Big] \delta v. \end{gathered} \tag{24}$$

As the functions v and $\overline{v}$ satisfy respectively the integral identities (in this case we use relation (20)), so their difference $\delta v = v^\varepsilon - \overline{v}$ also will satisfy the identity (20). Taking into account this fact, we have:

$$\begin{aligned} \triangle\ J = \int\limits_\Omega \Big[P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) \Big] \mathrm{d}x + \int\limits_\Omega P_v(x, \overline{v}, \overline{f})\delta v \mathrm{d}x + \\ + \langle \nu \triangle \varphi + \varphi \Big(M^\varepsilon - \overline{M} + \int\limits_\Omega k(g^\varepsilon - \overline{g}) \mathrm{d}y \Big), \delta v \rangle - \langle \varphi, \delta f \rangle + \int\limits_\Omega (r_1 + r_2) \mathrm{d}x. \end{aligned} \tag{25}$$

In the last expression, taking into account the conditions on M and g, also taking into account formulas (25) and (26) (see [1]), we rewrite the following:

$$\int\limits_\Omega \varphi \Big(M^\varepsilon - \overline{M} \Big) \mathrm{d}x = \int\limits_\Omega \varphi \Big(M(x, v^\varepsilon(x)) - \overline{M}(x, \overline{v}(x)) \Big) \mathrm{d}x =$$

$$= \int_\Omega \varphi \Big[\int_0^1 M_v(x, \overline{v} + \vartheta\delta v)\delta v d\vartheta \Big] \mathrm{d}x = \int_\Omega \varphi M_v(x, \overline{v}(x))\delta v(x)\mathrm{d}x + \int_\Omega \varphi(x) r_3 \mathrm{d}x =$$
$$= \int_\Omega \varphi(x)[N'(\overline{v})](x)\delta v(x)\mathrm{d}x + \int_\Omega \varphi(x) r_3 \mathrm{d}x, \quad (26)$$

where

$$r_3 = \int_0^1 \Big[M_v(x, \widehat{v}(x)) - M_v(x, \overline{v}(x)) \Big] \delta v(x)\mathrm{d}\vartheta, \quad \widehat{v} = \overline{v} + \vartheta\delta v, \quad (27)$$

and

$$\int_\Omega \varphi \Big[\int_\Omega k(g^\varepsilon - \overline{g})\mathrm{d}y \Big] \mathrm{d}x = \int_\Omega \varphi(x) \Big[\int_\Omega k(x,y) \Big(g(y, v^\varepsilon(y)) - g(y, \overline{v}(y)) \Big) \mathrm{d}y \Big] \mathrm{d}x =$$
$$= \int_\Omega \varphi(x) \Big[\int_\Omega k(x, y \Big[\int_0^1 g_v(y, \overline{v} + \vartheta\delta v)\delta v d\vartheta \Big] \mathrm{d}y \Big] \mathrm{d}x =$$
$$= \int_\Omega \varphi(x) \Big[\int_\Omega k(x,y) g_v(y, \overline{v}(y))\delta v(y)\mathrm{d}y \Big] \mathrm{d}x + \int_\Omega \varphi(x) \Big[\int_\Omega k(x,y) r_4 \mathrm{d}y \Big] \mathrm{d}x.$$

$$\int_\Omega \varphi(x) \Big[\int_\Omega k(x,y) g_v(y, \overline{v}(y))\delta v(y)\mathrm{d}y \Big] \mathrm{d}x = \int_\Omega \Big[\int_\Omega \varphi(x) k(x,y) g_v(y, \overline{v}(y))\delta v(y)\mathrm{d}y \Big] \mathrm{d}x =$$
$$= \int_\Omega \Big[\int_\Omega \varphi(x) k(x,y)\mathrm{d}x \Big] g_v(y, \overline{v}(y))\delta v(y)\mathrm{d}y = \int_\Omega \Big[\int_\Omega \varphi(y) k(y,x) g_v(x, \overline{v}(x))\mathrm{d}y \Big] \delta v(x)\mathrm{d}x =$$
$$= \int_\Omega (G'^*(\overline{v}) K^* \varphi^T)(x)\delta v(x)\mathrm{d}x,$$

where $(K^*\varphi^T)(x) = \int_\Omega k(y,x)\varphi^T(x)\mathrm{d}x$. In what follows, as a matter of convenience, we will simply write φ but not φ^T. Thus,

$$\int_\Omega \varphi(x) \Big[\int_\Omega k(x,y)(g^\varepsilon - \overline{g})\mathrm{d}y \Big] \mathrm{d}x = \int_\Omega (G'^*(v) K^* \varphi)(x)\delta v\mathrm{d}x + \int_\Omega \Big[\int_\Omega k(x,y) r_4 \mathrm{d}y \Big] \varphi(x)\mathrm{d}x.$$

where

$$r_4 = \int_0^1 \Big[g_v(y, \widehat{v}) - g_v(y, \overline{v}) \Big] \delta v \mathrm{d}\vartheta. \quad (28)$$

$$P(x, \overline{v}, f^\varepsilon) - P(x, \overline{v}, \overline{f}) = P(x, \overline{v}, \overline{f})\delta f + r_5, \quad (29)$$

where

$$r_5 = \int_0^1 \Big[P_f(x, \overline{v}, \widehat{f}) - P_f(x, \overline{v}, \overline{f}) \Big] \delta f \mathrm{d}\vartheta, \quad \text{with } \widehat{f} = \overline{f} + \vartheta\delta f. \quad (30)$$

Taking into account formulas (26), (27), (29) and (25) we obtain

$$\begin{aligned}\triangle J = \int\limits_{\Omega} P_f(x,\overline{v},\overline{f})\delta f \mathrm{d}x + \int\limits_{\Omega} P_v(x,\overline{v},\overline{f})\delta v \mathrm{d}x + \\ + \langle \nu \triangle \varphi + N'(x,\overline{v})\varphi + G'^{*}(v)K^{*}\varphi, \delta v\rangle - \langle \varphi, \delta f\rangle + \\ + \int\limits_{\Omega}(r_1+r_2+r_5)\mathrm{d}x + \int\limits_{\Omega} r_3\varphi \mathrm{d}x + \int\limits_{\Omega}\Bigg(\int\limits_{\Omega} kr_4 dy\Bigg)\varphi \mathrm{d}x. \quad (31)\end{aligned}$$

Remark 1. The transformations in the formula (31) are true only for the functions φ, sufficiently "smooth" and are issued only by the evidence of obtaining the conjugate form of the problem. For the following transformations, consider the following conjugate problem:

$$\nu \triangle \varphi + N'(v(x))\varphi + G'(v(y))K^{*}\varphi = -P_v. \quad (32)$$

From the existence (see Theorem 2 (see [3, p. 54] and Theorem of Hadamard) of the solution of the conjugate problem (32), we finally have the expression for $\triangle J$

$$\begin{aligned}\triangle J = \int\limits_{\Omega} P_f(x,\overline{v},\overline{f})\delta f \mathrm{d}x + \int\limits_{\Omega} \varphi\delta f \mathrm{d}x + \int\limits_{\Omega}(r_1+r_2+r_5)\mathrm{d}x + \\ + \int\limits_{\Omega} r_3\varphi \mathrm{d}x + \int\limits_{\Omega}\Bigg(\int\limits_{\Omega} kr_4 \mathrm{d}y\Bigg)\varphi \mathrm{d}x = \int\limits_{\Omega}\Big[P_f(x,\overline{v},\overline{f}) + \varphi\Big]\delta f \mathrm{d}x + R,\end{aligned}$$

where

$$R = \int\limits_{\Omega}(r_1+r_2+r_5)\mathrm{d}x + \int\limits_{\Omega}\Bigg(r_3 + \int\limits_{\Omega} kr_4 \mathrm{d}y\Bigg)\varphi \mathrm{d}x.$$

In assessing the balance of development, one can show that $R = o\Big(\|\delta f\|_{L_2(\Omega)}\Big)$. Due to the fact that P_v satisfies the Lipschitz condition with respect to the group of arguments (v, f) and using Theorem 1, we have

$$\begin{aligned}\left|\int\limits_{\Omega} r_1 \mathrm{d}x\right| = \left|\int\limits_{\Omega}\Bigg[\int\limits_0^1\Big(P_v(x,\widehat{v},f^{\varepsilon}) - P_v(x,\overline{v},f^{\varepsilon})\Big)\delta v \mathrm{d}\vartheta\Bigg]\mathrm{d}x\right| \leqslant \\ \leqslant \int\limits_{\Omega}\int\limits_0^1 |P_v(x,\widehat{v},f^{\varepsilon}) - P_v(x,\overline{v},f^{\varepsilon})|\,|\delta v|\,\mathrm{d}\vartheta\mathrm{d}x \leqslant \\ \leqslant \int\limits_{\Omega}\int\limits_0^1 L\Big(\|\widehat{v}-\overline{v}\|_{W_2^1(\Omega)} + \|f^{\varepsilon} - f^{\varepsilon}\|_{L_2(\Omega)}\Big)|\delta v|\,\mathrm{d}\vartheta\mathrm{d}x = \\ = \int\limits_{\Omega}\int\limits_0^1 L\vartheta\,\|\delta v\|_{W_2^1(\Omega)}\,|\delta v|\,\mathrm{d}x = \frac{1}{2}L\,\|\delta v\|_{W_2^1(\Omega)}\int\limits_{\Omega}|\delta v|\,\mathrm{d}x \leqslant \\ \leqslant \frac{1}{2}L(mes\Omega)^{\frac{1}{2}}\,\|\delta v\|_{W_2^1(\Omega)}\,\|\delta v\|_{W_2^1(\Omega)} = \frac{1}{2}L(mes\Omega)^{\frac{1}{2}}\,\|\delta v\|^2_{W_2^1(\Omega)} \leqslant \\ \leqslant \frac{1}{2}L(mes\Omega)^{\frac{1}{2}}c\,\|\delta v\|^2_{L_2(\Omega)} = o\big(\,\|\delta v\|_{L_2(\Omega)}\,\big),\end{aligned}$$

$$\left|\int_\Omega r_5 \mathrm{d}x\right| = \left|\int_\Omega \Big[\int_0^1 \Big(P_f(x,\overline{v},\widehat{f}) - P_f(x,\overline{v},\overline{f})\Big)\delta f \mathrm{d}\vartheta\Big] \mathrm{d}x\right| \leqslant$$

$$\leqslant \int_\Omega \int_0^1 \Big|P_f(x,\overline{v},\widehat{f}) - P_f(x,\overline{v},\overline{f})\Big| \left|\delta f\right| \mathrm{d}\vartheta \mathrm{d}x \leqslant$$

$$\leqslant \int_\Omega \int_0^1 L\Big(\left\|\overline{v} - \overline{v}\right\|_{W_2^1(\Omega)} + \left\|\widehat{f} - \overline{f}\right\|_{L_2(\Omega)} \Big) \left|\delta f\right| \mathrm{d}\vartheta \mathrm{d}x =$$

$$= \int_\Omega \int_0^1 L \left\|\widehat{f} - \overline{f}\right\|_{L_2(\Omega)} \left|\delta f\right| \mathrm{d}\vartheta \mathrm{d}x = \int_\Omega \int_0^1 L\vartheta \left\|\delta f\right\|_{L_2(\Omega)} \mathrm{d}\vartheta \mathrm{d}x =$$

$$= \frac{1}{2} L \left\|\delta f\right\|_{L_2(\Omega)} \int_\Omega \left|\delta f\right| \mathrm{d}x \leqslant \frac{1}{2} L(mes\Omega)^{\frac{1}{2}} c \left\|\delta f\right\|^2_{L_2(\Omega)} = o\big(\left\|\delta f\right\|_{L_2(\Omega)} \big),$$

and

$$\left|\int_\Omega r_2 \mathrm{d}x\right| = \left|\int_\Omega \Big(P_v(x,\overline{v},f^\varepsilon) - P_v(x,\overline{v},\overline{f})\Big)\delta v \mathrm{d}x\right| \leqslant$$

$$\leqslant \int_\Omega \left|P_v(x,\overline{v},f^\varepsilon) - P_v(x,\overline{v},\overline{f})\right| \left|\delta v\right| \mathrm{d}x \leqslant$$

$$\leqslant \int_\Omega L\Big(\left\|\overline{v} - \overline{v}\right\|_{W_2^1(\Omega)} + \left\|f^\varepsilon - \overline{f}\right\|_{L_2(\Omega)} \Big) \left|\delta v\right| \mathrm{d}x =$$

$$= \int_\Omega L \left\|f^\varepsilon - \overline{f}\right\|_{L_2(\Omega)} \left|\delta v\right| dx = L \int_\Omega \left\|\delta f\right\|_{L_2(\Omega)} \left|\delta v\right| \mathrm{d}x = L \left\|\delta f\right\|_{L_2(\Omega)} \int_\Omega \left|\delta v\right| \mathrm{d}x \leqslant$$

$$\leqslant \frac{1}{2} L(mes\Omega)^{\frac{1}{2}} \left\|\delta f\right\|_{L_2(\Omega)} \left\|\delta v\right\|_{W_2^1(\Omega)} \leqslant \frac{1}{2} L(mes\Omega)^{\frac{1}{2}} c \left\|\delta f\right\|^2_{L_2(\Omega)}.$$

The other members are evaluated in the same way. For the variation of the functional $\triangle\ J$, we have finally

$$\triangle\ J = \int_\Omega \Big[P_f(x,\overline{v},\overline{f}) + \varphi\Big]\delta f \mathrm{d}x + o\big(\left\|\delta f\right\|_{L_2(\Omega)} \big).$$

Let's introduce the following function

$$H(x, v(x), f(x), \varphi(x)) \stackrel{def}{=} H(x, v, f, \varphi) = P(x, v, f) + \varphi f.$$

In this case, the formula for the variation will take the following form

$$\triangle\ J = \int_\Omega \frac{\partial H}{\partial f}\big(x,\overline{v},\overline{f},\varphi\big)\big(f^\varepsilon - \overline{f}\big)\mathrm{d}x + o\big(\left\|\delta f\right\|_{L_2(\Omega)} \big).$$

So we've just proved the following theorem

Theorem 2. *Suppose that all the conditions of paragraph* 1 *[1] about functions* M *and* g *are satisfied, as well as the requirements of paragraph* 1 *about* P.

Then the functional $J(f)$ is differentiable with respect to f, and its derivatives at the point $\overline{f}$ are expressed by the formulae $J_f(\overline{f}) = \dfrac{\partial H}{\partial f}\left(x, \overline{v}, \overline{f}, \varphi\right).$

4. Necessary Conditions of Optimality

Let $\overline{f} = \overline{f}(x) \in U_0$, with $\overline{f}(x)$ an optimal control. Consider an arbitrary command $f(x)$, with $f = f(x) \in U_0$.

Let's find the variation f^ε of the optimal control $\overline{f}$ in the direction of $(f - \overline{f})$ as follows:

$$\begin{gathered} f^\varepsilon(x) = \overline{f}(x) + \varepsilon\big(f(x) - \overline{f}(x)\big) \\ \delta f = f^\varepsilon - \overline{f} = \varepsilon\big(f - \overline{f}\big), \end{gathered} \tag{33}$$

In the variations, ε is always the same and $f^\varepsilon \in U_0$. This is satisfied for example, when $\varepsilon \in [0, 1]$, because U_0 is a convex set.

4.1. First Variation of the Functionals

Consider a family of functions H_k, $k = \overline{0, s_1 + s_2}$, where

$$H_k(x, v, f, \varphi_k) = P_k(x, v, f) + \varphi_k f. \tag{34}$$

The functions φ_k are solutions of the conjugate problems. So

$$\triangle\ J = \int\limits_\Omega \frac{\partial H_k}{\partial f}\big(x, \overline{v}, \overline{f}, \varphi_k\big)\big(f^\varepsilon - \overline{f}\big)\mathrm{d}x + o\big(\, \|\delta f\|_{L_2(\Omega)}\,\big), \quad k = \overline{0, s_1 + s_2}, \tag{35}$$

the first variation δJ_k of the functional $J_k(f)$ at point $\overline{f}$ is determined as follows:

$$\delta J_k = \delta J_k(\overline{f}) = \lim_{\varepsilon \to 0} \frac{\triangle\ J_k(\overline{f})}{\varepsilon}. \tag{36}$$

As $f - \overline{f} = \varepsilon(f - \overline{f})$ and the norm $\|f - \overline{f}\|_{L_2(\Omega)}$ for any fixed f, are fixed finite quantities, then

$$\begin{aligned} \delta J_k = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int\limits_\Omega \frac{\partial H_k}{\partial f}\big(x, \overline{v}, \overline{f}, \varphi_k\big)\varepsilon(f - \overline{f})\mathrm{d}x + o(\varepsilon) \right\} = \\ = \int\limits_\Omega \frac{\partial H_k}{\partial f}\big(x, \overline{v}, \overline{f}, \varphi_k\big)(f - \overline{f})\mathrm{d}x \end{aligned} \tag{37}$$

where

$$\delta J_k = \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f - \overline{f})\mathrm{d}x, \tag{38}$$

where $\partial \overset{\circ}{H}_k$ is the function H_k with the arguments related to optimal control $\overline{f}$.

4.2. Establishment of the Necessary Conditions of Optimality

Let γ be a set of parameters:

$$\gamma = \big\{\lambda(f - \overline{f}),\ \lambda \geqslant 0,\ f \in U_0\big\} \tag{39}$$

Or simply , $\gamma = \big\{\lambda(f-\overline{f})\big\}$, giving the variation of the optimal control

$$f^\varepsilon = \overline{f} + \varepsilon\lambda(f-\overline{f}).$$

Then γ is the family of variations of the functionals $Z_\gamma = \big(\delta J_0^\gamma,\, \delta J_1^\gamma,\, \dots,\, \delta J_{s_1+s_2}^\gamma\big)$, where

$$\delta J_k = \lambda \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f-\overline{f})\mathrm{d}x, \quad k=\overline{0, s_1+s_2}. \tag{40}$$

All kinds of γ, whose form looks like the family of variation of the functional $\{Z_\gamma\} \overset{not}{=} K_1 \subset E^{s_1+s_2+1}$. Let's show that K_1 is a cone in $E^{s_1+s_2+1}$ with its apex at the zero point.

It is clear that $Z_\gamma = 0 \in E^{s_1+s_2+1}$ corresponds to the family $\gamma = \{0\}$, with $\lambda = 0$. We have implicitly $0 \in K_1$.

Consider the family $\gamma = \big\{\lambda(f-\overline{f})\big\}$. For that family, there is a vector of variation of the functionals $Z_\gamma = \big(\delta J_0^\gamma,\, \delta J_1^\gamma,\, \dots,\, \delta J_{s_1+s_2}^\gamma\big) \in K_1$.

Consider $\lambda Z_\gamma = \big(a\delta J_0^\gamma,\, a\delta J_1^\gamma,\, \dots,\, a\delta J_{s_1+s_2}^\gamma\big)$, where

$$a\delta J_k^\gamma = a\lambda \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f-\overline{f})\mathrm{d}x,\ a>0.$$

Consider the family $a\gamma = \{a\lambda(f-\overline{f})\}$ too.

Such a family is admissible, like the corresponding vector of variation of the functionals $Z_{a\gamma} \in K_1$. Moreover, it is clear that $Z_{a\gamma} = aZ_\gamma$, as $\delta J^{a\gamma} = a\delta J_k^\gamma$, we conclude that, $aZ_\gamma \in K_1$ and K_1 is a cone.

We now show that the cone K_1 is convex. For this it is sufficient to show that $\forall Z_{\gamma_1}, Z_{\gamma_2} \in K_1$ their sum $Z_{\gamma_1} + Z_{\gamma_2} \in K_1$. Consider Z_{γ_1} generated by the family $\gamma_1 = \big\{\lambda_1(f-\overline{f})\big\}$, and Z_{γ_1} generated by the family $\gamma_2 = \big\{\lambda_2(f-\overline{f})\big\}$. Consider the set

$$\gamma_1+\gamma_2 = \big\{\lambda(f-\overline{f})\big\}\,,\ \text{or}\ \lambda = \lambda_1+\lambda_2,$$
$$f = \vartheta_\lambda f^1 + (1-\vartheta_\lambda)f^2,\ 0 \leqslant \vartheta_\lambda = \frac{\lambda_1}{\lambda_1+\lambda_2} \leqslant 1.$$

As U_0 is convex, the set $\gamma_1+\gamma_2$ is admissible. So is the correspondent vector of variation of the functionals $Z_{\gamma_1+\gamma_2} \in K_1$.

In expression (40) for $\delta J_k^{\gamma_1+\gamma_2}$, $k=\overline{0, s_1+s_2}$, consider the expression

$$\lambda \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f-\overline{f})\mathrm{d}x = \Big(\lambda_1+\lambda_2\Big) \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{df}(\vartheta_\lambda f^1 - (1-\vartheta_\lambda)f^2 - \overline{f})\mathrm{d}x =$$
$$= \lambda_1 \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f^1-\overline{f})\mathrm{d}x + \lambda_2 \int\limits_\Omega \frac{\partial \overset{\circ}{H}_k}{\partial f}(f^2-\overline{f})\mathrm{d}x.$$

Thus, $\delta J_k^{\gamma_1+\gamma_2} = \delta J_k^{\gamma_1} + \delta J_k^{+\gamma_2}, \quad k=\overline{0, s_1+s_2}$, then $Z_{\gamma_1}+Z_{\gamma_2} = Z_{\gamma_1+\gamma_2} \in K_1$ and the cone K_1 is convex.

Definition 2. The contsraints at the point $\overline{f}$, part of restrictions $J_k(f) \leqslant 0$, for which $J_k(\overline{f}) = 0$ are called active. Those for which $J_k(\overline{f}) < 0$ are called inactive at that point.

To begin, suppose that all the restrictions (16) are active.
Consider the set

$$K_1^- = \left\{c \in E^{s_1+s_2+1} \,:\, c = (c_0, c_1, \ldots, c_{s_1}, 0, \ldots, 0),\ c_i < 0,\ i = \overline{0, s_1}\right\}$$

a negative angle in $E^{s_1+s_2+1}$. It is clear that K_1^- is a cone in $E^{s_1+s_2+1}$.

Lemma 1. *The cone K_1, built for optimal control and the cone K_1^- are divided in $E^{s_1+s_2+1}$ by the hyperplane Γ, defined by the nontrivial functional l^*:*

$$l^* = (l_0, l_1, \ldots, l_{s_1+s_2}) \in \left(E^{s_1+s_2+1}\right)^* = E^{s_1+s_2+1},\ \sum_{k=0}^{s_1+s_2} |l_k| > 0,\ \text{for } l_k \geqslant 0,\ k = \overline{0, s_1},$$

and the rests l_k, $k = \overline{s_1+1, s_1+s_2}$ may have any sign. The condition of separation of K_1 and K_1^- takes the following form

$$\langle l^*, Z_\gamma \rangle_{E^{s_1+s_2+1}} \geqslant \langle l^*, c \rangle_{E^{s_1+s_2+1}}, \quad \forall Z_\gamma \in K_1,\ \forall c \in K_1^-. \tag{41}$$

This follows from the known theorem (see [4], p.224 or [5], 3.1).

Theorem 3. *Let X be a normed space, U_1 a convex set in X, $u^* \in U_1$ a local minimum point in the problem*

$$J_0(u) \to \inf,$$
$$J_i(u) \leqslant 0, \quad i = \overline{1, s_1},$$
$$J_i(u) = 0, \quad i = \overline{s_1+1, s_1+s_2}, \quad u \in U_1,$$

where J_i, $i = \overline{0, s_1+s_2}$, s-differentiable at the point u^* and J_i, $i = \overline{s_1+1, s_1+s_2}$ continuous in the neighborhood of the point u^*.

Then there are numbers $l_0, l_1, l_2, \ldots, l_{s_1+s_2}$ such that

$$l^* = (l_0, l_1, l_2, \ldots, l_{s_1+s_2}) \neq 0, \quad l_0 \geqslant 0, \quad l_1 \geqslant 0, \ldots, l_{s_1} \geqslant 0,$$
$$\langle \mathcal{L}u(u^*, l^*), u - u^* \rangle \geqslant 0, \quad \forall u \in U_1, \quad l_i J_i(u^*) = 0, \quad i = \overline{1, s_1+s_2}$$

here $\mathcal{L}u(u^*, l^*) = l_0 J_0'(u^*) + l_1 J_1'(u^*) + \ldots + l_{s_1+s_2} J_{s_1+s_2}'(u^*)$ the gradient of the function $\mathcal{L}(u, l^*)$ with variable $u \in U_1$ at the point $u = u^*$.

Then, using inequality (41) in which $c \to 0$, we obtain

$$\langle l^*, Z_\gamma \rangle_{E^{s_1+s_2+1}} \geqslant 0,\ \forall Z_\gamma \in K_1, \tag{42}$$

That is, for any family γ like in (39).

Inequality (3) is well demonstrated, assuming that all restrictions (16) are actives.

Now consider the general case.

Let $I = \{k : 1 \leqslant k \leqslant s_1, J_k(\overline{f}) = 0\}$ be the set of all constraints at the point $\overline{f}$ among all the restrictions like (16). The other constraints in the formula (16) are inactive at the point $\overline{f}$, that is $J_k(\overline{f}) < 0$, $1 \leqslant k \leqslant s_1$, but $k \notin I$. So thanks to the continuity of the functional J_k with respect to the control f for small disrupted controls, non-active constraints are not affected. Therefore, we can not take account of them. In this case we will examine variations of functionals only for the active constraints and their vectorial variations

$$Z_\gamma = \left\{\delta J_0^\gamma, \{\delta J_k^\gamma\}_{k \in I}, \delta J_{s_1+s_2}^\gamma, \ldots, \delta J_{s_1+s_2}^\gamma\right\}.$$

Let's build the cone $K_1 = \{Z_\gamma\} \subset E^{dimI+s_1+1}$. The corresponding cone is

$$K_1^- = \left\{c \in E^{dimI+s_1+1} \,:\, c = (c_0, \{c_k\}_{k\in I}, 0, \ldots, 0),\ c_k < 0\right\},$$

and by taking the above steps till (42), we obtain

$$l_0\delta J_0^\gamma + \sum_{k\in I} l_k\delta J_k^\gamma + \sum_{k=s_1+1}^{s_1+s_2} l_k\delta J_k^\gamma \geqslant 0,\ \forall Z_\gamma \in K_1. \tag{43}$$

Let $l_k = 0$ for all $k : 1 \leqslant k \leqslant s_1,\ k \notin I$. Then (43) takes the form of (42).

Thus, for all inactive contraints $J_k(\overline{f}) < 0$ corresponding to $l_k = 0$, and for the active contraints $J_k(\overline{f}) = 0$,

$$l_k J_k(\overline{f}) = 0,\ k = \overline{1, s_1 + s_2} \tag{44}$$

so the condition (42) is verified too. From this we deduce the necessary conditions of optimality of the control.

Let's introduce the following function: $\Psi = \Psi(x) = \sum_{k=0}^{s_1+s_2} l_k\varphi_k(x)$, where $\varphi_k(x)$ is the solution of the conjugate problem (32). Multiplying (32) by l_k and making a summation for all $k = \overline{0, s_1 + s_2}$, then the function $\Psi(x)$ will be a solution of the problem:

$$\nu \vartriangle \Psi(x) - N'(x, \overline{v}(x))\Psi(x) + G'(y, \overline{v}(y))K^*\Psi = -\sum_{k=0}^{s_1+s_2} l_k P_{kv}. \tag{45}$$

Let's introduce the functions

$$\mathcal{H}(x, v, f, \Psi) = \sum_{k=0}^{s_1+s_2} l_k H_k(x, v(x), f(x), \Psi(x)). \tag{46}$$

Using the formula (32) for H_k and taking into account the introduced function $\psi(x)$, we can write formula (44) in expanded form:

$$\mathcal{H}(x, v, f, \Psi) \equiv \mathcal{H}(x, v(x), f(x), \Psi(x)) = \sum_{k=0}^{s_1+s_2} l_k P_k(x, v(x), f(x)) + \Psi(x)f(x). \tag{47}$$

Let's consider the family $\gamma = \{f - \overline{f}\}$ for all $f \in L_2(\Omega)$. To this family we associate the variation vector of the functional

$$Z_\gamma = \left(\delta J_0^\gamma,\, \delta J_1^\gamma,\, \ldots,\, \delta J_{s_1+s_2}^\gamma\right) \in K_1,$$

and inequality (43) persists: $\displaystyle\sum_{k=0}^{s_1+s_2} l_k\delta J_k^\gamma \geqslant 0.$

Replacing δJ_k^γ by their respective expressions from (40) using formulas (47), we obtain

$$\int_\Omega \frac{\partial\mathcal{H}}{\partial f}(x, \overline{v}(x), \overline{f}(x), \Psi(x))(f - \overline{f})\mathrm{d}x \geqslant 0, \quad \forall f \in U_0. \tag{48}$$

So we have just proved the following theorem:

Theorem 4 (The principle of linearized minimum). *Suppose that all of the conditions of theorems 1 and 2 are satisfied. Then, for the optimal control* $\overline{f}(x) \in U_0$

it is necessary that there exists a nontrivial vector

$$l^* = \bigl(l_0, l_1, \ldots, l_{s_1+s_2}\bigr), \quad \sum_{k=0}^{s_1+s_2} |l_k| > 0,$$

where $l_k \geqslant 0$ *for* $k = \overline{0, s_1}$ *and the conditions* (48) *as well as the conditions*

$$l_k J_k(\overline{f}) = 0 \quad k = \overline{1, s_1 + s_2} \quad \textit{(conditions (45))},$$

are satisfied; where function $v(x)$ *is the solution of the problem* (1)–(3), $\Psi(x)$ *is the solution of the conjugate problem* (46) *associated to* $\overline{f}(x)$ *and the function* $\mathcal{H}$ *is defined in* (47).

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УДК 517.917

Необходимые условия оптимальности для стационарной нелинейной возмущённой задачи гидродинамики в ограниченной области

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Цель работы состоит в том, чтобы установить оптимальные необходимые условия, которые могут позволить нам решить задачу относительно границ данной области. В предлагаемой статье исследуется частный случай, а именно, в линеаризованную систему добавлены нелинейные члены, позволяющие более точно описать движение жидкости, и вместе с тем допускающие однозначную разрешимость полученной нелинейной возмущённой краевой задачи.

Ключевые слова: необходимые условия оптимальности, команда, оптимальная команда, уникальность, возмущение, линеаризация, нелинейность.