Bulletin of PFUR Series *Mathematics. Information Sciences. Physics.* No 4. 2008. Pp. 5–19

Математика и теоретическая механика

UDC 517.95

Pontryagin's Principle of Maximum for Linear Optimal Control Problems with Phase Constraints in Infinite Dimensional Spaces

M. Longla

Department of Differential Equations and Mathematical Physics
Peoples' Friendship University of Russia
6, Miklukho-Maklaya str., Moscow, Russia, 117198

This paper presents the conditions of optimality for a problem with linear phase constraints in an infinite dimensional normal space with separated locally convex topology demonstrated using the works of *M.F. Sukhinin* in infinite dimensional normal spaces, his theory of differential equations in these spaces when functions are not Bochner-integrable and have no derivative of Gateaux. Problems with phase constraints were analyzed in finite spaces by many authors like *L.S. Pontryagin, L. Graves, V.G. Boltyanskiy, R.V. Gamkrelidze, A.A. Milyutin, A.V. Dmitruk, N.P. Osmolovskij and others.* Using the theory of differential equations of *Prof. M.F. Sukhinin* published in his monograph [1], applying the *Gamkrelidze* and *Pontryagin*'s method illustrated in book [2], we enounced and proved theorems for linear mixed constraint in the separated locally convex space X.

Key words and phrases: nonlinear optimization, topology, differential equations, constraint problems.

1. Integral, Differentiability and Properties

When functions of the type $f(t,x) = \sin(tx)$, were $x = x(\omega)$, $\omega \in [0,1]$ or $f(t,x(\omega),\omega) = \omega \sin(tx(\omega)/\omega)$ are used in a problem, one should ask what we mean talking of derivatives. The first function is nowhere differentiable by Freshet as a function from L_p to L_p, but is γ–differentiable as a function $f : L_p \to (L_p, \sigma)$. Here γ is a system of bounded subsets of L_p, σ is the weak topology. The second has no derivative of Gateaux at no point, but is γ–differentiable as a function $f : L_\infty \to (L_\infty, \sigma)$, were σ is the weak* topology. To use properly these functions and others with the same particularities, we need the following theory in infinite dimensional spaces.

1.1. Integral and Properties

Here X is a Banach space with an additional locally convex topology,
– $B(X)$ is the unite ball of X,
– $b(X)$ is the set of all bounded subsets of X,
– $c(X)$ is the set of all sequently compact subsets of X,
– $\wp(X)$ — is the set of seminorms, defining the topology θ,
– θ — a separated locally convex topology in X, satisfying the conditions:
1. $B(X)$ — is closed in X_θ,
2. $(B(X))_\theta$ — is sequently complete,
3. $b(X) \subset b(X_\theta)$.

Example. Let Y, Z — be Banach spaces, $X = \ell(Y,Z)$ with the strong operator's topology θ. Then the mentioned properties are satisfied.

Received 16$^{\text{th}}$ May, 2008.

Here the set $\ell_\gamma(X_\theta, X_\theta)$ is the space of linear sequently continue operators of X_θ into X_θ, with convergence by virtue of γ. $\ell(X_\theta, X_\theta)$ is the same set with strong convergence topology.

Let $I = [\alpha, \beta] \subset \mathbb{R}, E \subset M(I)$, $M(I)$ — is the set of measurable subsets of I. Let $\phi : E \to X_\theta$ be uniformly continuous, $\phi(E) \in b(X)$, $\alpha = t_1 < t_2 < \ldots < t_n = \beta$, $\xi_i \in E_i = I_i \cap E$. Let

$$Q_i = \begin{cases} \phi(\xi_i)\mu(E_i), & E_i \neq 0, \\ 0, & E_i = 0, \end{cases} \quad \text{and} \quad \int\limits_E \phi(t)\mathrm{d}t = \lim_{\max_i \Delta t_i \to 0} \sum_{i=1}^{n} Q_i$$

with respect to the topology θ. This limit exist and doesn't depend on the parameters.

The defined integral satisfies the next properties:

1. $\forall \psi \in \ell(X_\theta, \mathbb{R}) : \psi \int\limits_E \phi(t)\mathrm{d}t = \int\limits_E \psi\phi(t)\mathrm{d}t$.
2. $\forall \psi \in \ell(X_\theta, \mathbb{R}) : \left| \psi \int\limits_E \phi(t)\mathrm{d}t \right| \leqslant ||\psi|| \int\limits_E ||\phi(t)||\mathrm{d}t$.
3. $\forall \phi_1, \phi_2, \lambda_1, \lambda_2 \in \mathbb{R}, \int\limits_E (\lambda_1\phi_1(t) + \lambda_2\phi_2(t))\mathrm{d}t = \lambda_1 \int\limits_E \phi_1(t)\mathrm{d}t + \lambda_2 \int\limits_E \phi_2(t)\mathrm{d}t$.
4. $\Lambda_p(E, X_\theta) = \{\phi : E \to X_\theta | \phi$ is a class of equivalent measurable functions and $\int\limits_E ||\phi(t)||^p\mathrm{d}t < \infty\}$ for $1 \leqslant p < \infty$, $||\phi||_{\Lambda_p} = |||\phi|||_p = \left(\int\limits_E ||\phi(t)||^p\mathrm{d}t \right)^{1/p}$.
5. For $\phi \in \Lambda_1(E, X_\theta), \int\limits_E \phi(t)\mathrm{d}t = \lim_{n\to\infty} \int\limits_{K_n} \phi(t)\mathrm{d}t$, K_n — is compact in E, $\phi|_{K_n} : K_n \to X_\theta$ — is continuous, $\phi|_{K_n} : K_n \to X$ — is bounded, and $\mu(E \setminus K_n) \to 0$. The space of Bochner-integrable functions is a closed in $\Lambda_1(E, X_\theta)$.
6. For $\phi \in \Lambda_1(E, X_\theta)$, the function $\Phi : t \mapsto \Phi(t) = \int\limits_a^t \phi(s)\mathrm{d}s \in X_\theta$ — is differentiable almost everywhere and its derivative is $\Phi'(t) = \phi(t)$.
7. $\Phi(t)$ — is absolutely continuous as function of I into X.
8. $W_1^1(I, X_\theta) = \left\{ f \mid \exists a \in X, \exists \alpha \in I, \exists \phi \in \Lambda_1(I, X_\theta) : f(t) = a + \int\limits_\alpha^t \phi(s)\mathrm{d}s \right\}$.

The defined integral also satisfies other properties of the Lebesgue integral necessary in this paper (see [1, § 6]).

1.2. Differentiability and its Properties

Def 1. The function $r : U \to X$ is said to be γ-small at $x_0 \in U$, if

$$\forall p \in \wp(X) \quad \forall C \in \gamma \exists \delta > 0 \quad \forall h \in C \quad \forall |t| < \delta, \quad x_0 + th \in U : \quad p(r(x_0 + th)) \leqslant |t|.$$

Def 2. The function $f : U \to X$ is said to be γ-equivalent to the operator $A \in \ell(Y, X)$ at $x_0 \in U$, if $r(h) = f(x_0 + h) - f(x_0) - Ah$ is γ-small at 0. Moreover, if A is defined for all $h \in X$, then f is γ-differentiable and its γ-derivative at x_0 is A.

Def 3. f is sequently (γ, γ_1)-Lipshitzian at $x_0 \in U$, if $\forall C \subset \gamma \quad \forall \{h_n\} \subset C$ $\forall \{t_n\} \in c_0(\mathbb{R}), \quad t_n \neq 0, \quad x_0 + t_n h_n \in U : \quad \{t_n^{-1}[f(x_0 + t_n h_n) - f(x_0)]\} \in \gamma_1$.

Def 4. The mapping $f : X_\theta \to Y_\tau$ is said to be open at x_0 , if $\forall \Omega \subset X$, $x_0 \in int\Omega :$ $f(x_0) \in f(\Omega)$. If the contrary yields, we said that the mapping f is critical at x_0 (see [3, p. 781–839]).

Def 5. If X, Y are seminormal spaces, then the mapping $f : X \to Y$ is said to be correct (see [4, p. 223–228]) or have the covering property (see [5, p. 39–44], [6, p. 11–46]) at x_0 , if $\exists \varepsilon > 0 \ \forall \delta \in]0, \varepsilon[: f(x_0) + \varepsilon\delta B(Y) \subset f(x_0 + \delta B(X))$. If the contrary yields, we said that the mapping f is quasicritical at x_0.

Def 6. Let X, Y — be seminormal spaces. The mapping $f : X \to Y$ is lipschitzian at $x_0 \in X$, if $\exists r > 0$, $\exists \varepsilon > 0$ $\forall \delta \in]0, \varepsilon[$: $f(x_0 + \delta B(X)) \subset f(x_0) + r\delta B(Y)$.

Remark 1. Let X be a locally convex space. $A_1 : Z \to Y$ — is a linear operator, $A_2 \in l(Y, X)$, and $f : U \to Y$ and $g : f(U) \to X$ are respectively γ—equivalent to A_1 at x_0 and γ_1—equivalent to A_2 at $f(x_0)$. If f is sequently (γ, γ_1)–Lipshitzian at x_0, then $g \circ f : U \to X$ is γ-equivalent to $A_2 \circ A_1$ at x_0.

Remark 2. On the critical and quasicritical properties.

1. From the covering property of the mapping at a point follows its openness at this point, or in other words, from the criticity of the mapping at a given point follows its quasicriticity at this point.
2. If X_θ, Y_τ, Z_σ — are topological spaces, the mapping $f : X \to Y$ is continuous at $x_0 \in X$, and $g : f(X) \to Z$ is critical at $f(x_0)$, then $g \circ f : X \to Z$ is critical at x_0.
3. If X, Y, Z — are seminormal spaces, the mapping $f : X \to Y$ is lipschitzian at $x_0 \in X$, and the mapping $g : f(X) \to Z$ is quasicritical at $f(x_0)$, then $g \circ f : X \to Z$ is quasicritical at x_0.

2. Formulation of the Problem

2.1. General Settings

Let J be a convex functional, $A(t) \in \Lambda_1([t_0, t_1], l_\gamma(X_\theta, X_\theta))$, $-B(t) \in \Lambda_1([t_0, t_1], l(\mathbb{R}^r, X_\theta))$, $h^i(t) \in l(I, l(X_\theta, \mathbb{R}))$, $b^i(t) \in l(I, \mathbb{R}^r)$.

Let be defined the mapping $\bar{Q} : (x_0, x_1, t_0, t_1) \mapsto \bar{Q}(x_0, x_1, t_0, t_1) \in \mathbb{R}^N$. Let be given the equations:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad u \in U, \quad x \in X, \tag{1}$$

$$\text{we put } x(t_0) = x_0, \quad x(t_1) = x_1, \tag{2}$$

$$[h^i(t), x(t)] + b^i(t)u(t) \leqslant 0, \quad i = \overline{1, s}, \tag{3}$$

$$[\bar{h}^j(t), x(t)] + \bar{b}^j(t) \leqslant 0, \quad j = \overline{1, l}, \tag{4}$$

$$\bar{Q} \text{ — is quasicritical at } (x_0^*, x_1^*, t_0^*, t_1^*). \tag{5}$$

Question: Find de necessary conditions of existence of the solution of the system (1)–(4), for which at the given point Q is quasicritical.

In order to answer to this question, we check separately the subsystem (1)–(3), with (5) and the subsystem (1)–(2), (4)–(5). Having the results from the two subsystems, we combine them to find the answer to the question for the system (1)–(5).

Simultaneously, we use the answers to get the necessary conditions of existence of the optimal control for problems with linear constraints in the form of Pontryagin's principle of maximum in infinite dimensional spaces. To get this result, it suffices to take in the above problem $\bar{Q} = (Q, J)$, if we want to investigate in infinite dimensional banach space the case of the optimal control problem formed by (1)–(4) and the next equations:

$$Q(x_0, x_1, t_0, t_1) = 0, \tag{6}$$

$$J(x_0, x_1, t_0, t_1) \to \min. \tag{7}$$

Let set the next conditions on X, θ and γ, that allow us work with the defined integral and to differentiate by virtue of the topology the functions that we use:

$$L(f, B(X), D) = \sup_{x \in D} \overline{\lim_{t \to 0}} \sup_{k \in [t^{-1}(D-x) \cap B(X)]} \{|t|^{-1} ||f(x + tk) - f(x)||\},$$

$$Lip_b(D, X_\theta, \gamma) = \{f : X \to X_\theta | L(f, ||.||, B(X), D) < \infty\},$$

$$||f||_1 = ||f|| + L(f, B(X), D).$$

In the space $Lip_b(D, X_\theta, \gamma)$ is defined a topology with the next basis at 0:

$$\Xi(p, C, Q) = \{f | p(f(\tilde{x})) + L(f, p, C, Q) < 1\},$$
$$\text{where } p \in \wp(X_\theta), C \in \gamma, \text{ and } Q \in c(U).$$

1. $B(X)$ — is closed in X_θ, $(B(X))_\theta$ — is sequently complete,
2. $c(X) \subset \gamma \subset b(X_\theta)$, $\forall C \in \gamma\ \forall A \in \ell(X_\theta, X_\theta)\ \forall \{x_n\} \subset C$, $\{Ax_n\} \subset \gamma$,
3. $(C \in \gamma, C' \subset C) \Longrightarrow C' \in \gamma$,
4. $(C \in \gamma, M > 0) \Longrightarrow [-M, M]C \in \gamma$,
5. $(C \in \gamma, C' \in \gamma) \Longrightarrow C\ \mathrm{U}C' \in \gamma$.

2.2. Problem with Regular Constraints

As announced, let check the case of the subsystem (1)–(3), (5). Let $h(t) = (h^1(t), \ldots, h^s(t))^T \in [\ell(X_\theta, \mathbb{R})]^s$, $\bar{x} = (\tilde{x}, x)$, $b(t) = (b^1(t), \ldots, b^s(t))^T$, $\dot{\tilde{x}}(t) = [h(t), x(t)] + b(t)u(t)$. The solution $\{x^*, u^*, t_0^*, t_1^*, x_0^*, x_1^*\}$ of the given system is also solution of (II):

$$x(t) = \int_{t_0}^{t} A(s)x(s)\mathrm{d}s + \int_{t_0}^{t} B(s)u(s)\mathrm{d}s + x_0, \tag{8}$$

$$\tilde{x} = \int_{t_0}^{t} [h(s), x(s)]\mathrm{d}s + \int_{t_0}^{t} b(s)u(s)\mathrm{d}s. \tag{9}$$

The optimal solution of our system also satisfies the necessary conditions of optimality for the system (8)–(9) with the mapping (5), for which holds the proposition:

2.2.1. Existence of the Admissible Solutions

Proposition 1 (Existence of the admissible solutions)**.** *The system* (8)–(9) *has a solution* $\bar{x} = (\tilde{x}, x) \in \Lambda_1([t_0, t_1], \bar{X}_\theta \times \mathbb{R}^s)$ *for each* $u \in L_1(\mathbb{R}^r)$, $A \in \Lambda_1([t_0, t_1], \ell_\gamma(X_\theta, X_\theta))$ *that we can express by the formulas:*

$$x(t) = \Re(t, t_0)x_0 + \int_{t_0}^{t} \Re(t, l) \circ B(l)u(l)\mathrm{d}l, \tag{10}$$

$$\tilde{x}(t) = \int_{t_0}^{t} [h, x(l)]\mathrm{d}l + \int_{t_0}^{t} b(l)u(l)\mathrm{d}l. \tag{11}$$

This is a direct consequence of the next result from [1]:

Proposition 2 (Existence and uniqueness of the integral equation's solution)**.** *Let* $y \in \Lambda_\infty(I, X_\theta)$, $t_0 \in I$, $b(X) = b(X_\theta)$, X *— is an infinite dimensional normal space,* θ *— is a separated locally convex topology in* X, *and* $\gamma \subset b(X)$, $A(t) : ([t_0, t_1] \to \ell_\gamma(X_\theta, X_\theta))$ *is* θ*-integrable. Then the equation*

$$x(t) - \int_{t_0}^{t} A(s)x(s)\mathrm{d}s = y(t) \quad \textit{has} \quad x(t) = y(t) + \int_{t_0}^{t} \Re(t, s)A(s)y(s)\mathrm{d}s$$

as unique solution in $\Lambda_\infty(I, X_\theta)$*. Here* $W_1^1(I, \ell_\gamma(X_\theta, X_\theta)) \ni \Re : I \times I \to \ell_\gamma(X_\theta, X_\theta)$ *is the resolvent kernel of* $x' = A(t)x$*. We have*

$$\Re'_s(t, s) = -\Re(t, s) \circ A(s), \quad \Re'_t(t, s) = A(t) \circ \Re(t, s).$$

Proof. $\forall n \in \mathbb{N}, \quad \forall u \in U, \quad \exists \Pi_n(u) \in [t_0, t_1],\ u_n \in U, \quad A_n \in \Lambda_1(I, \ell_\gamma(X_\theta, X_\theta)),$ $B_n \in \Lambda_1(I, \ell_\gamma(\mathbb{R}^r, X_\theta)), \quad b_n \in L_1(I, \ell(\mathbb{R}^r, \mathbb{R}^s)):$

1. $\mu([t_0, t_1] \setminus \Pi_n(u)) < \frac{1}{n}$, $\Pi_n(u)$ — is compact;
2. u_n — is continuous on $\Pi_n(u)$ and $\lim\limits_{n\to\infty} u_n(t) \doteq u(t)$;
3. A_n — is continuous on $\Pi_n(u)$ and $\lim\limits_{n\to\infty} A_n(t) \doteq A(t)$;
4. B_n — is continuous on $\Pi_n(u)$ and $\lim\limits_{n\to\infty} B_n(t) \doteq B(t)$;
5. b_n — is continuous on $\Pi_n(u)$ and $\lim\limits_{n\to\infty} b_n(t) \doteq b(t)$; $\Pi_n(u) \subset \Pi_{n+1}(u)$.

Using the defined approximations for the given systems on $\Pi(u)$, we obtain a continuous solution x_n by the enounced proposition with $\Re_n(t_0, t)$. Taking into account the properties of θ, it easily comes out that $\Re_n(t_0, t) \to \Re(t_0, t)$ and $x_n \to x$ on $\Pi(u) = \bigcup_{n=1}^{\infty} \Pi_n(u)$. □

Let now study the problem on $\Pi_j(u)$ with the defined sequences. In order to reduce the quantity of indexes in this paper, we will denote the sequences just as their limits, as they can't be misunderstood in this case, keeping in mind that after all our operations the results should be turned to the limits of the used functions.

Let set $t_0(\varepsilon) = t_0 + \varepsilon\delta t_0$, $t_1(\varepsilon) = t_1 + \varepsilon\delta t_1$, $\bar{x}_0(\varepsilon) = \bar{x}_0^* + \varepsilon\delta\bar{x}_0$, $u(t, \varepsilon) = u^*(t) + \varepsilon\delta u(t)$,

$$\bar{x}_1(\varepsilon) = \bar{x}_1^* + \varepsilon\delta\bar{x}_1 + \gamma_0(\varepsilon\delta\bar{x}_0^*, \delta u). \tag{12}$$

Equations (10)–(11) define $f(\bar{x}_0, u, t_0, t_1) = \bar{x}(t_1)$, and (12) a continuous operator $\phi : \mathbb{R}_+ \to \vartheta(\bar{x}_0) \times \vartheta(\bar{x}_1) \times \vartheta(t_0) \times \vartheta(t_1)$.

For the case of the optimal control problem, let $c = J(x_0^*, x_1^*, t_0^*, t_1^*)$. Then for each solution of (8)–(9) we have $Q(\bar{x}_0, f(\bar{x}_0, u, t_0, t_1), t_0, t_1) = 0$ and $J(\bar{x}_0, f(\bar{x}_0, u, t_0, t_1), t_0, t_1) \geqslant c$. And, for the quasicritical mapping we have

$$\bar{Q} \circ \phi(\varepsilon)(\delta\bar{x}_0, \delta\bar{x}_1, \delta t_0, \delta t_1) - \text{is critical at } \varepsilon = 0.$$

From here we find some κ, for which $\kappa dQ \geqslant 0$. (see [1])

Therefore, the optimal solution of the initial problem should satisfy the necessary conditions of optimality given by the result of the next problem:

$$\bar{Q}(\varepsilon) = Q \circ \phi(\varepsilon)(\delta\bar{x}_0, \delta\bar{x}_1, \delta t_0, \delta t_1) = 0, \tag{13}$$

$$\bar{J}(\varepsilon) = J \circ \phi(\varepsilon)(\delta\bar{x}_0, \delta\bar{x}_1, \delta t_0, \delta t_1) \to \min. \tag{14}$$

For (13)–(14) and vectors from the above solution, as the critical value of ε is 0, we obtain for some $m \in \mathbb{R}^N$, $\quad n \in \mathbb{R} \setminus \{0\}$, $\quad \mu(t) : [t_0^*, t_1^*] \to \mathbb{R}^s$ the following:

$$mQ_{\bar{x}_0}\Delta\bar{x}_0 + mQ_{\bar{x}_1}\Delta\bar{x}_1 + mQ_{t_0}\delta t_0 + mQ_{t_1}\delta t + nJ_{\bar{x}_0}\Delta\bar{x}_0 + nJ_{\bar{x}_1}\Delta\bar{x}_1 + nJ_{t_0}\delta t_0 + nJ_{t_1}\delta t + \int_{t_0^*}^{t_1^*} [\psi(t), \dot{\delta x}(t) - A(t)\delta x(t) - B(t)\delta u(t)]\mathrm{d}t + \int_{t_0^*}^{t_1^*} [\mu(t), [h, \delta x(t)] + b(t)\delta u(t)]\mathrm{d}t \leqslant 0, \tag{15}$$

$$\Delta x(t) = \Re(t, t_0^*)\delta x_0 + \Re(t, t_1^*)(A(t)x^*(t) + B(t)u^*(t))\delta t - \Re(t, t_0^*)(A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*))\delta t_0 + \int_{t_0^*}^{t} \Re(t, l) \circ B(l)\delta u(l)\mathrm{d}l, \tag{16}$$

$$\delta\dot{\tilde{x}}(t) = ([h, x^*(t)] + b(t)u^*(t))\delta t - ([h, x_0] + b(t_0^*)u^*(t_0))\delta t_0 + [h, \delta x(t)] + b(t)\delta u(t). \quad (17)$$

From (15)–(16), we obtain the following:

$$\begin{aligned} &-[(mQ_{x_1} + nJ_{x_1} + \psi(t_1^*)) \circ \Re(t_1^*, t_0^*), A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] + \\ &\qquad + [mQ_{x_0} + nJ_{x_0}, A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] + mQ_{t_0} + nJ_{t_0} = 0, \quad (18) \end{aligned}$$

$$[mQ_{x_1} + nJ_{x_1}, A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] + nJ_{t_1} + mQ_{t_1} = 0, \quad (19)$$

$$(mQ_{x_1} + nJ_{x_1} + \psi(t_1^*)) \circ \Re(t_1^*, t_0^*) + mQ_{x_0} + nJ_{x_0} - \psi(t_0^*) = 0, \quad (20)$$

$$\left[mQ_{x_1} + nJ_{x_1} + \psi(t_1^*), \int_{t_0^*}^{t_1^*} \Re(t_1^*, l)B(l)\delta u(l)\mathrm{d}l \right] \leqslant 0. \quad (21)$$

Knowing from the above inequalities that $m(\psi) = \max_{u \in U} H(\psi, u) = H(\psi, u^*)$, $H(\psi, u) = [\psi(t), B(t)u]$, using the regularity of the set U, we find functions $\mu(t), \nu_1(t), \cdots, \nu_s(t)$, for which

$$\psi(t) \circ B(t) = \nabla_u H(\psi, u^*) = \mu(t) \circ b(t) + \sum_{\alpha=1}^{s} \nu_\alpha \nabla_u \tilde{q}_\alpha(u^*) \Rightarrow$$

$$\Rightarrow \mu(t)b(t)\delta u(t) = [\psi(t), B(t)\delta u(t)].$$

On the other hand, taking into account $\mathbf{rang}(b(t)) \doteq s$, we find a measurable $\Lambda(t)$, satisfying $[B(t) + \Lambda(t)b(t), \delta u(t)] = 0$. Then $[\psi(t), \Lambda(t)] = -\mu(t)$, and

$$\dot{\psi}(t) = -\psi(t) \circ A(t) + \mu(t)h,$$

$$\psi(t_1^*) = -mQ_{x_1} - nJ_{x_1}, \quad \psi(t_0^*) = \psi(t_1^*)\Re(t_1^*, t_0^*) - \int_{t_0^*}^{t_1^*} \mu(s)h \circ \Re(s, t_0^*)\mathrm{d}s.$$

Taking into account this fact, (21) vanishes and (18)–(20) become:

$$[\psi(t_0^*), A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] = -mQ_{t_0} - nJ_{t_0}, \quad (18')$$

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1} + nJ_{t_1}, \quad (19')$$

$$\psi(t_0^*) = mQ_{x_0} + nJ_{x_0}. \quad (20')$$

Let define $\tau_1, \cdots, \tau_k, \tau$: $t_0 < \tau_1, \tau_i \leqslant \tau_{i+1}$, $\tau = t_1$, $\tau_i \in \Pi_\alpha(u)$. Here $\Pi_\alpha(u) \subset M(I)$ with measure $\mu(\Pi_\alpha(u)) = 1/\alpha$, and the used functions are uniformly continuous on $\Pi_\alpha(u)$, for all $\alpha \in \mathbb{N}$. Such a subset exist according to [1, § 6.9.10].

Let $\delta t_1 \geqslant 0, \ldots, \delta t_k \geqslant 0$, $\delta t \in \mathbb{R}$ and $\{v_1, \ldots, v_k\} \subset U$, ($v_i = v_j$ is possible). Let $I_i = [\tau_i + \varepsilon l_i, \tau_i + \varepsilon(l_i + \delta t_i)]$, $i = \overline{1, k}$, for l_i defined as follows:

$$l_i = \begin{cases} \delta t - (\delta t_i + \cdots + \delta t_k), \text{ if } \tau_i = \tau; \\ -(\delta t_i + \cdots + \delta t_k), \text{ if } \tau_i = \tau_k < \tau; \\ -(\delta t_i + \cdots + \delta t_j), \text{ if } \tau_i = \tau_{i+1} = \cdots = \tau_j < \tau_{j+1} \quad (j < k). \end{cases}$$

We choose ε so that $I_i \cap I_j = \emptyset$, and $I_i \subset [t_0, t_1 + \varepsilon\delta t]$, and define

$$u(t) = \begin{cases} u^*(t), & \forall t \notin \Pi_\alpha(u) \cap \cup_1^k I_i, \\ v_i, & \forall t \in I_i \cap \Pi_\alpha(u) \end{cases} \quad \text{or} \quad \delta u(t) = \begin{cases} 0, & \forall t \notin \Pi_\alpha(u) \cap \cup_1^k I_i, \\ v_i - u^*(t), & \forall t \in I_i \cap \Pi_\alpha(u). \end{cases}$$

Considering in (15) that the integrals vanish, $\delta t_0 = 0$, $\delta x_0 = 0$, $\delta\tau_k = \delta t = 0$, we obtain $[mQ_{x_1} + nJ_{x_1},\ B(t_1^*)\delta u(t_1^*)] \geqslant 0$, what leads to

$$H(\psi(t_1^*), u(t_1^*)) \leqslant H(\psi(t_1^*), u^*(t_1^*)), \tag{21$'$}$$

$$[\mu(t) \circ h, x^*] + \mu(t) \circ b(t)u = 0, \tag{22}$$

$$\psi(t) \circ A(t) = -\dot{\psi}(t) + \mu(t) \circ h. \tag{23}$$

Using $\mu(t) = -[\psi(t), \Lambda(t)]$, we obtain

$$\dot{\psi}(t) = -\psi(t)(A(t) + \Lambda(t) \circ h) \text{ and}$$
$$\delta x(t) = \int_{t_0^*}^{t} (A(s) + \Lambda(s) \circ h)\delta x(s)\mathrm{d}s. \tag{23*}$$

Therefore, (16) becomes

$$\Delta x(t) = \sum_{i=0}^{k} \tilde{\Re}(t_1^*, \tau_i) \circ B(\tau_i)(v_i - u^*(\tau_i))\delta t_i, \tag{16$'$}$$

where $\tilde{\Re}(t, t_0^*)$ is the resolvent kernel of (23*). Coming back to (15) with the previous changes and (16$'$), we obtain

$$\psi(\tau_k) = \psi(t_1^*)\tilde{\Re}(t_1^*, \tau_k), \quad [\psi(\tau_k), B(\tau_k)\delta u(\tau_k)] \leqslant 0 \text{ or}$$
$$H(\psi(\tau_k), u(\tau_k)) \leqslant H(\psi(\tau_k), u^*(\tau_k)) \text{ for all } k. \tag{21$''$}$$

We can conclude that (21$''$) holds for all $t \in \Pi_\alpha(u)$. And taking the limit in the proved expressions, we obtain the necessary conditions for the optimal control and (21$''$) holds on $\Pi(u) = \bigcup_{\alpha=1}^{\infty} \Pi_\alpha(u)$.

The result makes sense only if $\psi(t_1^*) = mQ_{x_1} + nJ_{x_1} \neq 0$. This condition is guaranteed if for some i_0, $\{Q_{x_1}^{i_0}, J_{x_1}\}$ is linearly independent. The satisfaction of the above condition and those of the problem formulation leads to the following theorems.

2.3. Theorems

Here we enounce theorems for different cases, taking into account the above transformation.

Theorem 1 (Analog of the maximum principle)**.** *Let* $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ *— be the measurable optimal solution of* (1)–(3), (6)–(7). *Let* $x^*(t) \in W_1^1(I, X_\theta)$ *and* $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. *Let* $A \in \Lambda_1(I, X_\theta)$. Q *is* $b(X_\theta^2) \times b(\mathbb{R}^2)$ *— differentiable at* $(x_0^*, x_1^*, t_0^*, t_1^*)$. *If* $Q : \vartheta(x_0^*) \times \vartheta(x_1^*) \times \vartheta(t_0^*) \times \vartheta(t_1^*) \to \mathbb{R}^N$ *is continuous in* $\vartheta((x_0^*, x_1^*))$, $\mathrm{rang}\,(b(t)) \doteq s$ *and for some* i_0, $\{Q_{x_1}^{i_0}(x_0^*, x_1^*, t_0^*, t_1^*), J_{x_1}(x_0^*, x_1^*, t_0^*, t_1^*)\}$ *is linearly independent, then* $\exists \psi(t) \in W_1^1([t_0^*, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, *and* $\mu(t) \in L_1(I, \mathbb{R}^s)$, *for which holds:* $\forall u \in U \quad \exists \Pi(u) : \bar{\mu}(\Pi(u)) = t_1^* - t_0^*$ *and* $\forall t \in \Pi(u)$,

$$\dot{x}^*(t) - A(t)x^*(t) = H_\psi(\psi(t), u^*(t)) \equiv B(t)u^*(t), \tag{24}$$

$$\dot{\psi}(t) = -\psi(t)A(t) + \mu(t) \circ h(t), \tag{25}$$

$$H(\psi(t), u(t)) \leqslant H(\psi(t), u^*(t)), \quad [\mu(t) \circ h(t), x^*(t)] + \mu(t) \circ b(t)u^*(t) = 0, \tag{26}$$

where $\mu(t)$ comes from the maximum's condition;

$$[\psi(t_0^*), A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] = -mQ_{t_0} - nJ_{t_0}, \quad \psi(t_0^*) = mQ_{x_0} + nJ_{x_0}, \tag{27}$$

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1} + nJ_{t_1}, \quad \psi(t_1^*) = -mQ_{x_1} - nJ_{x_1}. \tag{28}$$

From this theorem follows:

Corollary 1 (Integral form of the maximum principle)**.** *Let $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ — be the measurable optimal solution of the given system. Let $x^*(t) \in W_1^1(I, X_\theta)$ and $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. Let $A \in \Lambda_1(I, X_\theta)$. Q is $b(X_\theta^2) \times b(\mathbb{R}^2)$ — differentiable at $(x_0^*, x_1^*, t_0^*, t_1^*)$. If $Q : \vartheta(x_0^*) \times \vartheta(x_1^*) \times \vartheta(t_0^*) \times \vartheta(t_1^*) \to \mathbb{R}^N$ is continuous in $\vartheta((x_0^*, x_1^*))$,* rang $(b(t)) \doteq s$ *and for some i_0, $\{Q_{x_1}^{i_0}, J_{x_1}\}$ is linearly independent, then $\exists \psi(t) \in W_1^1([t_0^*, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, and $\mu(t) \in L_1(I, \mathbb{R}^s)$, for which holds : $\forall u \in U$,*

$$x^*(t) = x_0^* + \int_{t_0^*}^{t} A(s)x^*(s)\mathrm{d}s, \quad \dot{\psi} = -\psi(t)A(t) + \mu(t) \circ h(t), \tag{29}$$

$$\int_{t_0^*}^{t_1^*} H(\psi(t), u(t))\mathrm{d}t \leqslant \int_{t_0^*}^{t_1^*} H(\psi(t), u^*(t))\mathrm{d}t, \tag{30}$$

where $\mu(t)$ comes from the maximum's condition;

$$[\psi(t_0^*), A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] = -mQ_{t_0} - nJ_{t_0}, \quad \psi(t_0^*) = mQ_{x_0} + nJ_{x_0}, \tag{31}$$

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1} + nJ_{t_1}, \psi(t_1^*) = -mQ_{x_1} - nJ_{x_1}, \tag{32}$$

$$[\mu(t) \circ h(t), x^*(t)] + \mu(t) \circ b(t)u^*(t) = 0. \tag{33}$$

If $Q = Q(x_1, t_0, t_1)$ and $J = J(x_1, t_0, t_1)$, and x_0 is a known vector, then the variation vanishes at this point, and we easily get from the proof of the *theorem 1 and corollary 1*:

Corollary 2 (Case of fixed initial point)**.** *Let $\{x^*(t),\ u^*(t),\ x_1^*,\ t_0^* \leqslant t \leqslant t_1^*\}$ — be the measurable optimal solution of the problem. Let $x^*(t) \in W_1^1(I, X_\theta)$ and $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. Let $A \in \Lambda_1(I, X_\theta)$. Q is $b(X_\theta) \times b(\mathbb{R}^2)-$ differentiable at (x_1^*, t_0^*, t_1^*). If $Q : \vartheta(x_1^*) \times \vartheta(t_0^*) \times \vartheta(t_1^*) \to \mathbb{R}^N$ is continuous in $\vartheta(x_1^*)$ for fixed values t_0^*, t_1^*,* rang $(b(t)) \doteq s$ *and for some i_0 $\{Q_{x_1}^{i_0}(x_1^*, t_0^*, t_1^*), J_{x_1}(x_1^*, t_0^*, t_1^*)\}$ is linearly independent, then there exist $\psi(t) \in W_1^1([t_0^*, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, and $\mu(t) \in L_1(I, \mathbb{R}^s)$, for which holds: $\forall u \in U \quad \exists \Pi(u) : \bar{\mu}(\Pi(u)) = t_1^* - t_0^*$,*

$$\dot{\psi}(t) = -\psi(t)A(t) + \mu(t) \circ h(t), \tag{34}$$

$$H(\psi(t), u(t)) \leqslant H(\psi(t), u^*(t)), \quad [\mu(t) \circ h, x^*(t)] + \mu(t) \circ b(t)u^*(t) = 0, \tag{35}$$

where $\mu(t)$ comes from the maximum's condition;

$$[\psi(t_0^*), A(t_0^*)x_0^* + B(t_0^*)u^*(t_0^*)] = -mQ_{t_0} - nJ_{t_0}, \tag{36}$$

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1} + nJ_{t_1}, \quad \psi(t_1^*) = -mQ_{x_1} - nJ_{x_1}. \tag{37}$$

If x_0, t_0 are known parameters, then $\delta t_0 = 0$, $\delta x_0 = 0$ and the next corollary hold:

Corollary 3 (Case of fixed initial point and initial time). *Let $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ — be the measurable optimal solution of the problem. Let $x^*(t) \in W_1^1(I, X_\theta)$ and $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. Let $A \in \Lambda_1(I, X_\theta)$. Q is $b(X_\theta) \times b(\mathbb{R})-$ differentiable at (x_1^*, t_1^*). If $Q : \vartheta(x_1^*) \times \vartheta(t_1^*) \to \mathbb{R}^N$ is continuous in $\vartheta(x_1^*)$ for the fixed value t_1^*, $\operatorname{rang}(b(t)) \doteq s$ and for some i_0 $\{Q_{x_1}^{i_0}(x_1^*, t_1^*), J_{x_1}(x_1^*, t_1^*)\}$ is linearly independent, then there exist $\psi(t) \in \psi([t_0, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, and $\mu(t) \in L_1(I, \mathbb{R}^s)$, for which holds: $\forall u \in U \quad \exists \Pi(u) : \bar{\mu}(\Pi(u)) = t_1^* - t_0$, $\forall t \in \Pi(u)$,*

$$\dot{\psi}(t) = -\psi(t)A(t) + \mu(t) \circ h(t), \tag{38}$$

$$H(\psi(t), u(t)) \leqslant H(\psi(t), u^*(t)), \quad [\mu(t) \circ h(t), x^*(t)] + \mu(t) \circ b(t)u^*(t) = 0, \tag{39}$$

where $\mu(t)$ comes from the maximum's condition;

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1} + nJ_{t_1}, \quad \psi(t_1^*) = -mQ_{x_1} - nJ_{x_1}. \tag{40}$$

Corollary 4 (Integral form for fixed initial point and time). *Let $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ — be the measurable solution of the problem for which $\bar{Q}$ is quasicritical. Let $x^*(t) \in W_1^1(I, X_\theta)$ and $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. Let $A \in \Lambda_1(I, X_\theta)$. $\bar{Q}$ is $b(X_\theta) \times b(\mathbb{R})-$ differentiable at (x_1^*, t_1^*). If $\bar{Q} : \vartheta(x_1^*) \times \vartheta(t_1^*) \to \mathbb{R}^N$ is continuous in $\vartheta(x_1^*)$ for the fixed value t_1^* , $\operatorname{rang}(b(t)) \doteq s$ and for some i_0, $\bar{Q}_{x_1}^{i_0}(x_1^*, t_1^*) \neq 0$, then there exist $\psi(t) \in W_1^1([t_0, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, and $\mu(t) \in L_1(I, \mathbb{R}^s)$, for which holds $\forall u \in U$:*

$$x^*(t) = x_0^* + \int_{t_0}^{t} (A(s)x^*(s) + B(s)u^*(s))\mathrm{d}s, \ \dot{\psi}(t) = -\psi(t)A(t) + \mu(t) \circ h(t), \tag{41}$$

$$\int_{t_0}^{t_1^*} H(\psi(s), u(s))\mathrm{d}s \leqslant \int_{t_0^*}^{t_1^*} H(\psi(s), u^*(s))\mathrm{d}s, \tag{42}$$

$$[\mu(t) \circ h(t), x^*(t)] + \mu(t) \circ b(t)u^*(t) = 0,$$

where $\mu(t)$ comes from the maximum's condition;

$$[\psi(t_1^*), A(t_1^*)x_1^* + B(t_1^*)u^*(t_1^*)] = mQ_{t_1}, \quad \psi(t_1^*) = -mQ_{x_1}. \tag{43}$$

2.4. Case of Irregular Phase Constraints

Now let be given the system $(I)^*$ that consist of (1)–(2), (4)–(5). For this system, we suppose that for each α the constraints' mapping have almost everywhere k_α derivatives. We set $k = \max\{k_\alpha\}$, where k_α is the least number for which

$$\frac{\mathrm{d}^{k_\alpha}}{\mathrm{d}t^{k_\alpha}} \left([\bar{h}_\alpha(t), x(t)] + \bar{b}_\alpha(t)\right) |_{(1)} = [h_\alpha(t), x(t)] + c_\alpha(t)u(t) + \bar{\bar{b}}_\alpha(t),$$

$$c_\alpha(t) \neq 0, \quad c(t) = (c_1, \ldots, c_l), \quad \operatorname{rang}(b(t)) \doteq l.$$

Let define the next variables:

$$y_{\alpha,i}(t) = -\frac{\mathrm{d}^{k_\alpha - i}}{\mathrm{d}t^{k_\alpha - i}} \left([\bar{h}_\alpha(t), x(t)] + \bar{b}_\alpha(t)\right) |_{(1)}, \ i = \overline{1, k_\alpha - 1},$$

$$y_{\alpha,i}(t) = 0, \ i = \overline{k_\alpha, k - 1},$$

$$y_{\alpha,k}^2(t) = -\left([\bar{h}_\alpha(t), x(t)] + \bar{b}_\alpha(t)\right),$$

$$y_i = (y_{1,i}, \ldots, y_{l,i}),$$

$$y_i \dot{y}_j = (y_{1,i}\dot{y}_{1,j}, \ldots, y_{l,i}\dot{y}_{l,j}).$$

Using the properties of the defined parameters, we have

$$\begin{gathered} \dot{y_1}(t) = -[h(t), x(t)] - c(t)u(t) - \bar{\bar{b}}(t), \\ \dot{y_i}(t) = y_{i-1}(t), \quad i = \overline{2, k-1}, \\ 2y_k(t)\dot{y_k}(t) = (y_{1,k_1-1}, \ldots, y_{l,k_l-1}), \\ \bar{x}(t) = (x(t), y_1(t), \ldots, y_k(t)). \end{gathered}$$

The system $(I)^*$ equivalent to a system of the kind (I) with a new

$$\bar{Q}(\bar{x}_0, \bar{x}_1, t_0, t_1) = \begin{pmatrix} Q(x_0, x_1, t_0, t_1) \\ y_{\alpha,1}(t) + \frac{d^{k_\alpha - 1}}{dt^{k_\alpha - 1}}([\bar{h}_\alpha(t), x(t)] + \bar{b}_\alpha(t)) \,|_{(1)}^{t_1} \\ \vdots \\ y_{\alpha,k_\alpha-1}(t) + \frac{d}{dt}([\bar{h}_\alpha(t), x(t)] + \bar{b}_\alpha(t)) \,|_{(1)}^{t_1} \\ \alpha = \overline{1, l} \end{pmatrix}.$$

Hence holds an equivalent of the expression (15) for all variations with some $\rho^1_{\alpha,\beta} \in \mathbb{R}$, $m \in \mathbb{R}^N \setminus \{0\}$ and $\bar{m} = (m, \rho^1_{1,1}, \ldots, \rho^1_{1,k_\alpha}, \ldots, \rho^1_{l,1}, \ldots, \rho^1_{l,k_\alpha})$:

$$\begin{gathered} (\bar{m}\bar{Q}_{x_0})\Delta x_0 + (\bar{m}\bar{Q}_{x_1})\Delta x_1 + (\bar{m}\bar{Q}_{t_0})\delta t_0 + (\bar{m}\bar{Q}_{t_1})\delta t_1 + \\ + \sum_{\alpha=1}^{l} \int_{t_0}^{t_1} \mu_{\alpha,1}(\dot{\delta} y_{\alpha,1} + [h_\alpha(t), \delta x(t)] + c_\alpha(t)\delta u(t))\mathrm{d}t + \\ + \sum_{\alpha=1}^{l} \int_{t_0}^{t_1} \mu_{\alpha,k_\alpha}(2y_{\alpha,k_\alpha}\dot{y}_{\alpha,k_\alpha} - \delta y_{\alpha,k_\alpha-1})\mathrm{d}t + \sum_{\alpha=1}^{l} \sum_{i=2}^{k_\alpha-1} \int_{t_0}^{t_1} \mu_{\alpha,i}(\dot{\delta} y_{\alpha,i} - \delta y_{\alpha,i-1})\mathrm{d}t + \\ + \int_{t_0^*}^{t_1^*} [\psi(t), \dot{\delta} x(t) - A(t)\delta x(t) - B(t)\delta u(t)]\mathrm{d}t \geqslant 0. \end{gathered} \tag{15*}$$

Therefore, we obtain for the optimal control problem the conditions:

$$H(t_1^*) = mQ_{t_1} + nJ_{t_1} + \sum_{\alpha=1}^{l} \sum_{i=1}^{k_\alpha} \rho^1_{\alpha,i} \left(\frac{\partial}{\partial t} \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}} \left[\bar{h}_\alpha(t), x^*(t)\right] - \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left[\bar{h}_\alpha(t), x^*(t)\right] \right) \Bigg|_{(1)}^{t_1^*},$$

$$H\left(t_0^*\right) = -mQ_{t_0} - nJ_{t_0} - \sum_{\alpha=1}^{l} \sum_{i=1}^{k_\alpha} \rho^1_{\alpha,i} \left(\frac{\partial}{\partial t} \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left[\bar{h}_\alpha(t), x^*(t)\right] \right) \Bigg|_{(1)}^{t_0^*},$$

$$\dot{\psi}(t) = -\psi(t)A(t) + \sum_{\alpha=1}^{l} \mu_{\alpha,1} h_\alpha(t), \quad \psi(t_0^*) = mQ_{x_0} + nJ_{x_0}, \tag{44}$$

$$\psi(t_1^*) = -mQ_{x_1} - nJ_{x_1} - \sum_{\alpha=1}^{l} \sum_{i=1}^{k_\alpha} \rho^1_{\alpha,i} \frac{\partial}{\partial x} \frac{d^{i-1}}{dt^{i-1}} [\bar{h}_\alpha(t), x^*(t)] \Bigg|_{(1)}^{t_1^*}, \tag{45}$$

$$\begin{aligned} H\left(u\left(t_1^*\right)\right) - \sum_{\alpha=1}^{l} \mu_{\alpha,1}\left(t_1^*\right) \left(c_\alpha\left(t_1^*\right) u\left(t_1^*\right) + \bar{\bar{b}}\left(t_1^*\right) \right) \leqslant \\ \leqslant H\left(u^*\left(t_1^*\right)\right) - \sum_{\alpha=1}^{l} \mu_{\alpha,1}\left(t_1^*\right) \left(c_\alpha\left(t_1^*\right) u^*\left(t_1^*\right) + \bar{\bar{b}}\left(t_1^*\right) \right), \end{aligned} \tag{46}$$

$$\dot{\mu}_{\alpha,i}(t) = -\mu_{\alpha,i+1}(t), \quad \mu_{\alpha,i}\left(t_1^*\right) = -\rho^1_{\alpha,i}, \quad \mu_{\alpha,i}\left(t_0^*\right) = 0, \quad i = \overline{1, k-1}, \tag{47}$$

$$\mu_{\alpha,1}^{(k_\alpha)}(t)\left([h_\alpha(t),x^*(t)]+c_\alpha(t)u^*(t)+\bar{b}_\alpha(t)\right)=0,\quad \alpha=\overline{1,l}. \tag{48}$$

Varying now the points of contact with the boundary set

$$G=\{x\in X:[\bar{h}_\alpha(t),x(t)]+\bar{b}_\alpha(t)=0\},$$

by setting the points $\tau_1=t_{en},\tau_2=t_{ex}$, where t_{en} — is the point of entry and t_{ex} — is the point of exit, we obtain a varied solution that differs from the optimal solution only in the small neighborhoods of t_{en} and t_{ex}. Then, we set $\Delta x_0=\Delta x(t_{en}^-)=0$, $\Delta x_1=\Delta x(t_{en}^+)$, $\delta y_{\alpha,i}(t_{en}^+)=0$, $\delta y_{\alpha,i}(t_{ex}^-)=0$, $\Delta y_{\alpha,i}(t_{ex}^+)=0$, $i=\overline{2,k_\alpha}$, $\alpha=\overline{1,l}$. Using these considerations in (15*), we obtain:

$$H(t_{en}^+)=H(t_{en}^-)-\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t_{en}^-)\left(\left[h_\alpha\left(t_{en}^-\right),x^*\left(t_{en}^-\right)\right]+c_\alpha\left(t_{en}^-\right)u^*\left(t_{en}^-\right)+\bar{\bar{b}}\left(t_{en}^-\right)\right), \tag{49}$$

$$H\left(t_{ex}^-\right)=H\left(t_{ex}^+\right)-\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{ex}^+\right)\left(\left[h_\alpha\left(t_{ex}^+\right),x^*\left(t_{ex}^+\right)\right]+c_\alpha\left(t_{ex}^+\right)u^*\left(t_{ex}^+\right)+\bar{\bar{b}}\left(t_{ex}^+\right)\right). \tag{50}$$

Therefore, we can introduce the next conditions on the used functions:

Condition 1. $A:I\to\ell(X_\theta,X_\theta)$ *– is measurable, and*

$$A(.)\in W_1^{k-1}(I,\ell(X_\theta,X_\theta)),$$

Condition 2. $B:I\to\ell(\mathbb{R}^r,X_\theta)$ *– is measurable, and*

$$B(.)\in\tilde{\Lambda}_1(I,\ell(\mathbb{R}^r,X_\theta)),$$

Condition 3. $\bar{h}:I\to\ell(X_\theta,\mathbb{R}^l)$ *– is measurable, and* $\bar{h}(.)\in W_1^k(I,\ell(,X_\theta,\mathbb{R}^l))$,

Condition 4. $\bar{b}:I\to\ell(\mathbb{R}^r,\mathbb{R}^l)$ *– is measurable, and* $\bar{b}(.)\in W_1^k(I,\mathbb{R}^l)$,

Condition 5. *Q and J -* $b(X_\theta^2\times\mathbb{R}^2)$*-differentiable at* $(x_0^*,x_1^*,t_0^*,t_1^*)$ *and continuous in its neighborhood, where* $(x_0^*,x_1^*,t_0^*,t_1^*)$ *— is the quasicritical (optimal parameter) of the problem.*

Theorem 2 (Irregular case)**.** *Let hold conditions 1–5. Let* $\{x^*(t),u^*(t),x_0^*,x_1^*,t_0^*\leqslant t\leqslant t_1^*\}$ *— be the measurable optimal solution of the problem. Let* $x^*(t)\in W_1^1(I,X_\theta)$ *and* $u^*(t)\in L_\infty(I,\mathbb{R}^r)$*. If* $\operatorname{rang}(c(t))\dot{=}l$ *and for some* i_0, $\left\{Q_{x_1}^{i_0}\left(x_0^*,x_1^*,t_0^*,t_1^*\right),J_{x_1}\left(x_0^*,x_1^*,t_0^*,t_1^*\right)\right\}$ *is linearly independent, then* $\exists\psi(t)\in W_1^1\left(\left[t_0^*,t_1^*\right],\ell_\gamma\left(X_\theta,\mathbb{R}\right)\right)$*, and* $\mu_{\alpha,i}(t)\in W_1^1(I,\mathbb{R})$*, for which holds :* $\forall u\in U$ $\exists\Pi(u):\bar{\mu}(\Pi(u))=t_1^*-t_0^*$*,*

$$\dot{x}^*(t)=A(t)x^*+B(t)u^*(t), \tag{51}$$

$$\dot{\psi}(t)=-\psi(t)A(t)+\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)\circ h_\alpha(t), \tag{52}$$

$$H(u(t))-\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u(t)\leqslant H(u^*(t))-\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u^*(t), \tag{53}$$

where $\mu_{\alpha,1}(t)$ *comes from the maximum's condition;*

$$H(t_{en}^+)-H\left(t_{en}^-\right)=-\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{en}^-\right)\left(\left[h_\alpha\left(t_{en}^-\right),x^*\left(t_{en}^-\right)\right]+c_\alpha\left(t_{en}^-\right)u^*\left(t_{en}^-\right)+\bar{\bar{b}}\left(t_{en}^-\right)\right),$$

$$H\left(t_{ex}^-\right)-H\left(t_{ex}^+\right)=-\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{ex}^+\right)\left(\left[h_\alpha\left(t_{ex}^+\right),x^*\left(t_{ex}^+\right)\right]+c_\alpha\left(t_{ex}^+\right)u^*\left(t_{ex}^+\right)+\bar{\bar{b}}\left(t_{ex}^+\right)\right),$$

$$H\left(t_0^*\right) = -mQ_{t_0} - nJ_{t_0} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t), x^*(t)\right]\right)\Bigg|_{(1)}^{t_0^*}, \tag{54}$$

$$H\left(t_1^*\right) = mQ_{t_1} + nJ_{t_1} + \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t), x^*(t)\right] - \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t), x^*(t)\right]\right)\Bigg|_{(1)}^{t_1^*}, \tag{55}$$

$$\psi\left(t_0^*\right) = mQ_{x_0} + nJ_{x_0} \quad \textit{and hold (47) and (48),} \tag{56}$$

$$\psi\left(t_1^*\right) = -mQ_{x_1} - nJ_{x_1} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\frac{\partial}{\partial x}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t), x^*(t)\right]\Bigg|_{(1)}^{t_1^*}. \tag{57}$$

We say that a solution of the system of equations is quasicritical for the mapping $\bar{Q}$, if it offers the quasicritical point of this mapping. As consequence of the above theorem we have the integral form of the theorem:

Corollary 5 (Integral form of the conditions of singularity)**.** *Let hold the conditions 1–5. Let* $\{x^*(t),\ u^*(t),\ x_0^*,\ x_1^*,\ t_0^* \leqslant t \leqslant t_1^*\}$ *— be the measurable quasicritical solution of the problem. Let* $x^*(t) \in W_1^1(I, X_\theta)$ *and* $u^*(t) \in L_\infty(I, \mathbb{R}^r)$*. If* $\operatorname{rang}(c(t)) \doteq l$ *and for some* i_0*,* $Q_{x_1}^{i_0} \neq 0$*, then* $\exists\psi(t) \in W_1^1([t_0^*, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$*, and* $\mu_{\alpha,i}(t) \in W_1^1(I, \mathbb{R})$*, for which holds:* $\forall u \in U$*,*

$$x^*(t) = x_0^* + \int_{t_0^*}^{t}\left(A(s)x^*(s) + B(s)u^*(s)\right)\mathrm{d}s, \tag{58}$$

$$\dot{\psi} = -\psi(t)A(t) + \sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)\circ h_\alpha(t). \tag{59}$$

$$\int_{t_0^*}^{t_1^*}\left(H\left(u(t)\right) - \sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u(t)\right)\mathrm{d}t \leqslant \int_{t_0^*}^{t_1^*}\left(H\left(u^*(t)\right) - \sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u^*(t)\right)\mathrm{d}t, \tag{60}$$

where $\mu_{\alpha,1}(t)$ *comes from the maximum's condition;*

$$H\left(t_{en}^+\right) - H\left(t_{en}^-\right) = -\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{en}^-\right)\left(\left[h_\alpha\left(t_{en}^-\right), x^*\left(t_{en}^-\right)\right] + c_\alpha\left(t_{en}^-\right)u^*\left(t_{en}^-\right) + \bar{\bar{b}}\left(t_{en}^-\right)\right),$$

$$H\left(t_{ex}^-\right) - H\left(t_{ex}^+\right) = -\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{ex}^+\right)\left(\left[h_\alpha\left(t_{ex}^+\right), x^*\left(t_{ex}^+\right)\right] + c_\alpha\left(t_{ex}^+\right)u^*\left(t_{ex}^+\right) + \bar{\bar{b}}\left(t_{ex}^+\right)\right),$$

$$H\left(t_0^*\right) = -mQ_{t_0} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t), x^*(t)\right]\right)\Bigg|_{(1)}^{t_0^*}, \tag{61}$$

$$H\left(t_1^*\right) = mQ_{t_1} + \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t), x^*(t)\right] - \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t), x^*(t)\right]\right)\Bigg|_{(1)}^{t_1^*}, \tag{62}$$

$$\psi\left(t_0^*\right) = mQ_{x_0}, \quad \textit{and hold (47) and (48),} \tag{63}$$

$$\psi\left(t_1^*\right) = -mQ_{x_1} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial x}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t), x^*(t)\right]\right)\Bigg|_{(1)}^{t_1^*}. \tag{64}$$

2.5. General Case of Linear Mixed Constraints

It's easy to prove, that other versions of the two theorems also hold for this case. For the case when all the equations are taken into account, the irregular constraints influence the maximums' condition, and using the first two cases and the condition of regularization of the general system, we can state the following theorem:

Theorem 3 (General case). *Let hold the conditions 1–5. Let* $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ *— be the measurable quasicritical solution of the problem* (1)–(5). *Let* $x^*(t) \in W_1^1(I, X_\theta)$ *and* $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. *If* $\operatorname{rang}(b(t), c(t)) \doteq s + l$. *Then* $\exists \psi(t) \in W_1^1([t_0^*, t_1^*], \ell_\gamma(X_\theta, \mathbb{R}))$, *and* $\mu_{\alpha,i}(t) \in W_1^1(I, \mathbb{R})$, $\mu(t) \in L_1(I, \mathbb{R}^s)$, *for which holds:* $\forall u \in U$,

$$x^*(t) = x_0^* + \int_{t_0^*}^{t} (A(s)x^*(s) + B(s)u^*(s))\,\mathrm{d}s, \tag{65}$$

$$\dot{\psi} = -\psi(t)A(t) + \sum_{\alpha=1}^{l} \mu_{\alpha,1}(t) \circ h_\alpha(t) + \mu(t) \circ h(t), \tag{66}$$

$$\int_{t_0^*}^{t_1^*} (H(u(t)) - \sum_{\alpha=1}^{l} \mu_{\alpha,1}(t)c_\alpha(t)u(t))\mathrm{d}t \leqslant \int_{t_0^*}^{t_1^*} (H(u^*(t)) - \sum_{\alpha=1}^{l} \mu_{\alpha,1}(t)c_\alpha(t)u^*(t))\mathrm{d}t, \tag{67}$$

where $\mu_{\alpha,1}(t)$, $\mu(t)$ *comes from the maximum's condition;*

$$H(t_0^*) = -m\bar{Q}_{t_0} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha} \rho_{\alpha,i}^1 \left(\frac{\partial}{\partial t} \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left[\bar{h}_\alpha(t), x^*(t)\right] \right)\Bigg|_{(1)}^{t_0^*}, \tag{68}$$

$$H(t_1^*) = m\bar{Q}_{t_1} + \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha} \rho_{\alpha,i}^1 \left(\frac{\partial}{\partial t} \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}} \left[\bar{h}_\alpha(t), x^*(t)\right] - \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left[\bar{h}_\alpha(t), x^*(t)\right] \right)\Bigg|_{(1)}^{t_1^*}, \tag{69}$$

$$\psi(t_0^*) = mQ_{x_0}, \tag{70}$$

$$\psi(t_1^*) = -m\bar{Q}_{x_1} - \sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha} \rho_{\alpha,i}^1 \left(\frac{\partial}{\partial x} \frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}} \left[\bar{h}_\alpha(t), x^*(t)\right] \right)\Bigg|_{(1)}^{t_1^*}, \tag{71}$$

$$H(t_{en}^+) - H(t_{en}^-) = -\sum_{\alpha=1}^{l} \mu_{\alpha,1}(t_{en}^-) \left([h_\alpha(t_{en}^-), x^*(t_{en}^-)] + c_\alpha(t_{en}^-)u^*(t_{en}^-) + \bar{\bar{b}}(t_{en}^-) \right),$$

$$H(t_{ex}^-) - H(t_{ex}^+) = -\sum_{\alpha=1}^{l} \mu_{\alpha,1}(t_{ex}^+) \left([h_\alpha(t_{ex}^+), x^*(t_{ex}^+)] + c_\alpha(t_{ex}^+)u^*(t_{ex}^+) + \bar{\bar{b}}(t_{ex}^+) \right),$$

$$[\mu(t)h(t), A(t)x(t)] + \mu(t)b(t)u^*(t) = 0,$$

and hold (47) and (48).

As a consequence we have the Pontryagin's principle of maximum for linear optimal control problems:

Corollary 6 (General Case for the optimal control problem). *Let hold the conditions 1–5. Let* $\{x^*(t), u^*(t), x_0^*, x_1^*, t_0^* \leqslant t \leqslant t_1^*\}$ *— be the measurable optimal solution of the general problem. Let* $x^*(t) \in W_1^1(I, X_\theta)$ *and* $u^*(t) \in L_\infty(I, \mathbb{R}^r)$. *If* $\operatorname{rang}(b(t), c(t)) \doteq s + l$ *and for some* i_0, $\{Q_{x_1}^{i_0}(x_0^*, x_1^*, t_0^*, t_1^*), J_{x_1}(x_0^*, x_1^*, t_0^*, t_1^*)\}$ *is* *linearly independent, then* $\exists\psi(t)\in W_1^1([t_0^*,t_1^*],\ell_\gamma(X_\theta,\mathbb{R}))$, *and* $\mu_{\alpha,i}(t)\in W_1^1(I,\mathbb{R})$, $\mu(t)\in L_1(I,\mathbb{R}^s)$, *for which holds:* $\forall u\in U \quad \exists\Pi(u):\bar{\mu}(\Pi(u))=t_1^*-t_0^*$,

$$\dot{x}^*(t)=A(t)x^*(t)+B(t)u^*(t), \tag{72}$$

$$\dot{\psi}(t)=-\psi(t)A(t)+\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)\circ h_\alpha(t)+\mu\circ h(t), \tag{73}$$

$$H(u(t))-\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u(t)\leqslant H(u^*(t))-\sum_{\alpha=1}^{l}\mu_{\alpha,1}(t)c_\alpha(t)u^*(t), \tag{74}$$

where $\mu_{\alpha,1}(t)$, $\mu(t)$ *come from the maximum's condition, hold (47) and (48);*

$$H\left(t_{en}^+\right)-H\left(t_{en}^-\right)=-\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{en}^-\right)\left(\left[h_\alpha\left(t_{en}^-\right),x^*\left(t_{en}^-\right)\right]+c_\alpha\left(t_{en}^-\right)u^*\left(t_{en}^-\right)+\bar{\bar{b}}\left(t_{en}^-\right)\right),$$

$$H\left(t_{ex}^-\right)-H\left(t_{ex}^+\right)=-\sum_{\alpha=1}^{l}\mu_{\alpha,1}\left(t_{ex}^+\right)\left(\left[h_\alpha\left(t_{ex}^+\right),x^*\left(t_{ex}^+\right)\right]+c_\alpha\left(t_{ex}^+\right)u^*\left(t_{ex}^+\right)+\bar{\bar{b}}\left(t_{ex}^+\right)\right),$$

$$H\left(t_0^*\right)=-mQ_{t_0}-nJ_{t_0}-\sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t),x^*(t)\right]\right)\Bigg|_{(1)}^{t_0^*}, \tag{75}$$

$$H\left(t_1^*\right)=mQ_{t_1}+nJ_{t_1}+\sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial t}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t),x^*(t)\right]-\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\bar{h}_\alpha(t),x^*(t)\right]\right)\Bigg|_{(1)}^{t_1^*}, \tag{76}$$

$$\psi\left(t_0^*\right)=mQ_{x_0}+nJ_{x_0},\quad [\mu(t)h(t),A(t)x(t)]+\mu(t)b(t)u^*(t)=0, \tag{77}$$

$$\psi\left(t_1^*\right)=-mQ_{x_1}-nJ_{x_1}-\sum_{\alpha=1}^{l}\sum_{i=1}^{k_\alpha}\rho_{\alpha,i}^1\left(\frac{\partial}{\partial x}\frac{\mathrm{d}^{i-1}}{\mathrm{d}t^{i-1}}\left[\bar{h}_\alpha(t),x^*(t)\right]\right)\Bigg|_{(1)}^{t_1^*}. \tag{78}$$

In addition to this, one can obviously show that, if U is a bounded convex set, then the optimal control $u^*(t)$ takes values on its boundary, precisely on the intersections of consecutive components of this boundary [2, § 17].

The results of this paper also hold if in (1)–(5) the equation of the trajectory has the form $\dot{x}=A(t)x(t)+u(t)B(t)x(t)+C(t)u(t)$ and the constraints are of the form $[h^i(t)+u(t)k^i(t),x(t)]+b(t)u(t)\geqslant 0$. The only difference between this version and the detailed one is given by the conditions on the constraints.

In this work $H^*(t)$, $H(u^*(t))$ denotes $H(\psi(t),u^*(t))$.

Remark 3. The theory seems to be a familiar one, but if one doesn't understand the meaning of the sequently continuity, the derivation by virtue of the system of bounded subsets γ, the operators' equivalency in topological Banach spaces, the integrability with respect to the topology, he will be unable to value this paper, as it's very easy to get lost thinking of the usual problems [1, § 6.1].

References

1. *Сухинин М. Ф.* Избранные главы нелинейного анализа. — М.: Изд-во РУДН, 1992.
2. *Понтрягин Л. С.* Математическая теория оптимальных процессов. — М.: Наука, 1976.
3. *Гамкрелидзе Р. В., Харатишвили Г. Л.* Экстремальные задачи в линейных топологических пространствах // Известия АН СССР. Сер. Мат. — Т. 33, № 4. — 1969. — С. 781–839.
4. *Сухинин М. Ф.* Об ослабленным варианте правила множителей Лагранжа в банаховом пространстве // Математические заметки. — Т. 21, № 2. — 1977.

5. *Дмитрук А. В.* К обоснованию метода скользящих режимов для задач оптимального управления со смешанными ограничениями // Функциональный анализ и его приложения. — Т. 10, № 3. — 1976.
6. *Дмитрук А. В., Милютин А. А., Осмоловский Н. П.* Теорема Люстерника и теория экстремума // Успехи математических наук. — Т. 35, № 6. — 1980.
7. *Колмогоров А. Н., Фомин С. В.* Элементы теории функций и функционального анализа. — М.: Наука, 1972.
8. *Алексеев В. М., Тихомиров В. М., Фомин С. В.* Оптимальное управление. — М.: Наука, 1979.
9. *Васильев Ф. П.* Методы решений экстремальных задач. — М.: Наука, 1981.
10. *Лонгла М.* Условия оптимальности в бесконечномерном пространстве. — М.: ВИНИТИ № 412–В2008, 2008.

УДК 517.95

Принцип максимума Понтрягина в линейных задачах со смешанными ограничениями в бесконечномерном пространстве

М. Лонгла

Кафедра дифференциальных уравнений и математической физики
Российский университет дружбы народов
ул. Миклухо-Маклая, 6, Москва, Россия, 117198

Выведены необходимые условия оптимальности в некоторых задачах с линейными регулярными и нерегулярными ограничениями в нормированном пространстве с особой отделимой локально выпуклой топологией, основываясь на трудах М.Ф. Сухинина. Используемые функции могут не быть интегрируемыми по Бохнеру и не быть дифференцируемыми по Гато в обычном смысле. Здесь изложена попытка обобщать результаты, полученные в конечномерных пространствах Л. Грейвзом, Л.С. Понтрягиным, В.Г. Болтянским, Р. В. Гамкрелидзе, А.В. Дмитруком, А.А. Милютиным, Е.Ф. Мищенко, Мак-Шейном и др. Не исследованные задачи описанного выше типа рассматриваются в данной работе, опираясь на теории дифференцирования по системе подмножеств, эквивалентности функций и операторов в локально выпуклом банаховом пространстве, и интегрирования по локально выпуклой топологии, изложенной М.Ф. Сухининым в своей монографии [1]. Сформулированы и доказаны теоремы для случая, когда фазовые ограничения и смешанные ограничения суть линейные функции траектории и управления в бесконечномерном локально выпуклом отделимом пространстве с нормой.