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Stabilization of Redundantly Constrained Dynamic System

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This article addresses the issue of constraint stabilization in a dynamic system. The well known Lagrange's equation of motion of second order is used for modelling the dynamics of a mechanical systems considered in this paper. It is known that Baumgarte's method of constraint stabilization does not avoid the problem of singularity of mass matrices that may result from redundancy of constraints and as a result it fails to run simulations near and at singularity points. A generalized Baumgarte's method of constraint stabilization is developed and the stability of the developed method is ascertained by Lyapunov's direct method. The developed method avoids using the same correction parameters for all constraints under discussion. The usual Baumgarte's method, which uses the same correction parameters, becomes a particular case of the one developed in this article. Moreover, a modified Lagrange's equation is constructed in a way that explains all the details of its derivation. The modified Lagrange's equation improves Lagrange's equation of motion in such a way that, it addresses the issue of redundant constraints and singular mass matrices. As it is the case in Baumgarte's method, the usual Lagrange's equation is a particular case of the improved method developed in this paper. Besides, a numerical example is provided in order to demonstrate the effectiveness of the methods developed. Finally, the carried out simulations show asymptotic stability of the trajectories and run without problem at singularity points.

Key words and phrases: stability, generalized Baumgarte's method, modified Lagrange's equation, singular mass matrices, redundant constraints, Lyapunov's direct method.

1. Introduction

One of the commonly used method of modeling Dynamics of Constrained mechanical systems is Lagrange's equation of motion [1–3]. This method of constructing motion equations of a mechanical system results in a set of Index 3 Differential Algebraic Equations (DAE). These set of DAE of motion does not use explicitly the position and velocity equations associated to the constraints, as a result of which the problem of stability in the position and velocity level came into being. To overcome these deviations of the trajectory of the system from the constraints, we need to use constraint stabilization methods. The strategies generally used to overcome this problem are the Baumgarte's stabilization method [4] and penalty method [5, 6].

Moreover, in addition to the stability problems that may happen in mechanical systems, the presence of redundant constraints is also unavoidable in practice. In the presence of more equations than strictly needed the Jacobian matrix becomes rank deficient. This is reflected by the fact that some of the equations are dependent on the remaining ones. The Jacobian matrix can also be rank deficient when the mechanical system reaches a configuration in which there is a sudden change in the number of degree of freedom. For instance, a slider crank mechanism [5] reaches a singular configuration when both two links are at vertical position. In this position both links overlap and the mechanism has not one but two degree of freedom that corresponds to two possible motions that the mechanism can undergo. This makes the Jacobian matrix singular.

It is also known that Baumgarte's method of constraint stabilization does not avoid the problem of redundancy explained above, as it may fail to run around singularity

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points and in the presence of redundant constraints [5, 6]. Moreover, it can be observed that since the same correction parameters say, α and β, in the sense that, the same α and the same β for all the constraints are used in Baumgarte's stabilization method, some violations will be eliminated while others may not be. The so called penalty method discussed in [5, 6] explicitly addresses the problem of redundancy in contrary to Baumgarte's method. But in this method, like in Baumgarte's method, the same correction parameters are used, as a result of which some violations will be eliminated while others may not be. The challenge of choosing the correct parameters for stabilization is a common problem to both Baumgarte's and penalty methods.

Based on the analysis given above, this paper generalizes both the Baumgarte's and penalty methods. A modified form of Lagrange's equation of motion is developed in a way that keeps the advantage of addressing the issue of redundant constraints and singular mass matrices. The generalization of the above methods is done in the sense that, the problem of using the same correction parameters for all the constraints is avoided. As a result, we can choose different constants as needed while making experimentation by simulation to stabilize the system.

The organization of the paper is as follows: In section 2, Lagrange's equation of motion is revisited in a way that it addresses problems related to singular mass matrices and singular Jacobian matrix. This section is mainly based on [7]. Section 3 has two parts: In section 3.1, Generalization of Baumgarte's method of constraint stabilization and stability of the generalized method based on Lypunov's direct method is developed. In section 3.2 Modified Lagrange's equation is developed in such a way that the constructed formula improves Lagrange's equation of motion and generalizes penalty method. In section 4 a numerical example is given to show the effectiveness of the methods developed in simulating dynamic systems.

2. Dynamic Equations of Motion

Dynamics of mechanical systems can be described by second order Lagrange's equations of motion. Applying the Lagrange multiplier method, the dynamic equation of n-generalized coordinate mechanical system subjected to m ($m < n$) holonomic constraints [1–3] can be written in descriptor form as:

$$M(\mathbf{q})\ddot{\mathbf{q}} + \Theta_{\mathbf{q}}^T \lambda = Q(\mathbf{q}, \dot{\mathbf{q}}, t), \tag{1}$$

$$\Theta(\mathbf{q}, t) = 0. \tag{2}$$

Eq. (2) denotes the holonomic constraint equations, $M \in \Re^{n\times n}$ is the mass matrix, $\lambda \in \Re^m$ is the Lagrangian multipliers, $\Theta_{\mathbf{q}} \in \Re^{m\times n}$ is the Jacobian matrix of the constraints. $Q \in \Re^n$ is the generalized force, $\mathbf{q} \in \Re^n$ is the generalized coordinates of the system.

Therefore the governing equations of a constrained mechanical system can be described by a set of n differential equations (1) and m algebraic equations (2). To eliminate the Lagrange multipliers, one can differentiate Eq. (2) with respect to time, thus yielding,

$$\dot{\Theta}(\mathbf{q}, \dot{\mathbf{q}}, t) = \Theta_{\mathbf{q}}\dot{\mathbf{q}} - \eta = 0. \tag{3}$$

Equation (3) represents the velocity constraint equations, where $\eta = -\Theta_t$. Differentiating Eq. (3) with respect to time again, one can obtain,

$$\ddot{\Theta}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, t) = \Theta_{\mathbf{q}}\ddot{\mathbf{q}} - \xi. \tag{4}$$

Equation (4) is the acceleration constraint equation, where

$$\xi = -(\Theta_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}} - 2\Theta_{\mathbf{q}t}\dot{\mathbf{q}} - \Theta_{tt}. \tag{5}$$

Here the holonomic constraints may be dependent or independent. Accordingly the inverse of an $m \times m$ matrix $A = \Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^{T}$ may or may not exist. Hence in general we have from Eq. (1) and Eq. (4),

$$\lambda = A^{+}(\Theta_{\mathbf{q}} M^{-1} Q - \xi) + (I - A^{+}A)u, \tag{6}$$

where A^{+} is the Generalized Inverse of matrix A , u is an arbitrary n-vector and I is an identity matrix of appropriate size. Hence the condensed form of equations (1) and (2) can be written as:

$$M\ddot{\mathbf{q}} = Q^{*}, \tag{7}$$

where $Q^{*} = Q - \Theta_{\mathbf{q}}^{T}[A^{+}(\Theta_{\mathbf{q}} M^{-1} Q - \xi) + (I - A^{+}A)]u$.

It can be observed that the quantity in Eq. (7) given by:

$$Q^{c} \equiv \Theta_{\mathbf{q}}^{T}[A^{+}(\Theta_{\mathbf{q}} M^{-1} Q - \xi) + (I - A^{+}A)]u \tag{8}$$

is the constraint force of the system. The constraint force is given as a sum of two components [7]. The first component is the extent to which the acceleration of the unconstrained system deviates from the acceleration of the constrained system with constant of proportionality $\Theta_{\mathbf{q}}^{T} A^{+}$ and the second component is proportional to n-vector u with constant of proportionality $\Theta_{\mathbf{q}}^{T}[(I - A^{+}A)]$.

Moreover, referring to the dynamic equation of the constraints given by Eq. (4), a non-zero virtual displacement vector δr such that $\Theta_{\mathbf{q}} \delta r = 0$ (δr is in the null space of $\Theta_{\mathbf{q}}$) at time t is said to be a virtual displacement [7]. Suppose at a particular instance of time t, the virtual work done by the constraint force, $(\delta r)^{T} Q^{c} = 0$. At this instance of time we have:

$$0 = (\delta r)^{T} Q^{c} = (\delta r)^{T} \Theta_{\mathbf{q}}^{T} A^{+}(\Theta_{q} M^{-1} Q - \xi) + (\delta r)^{T} \Theta_{\mathbf{q}}^{T}(I - A^{+}A)u. \tag{9}$$

For a virtual displacement $\delta\mu$ put $\delta r = M^{-1}\Theta_{\mathbf{q}}\delta\mu$. Then $\Theta_{\mathbf{q}}\delta r = \Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^{T} \delta\mu = A\delta\mu = 0$. This indicates that, if δr is in the Null space of $\Theta_{\mathbf{q}}$, then $\delta\mu$ is in the Null space of $A = \Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^{T}$. The general solution of $A\delta\mu = 0$ is $= (I - A^{+}A)\delta w$ for an n vector w. This in turn indicates that:

$$\delta r = M^{-1} \Theta_{\mathbf{q}}^{T}(I - A^{+}A)\delta w. \tag{10}$$

Substituting Eq. (10) into Eq. (9) we have:

$$\begin{aligned} 0 &= (\delta r)^{T} Q^{c} = \\ &= (\delta w)^{T} M^{-1} \Theta_{\mathbf{q}}^{T}(I - A^{+}A)\Theta_{\mathbf{q}}^{T}[A^{+}(\Theta_{\mathbf{q}} M^{-1} Q - \xi)] + (\delta r)^{T} \Theta_{\mathbf{q}}^{T}(I - A^{+}A)u = \\ &= (\delta w)^{T} A(I - A^{+}A)u. \end{aligned}$$

From which it follows that:

$$(I - A^{+}A)u = 0. \tag{11}$$

We can conclude from the equations (9), (10) and (11) and the discussions then made that the equation given by Eq. (8) is the sum of two constraint forces resulting from the ideal and non-ideal nature of the constraints. As it is shown above using principle of virtual displacement the constraint force due to the ideal nature of the constraints is given by

$$Q^{ic} \equiv \Theta_{\mathbf{q}}^{T} A^{+}(\Theta M^{-1} Q - \xi) \tag{12}$$

and the constraint force resulting from non-ideal nature of the constraints is given by:

$$Q^{nic} \equiv \Theta_{\mathbf{q}}^T (I - A^+ A) u. \tag{13}$$

Remark 1. Equation (11) shows that when the constraint is ideal, then $(I - A^+ A)u = 0$. As a result we can have the following proposition.

Proposition 1. A holonomic constraint (2) applied to a Lagrange's Mechanical system given by equation (1) is said to be Ideal constraint if and only if $(I - A^+ A)u = 0$ where $A = \Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^T$ and u is arbitrary vector. This theorem is an extension of the definition of Ideal constraint given by R. G. Mykharlyamov in [2, P. 81].

Remark 2. If the holonomic constraints are assumed to be independent, then the inverse of the $m \times m$ matrix, $A^{-1} = (\Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^T)^{-1}$ exists and then, since in this case $A^{-1} = A^+$, the governing equation of the system indicated in Eq. (7) reduces to:

$$M\ddot{\mathbf{q}} = Z^*, \tag{14}$$

where $Z^* = Q - \Theta_{\mathbf{q}}^T (\Theta_{\mathbf{q}} M^{-1} \Theta_{\mathbf{q}}^T)^{-1} (\Theta_{\mathbf{q}} M^{-1} Q - \xi)$.

It also needs to be noted that in the case when $A^{-1} = A^+$ we have only Ideal constrains, since $(I - A^+ A) = 0$.

3. Stabilization of the Constraint Equations

When we intend to solve the dependent acceleration from Eq. (14) using standard ODE solvers, Eq. (2) and (3) may not be satisfied. This is because Eq. (4) which are obtained by differentiating Eq. (2) twice with respect to time is used. This in turn implies that the system of equation given by Eq. (4) is unstable. This requires the incorporation of stabilization to the system as discussed below.

3.1. Generalization of Baumgarte's Constraint Violation Stabilization Method

When the violations in Eq. (2) and Eq. (3) are beyond a specified error tolerance, Baumgarte's stabilization method [4] replaces $\ddot{\Theta}$ in Eq. (4) by:

$$\ddot{\Theta} + 2\alpha\dot{\Theta} + \beta^2\Theta, \tag{15}$$

where α and β are appropriately chosen correction parameters to make the origin asymptotically stable. It can be observed that different types of violations in Eq. (2) and Eq. (3) are not considered by Eq. (15) since the same correction parameters α and β are used. Hence some violations will be eliminated while others may not be. Therefore, let us consider a general case of (15) by replacing it with:

$$\ddot{\Theta} + K_D\dot{\Theta} + K_P\Theta, \tag{16}$$

where K_D and K_P are constant positive definite symmetric matrices. Equation (16) may be expressed in terms of $[\Theta^T, \dot{\Theta}^T]$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\Theta \\ \dot{\Theta}\end{pmatrix} = \begin{pmatrix}\dot{\Theta} \\ -K_D\dot{\Theta} - K_P\Theta\end{pmatrix} = \begin{pmatrix}0 & I \\ -K_P & -K_D\end{pmatrix}\begin{pmatrix}\Theta \\ \dot{\Theta}\end{pmatrix}, \tag{17}$$

where I is the identity matrix of size m.

Let us investigate the stability of the origin in (17): The immediate Lyapunaov function candidate is:

$$V(\Theta,\dot\Theta)=\frac{1}{2}\begin{pmatrix}\Theta\\ \dot\Theta\end{pmatrix}^T\begin{pmatrix}K_p+\epsilon K_D & \epsilon I\\ \epsilon I & I\end{pmatrix}\begin{pmatrix}\Theta\\ \dot\Theta\end{pmatrix}=$$

$$=\frac{1}{2}\left(\dot\Theta+\epsilon\Theta\right)^T\left(\dot\Theta+\epsilon\Theta\right)+\frac{1}{2}\Theta^T[K_P+\epsilon K_D-\epsilon^2 I]\Theta, \quad (18)$$

where the constant ϵ satisfies:

$$K_D-\epsilon I>0,$$

$$K_P+\epsilon K_D-\epsilon^2 I>0.$$

Evaluating the total time derivative of $V(\Theta,\dot\Theta)$ we obtain:

$$\dot V(\Theta,\dot\Theta)=-\begin{bmatrix}\Theta\\ \dot\Theta\end{bmatrix}\begin{bmatrix}\epsilon K_P & 0\\ 0 & K_D-\epsilon I\end{bmatrix}\begin{bmatrix}\Theta\\ \dot\Theta\end{bmatrix}. \quad (19)$$

Equation (19) is globally negative definite and as a result we conclude that $(\Theta,\dot\Theta)=(0,0)$ is globally asymptotically stable.

Remark 3. For practical purposes we can choose $K_D=diag(2k_1,2k_2,\ldots,2k_m)$ and $K_P=diag(k_1^2,k_2^2,\ldots,k_m^2)$.

Now ξ of Eq. (5) have a new form given as:

$$\Xi=-(\Theta_{\mathbf{q}}\dot{\mathbf{q}})_{\mathbf{q}}\dot{\mathbf{q}}-2\Theta_{\mathbf{q}t}\dot{\mathbf{q}}-\Theta_{tt}-K_D\dot\Theta-K_p\Theta. \quad (20)$$

We can now write Eq. (14), after including Generalized Baumgarte's constraint stabilization method we developed, as follows

$$M\ddot q=Q-\Theta_{\mathbf{q}}^T(\Theta_{\mathbf{q}}M^{-1}\Theta_{\mathbf{q}}^T)^{-1}(\Theta_{\mathbf{q}}M^{-1}Q-\Xi), \quad (21)$$

where Ξ is given in (20).

Equation (14) can then be integrated numerically as before with the modified Ξ. One of the advantages of using Generalized Baumgarte's constraint stabilization method, as can be seen above is, it reduces the Index 3 DAE given by Eqs. (1) and (2) to DAE of index 0, which is purely a Differential Equation. Hence we can use any of the standard numerical methods to solve the resulting differential equation such as Runge–Kutta method. The other advantage is, it uses different correction parameters for different constraints if need be, that may help to reduce all the constraint violations in contrary to Eq. (15).

However there are some drawbacks in this method:

a) the selection of correction parameters in K_D and K_P has no certain rules to follow;
b) the other problem with Generalized Baumgarte's stabilization developed in this paper is that, it does not solve all possible instabilities, such as near kinematic singular configurations.

These difficulties work in favor of other constraint stabilization method developed below.

3.2. Modified Lagrange's Equation Technique of Constraint Stabilization

As it is stated in [8] in order to stabilize the constraints in (2) and (3) it is necessary to take account of the deviation from equations and introduce a corresponding correction to the dynamic equation of the system. Let the deviation of the constraints be denoted by vectors $\mathbf{y}$ and $\dot{\mathbf{y}}$ whose coordinates are called excess variables such that:

$$\Theta(\mathbf{q}, t) = \mathbf{y}, \tag{22}$$

$$\Theta_{\mathbf{q}}\dot{\mathbf{q}} + \dot{\Theta}_t = \dot{\mathbf{y}}, \tag{23}$$

where $\mathbf{y} = (y^1, y^2, ..., y^m)$ and $\dot{\mathbf{y}} = (\dot{y}^1, \dot{y}^2, ..., \dot{y}^m)$.

Now considering the excess variables, the mechanical system is determined by the generalized coordinates, $\mathbf{u} = \{\mathbf{q}, \mathbf{y}\}$ and the generalized velocities $\dot{\mathbf{u}} = \{\dot{\mathbf{q}}, \dot{\mathbf{y}}\}$. Let T^0 the kinetic energy, V^0 the potential energy, D^0 the dissipative function of the system before it is subjected to the m-constraints such that,

$$T^0 = T^0(\mathbf{q}, \dot{\mathbf{q}}), \quad V^0 = V^0(\mathbf{q}), \quad D^0 = D^0(\mathbf{q}, \dot{\mathbf{q}})$$

Let T the kinetic energy, V the potential Energy, D the dissipative function of the constrained system subjected to the m-constraints such that,

$$T = T(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{y}, \dot{\mathbf{y}}), \quad V = V(\mathbf{q}, \mathbf{y}), \quad D = D(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{y}, \dot{\mathbf{y}}).$$

We now expand the constrained mechanical system by Taylor's expansion method, around $\mathbf{y} = 0$, $\dot{\mathbf{y}} = 0$. Assume T, V and D are at least twice differentiable with respect to all the variables and the system is at an equilibrium position for $\mathbf{y} = 0$, $\dot{\mathbf{y}} = 0$.

Putting $T^0 = T^0(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum\limits_{i,j=1}^{m}(m_{ij})\dot{q}^i\dot{q}^j$ we obtain:

$$T = \frac{1}{2}\sum_{i,j=1}^{m}(m_{ij})\dot{q}^i\dot{q}^j + \frac{1}{2}\sum_{i,j}^{m}(a_{ij})\dot{y}^i\dot{y}^j) + T^3,$$

where $a_{ij} = \dfrac{\partial^2 T(\mathbf{q}, \dot{\mathbf{q}}, 0, 0)}{\partial y^i \partial y^j}$ which is symmetric in the indices i and j. Similarly,

$$V = V^0(\mathbf{q}) + \frac{1}{2}\sum_{i,j=1}^{m}(v_{ij})y^iy^j + V^3,$$

where $v_{ij} = \dfrac{\partial^2 V(\mathbf{q}, \dot{\mathbf{q}}, 0, 0)}{\partial y^i \partial y^j}$ which is symmetric in the indices i and j. And

$$D = D^0(\mathbf{q}, \dot{\mathbf{q}}) + \frac{1}{2}\sum_{i,j}^{m}(c_{ij})\dot{y}^i\dot{y}^j + D^3,$$

where $c_{ij} = \dfrac{\partial^2 D(\mathbf{q}, \dot{\mathbf{q}}, 0, 0)}{\partial y^i \partial y^j}$ which is symmetric in the indices i and j. Hence, ignoring higher order terms we have:

$$\begin{cases} T = \frac{1}{2}\sum\limits_{i,j=1}^{n}(m_{ij})\dot{q}^i\dot{q}^j + \frac{1}{2}\sum\limits_{i,j=1}^{m}(a_{ij})\dot{y}^i\dot{y}^j, \\ V = V^0(\mathbf{q}) + \frac{1}{2}\sum\limits_{i,j=1}^{m}(v_{ij})y^iy^j, \\ D = D^0(\mathbf{q}, \dot{\mathbf{q}}) + \frac{1}{2}\sum\limits_{ij}(c_{ij})\dot{y}^i\dot{y}^j, \end{cases} \tag{24}$$

which is the result obtained in [8].

Let $M = \sum\limits_{i,j=1}^{n}(m_{ij})$, $A = \sum\limits_{i,j=1}^{m}(a_{ij})$, $K_P = \sum\limits_{i,j=1}^{m}(v_{ij})$, $K_D = \sum\limits_{i,j=1}^{m}(c_{ij})$. Note also that the coefficients a_{ij}, v_{ij} and c_{ij} in general depend on the generalized coordinates $\mathbf{q}$ and $\dot{\mathbf{q}}$. In this paper they are assumed to be constants. It is also assumed that matrices M, A, K_P and K_D, that contain the value of the coefficients for the different constraint equations, do not account for the coupling between the different constraints and hence the entries off the diagonals can be taken to be zero and therefore the matrices in this case are each diagonal matrices. Assume also that the matrices are each positive definite. Now Eq. (24) can be written in the form:

$$\begin{cases} T = \dfrac{1}{2}\dot{\mathbf{q}}^T M\dot{\mathbf{q}} + \dfrac{1}{2}(\dot{\mathbf{y}})^T A\dot{\mathbf{y}}, \\ V = V^0(\mathbf{q}) + \dfrac{1}{2}(\mathbf{y})^T K_P\mathbf{y}, \\ D = D^0(\mathbf{q},\dot{\mathbf{q}}) + \dfrac{1}{2}(\dot{\mathbf{y}})^T K_D\dot{\mathbf{y}}. \end{cases} \tag{25}$$

The perturbation expression can be seen to be given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\mathbf{y}}}\left(\frac{1}{2}\dot{\mathbf{y}}^T A\dot{\mathbf{y}}\right) - \frac{\partial}{\partial\mathbf{y}}\left(\frac{1}{2}\dot{\mathbf{y}}^T A\dot{\mathbf{y}}\right) + \frac{\partial}{\partial\mathbf{y}}\left(\frac{1}{2}\mathbf{y}^T K_P\mathbf{y}\right) + \frac{\partial}{\partial\dot{\mathbf{y}}}\left(\frac{1}{2}\dot{\mathbf{y}}^T K_D\dot{\mathbf{y}}\right). \tag{26}$$

Each of the expressions in Eq. (26) can be simplified as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{\mathbf{y}}}\left(\frac{1}{2}\dot{\mathbf{y}}^T A\dot{\mathbf{y}}\right) = \left((\mathbf{y}^T)_{\mathbf{y}}\right) A\ddot{\mathbf{y}} + \left((\dot{\mathbf{y}}^T)_{\mathbf{y}} A\dot{\mathbf{y}}\right),$$

$$\frac{\partial}{\partial\mathbf{y}}\left(\frac{1}{2}\mathbf{y}^T K_P\mathbf{y}\right) = \left((\dot{\mathbf{y}}^T)_{\mathbf{y}}\right) A\dot{\mathbf{y}},$$

$$\frac{\partial}{\partial\dot{\mathbf{y}}}\left(\frac{1}{2}\dot{\mathbf{y}}^T\dot{\mathbf{y}}\right) = \left((\mathbf{y}^T)_{\mathbf{y}}\right) K_D\dot{\mathbf{y}}.$$

Combining all the above terms yields a relatively simple form of the perturbation expression given by:

$$(\mathbf{y}^T)_{\mathbf{y}}(A\ddot{\mathbf{y}} + K_D\dot{\mathbf{y}} + K_P\mathbf{y}). \tag{27}$$

Expression (27) is said to be the arising force that resist the constraint violation where A is the value of the constraining masses, K_D is the damping coefficient and K_P is stiffness coefficient [9].

The dynamic system is achieved by combining the equation for the unconstrained system (taking $D^0(\mathbf{q},\dot{\mathbf{q}}) = 0$) perturbation expression and is given in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T^0}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial T^0}{\partial\mathbf{q}} + \frac{\partial V^0}{\partial\mathbf{q}} = Q_{ex} - (\mathbf{y}^T)_{\mathbf{y}} A(\ddot{\mathbf{y}} + K_D\dot{\mathbf{y}} + K_P\mathbf{y}), \tag{28}$$

where Q_{ex} is an external force applied to the system. Now from $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial T^0}{\partial\dot{\mathbf{q}}}\right) - \dfrac{\partial T^0}{\partial\mathbf{q}} + \dfrac{\partial V^0}{\partial\mathbf{q}}$ we obtain $M\ddot{\mathbf{q}} = Q$, where $Q = L_{\mathbf{q}} + Q_{ex} - \dot{M}\dot{\mathbf{q}}$ and $L^0_{\mathbf{q}} = \dfrac{\partial(T^0 - V^0)}{\partial\mathbf{q}}$. Therefore Eq. (28) can be written as:

$$M\ddot{\mathbf{q}} + \left((\mathbf{y}^T)_{\mathbf{y}}\right) A(\ddot{\mathbf{y}} + K_D\dot{\mathbf{y}} + K_P\mathbf{y}) = Q. \tag{29}$$

Substituting $\ddot{\mathbf{y}}, \dot{\mathbf{y}}$ and $\mathbf{y}$ from Eq. (22) and Eq. (23) one obtains from Eq. (29)

$$M\ddot{q} + \Theta_{\mathbf{q}}^T A \left[\Theta_{\mathbf{q}}\ddot{\mathbf{q}} + \dot{\Theta}_{\mathbf{q}}\dot{\mathbf{q}} + \dot{\Theta}_t + K_D\dot{\Theta} + K_P\Theta\right] = Q.$$

This can also be written as:

$$\left(M + \Theta_{\mathbf{q}}^T A \Theta_{\mathbf{q}}\right)\ddot{\mathbf{q}} = Q - \Theta_{\mathbf{q}}^T A \left(K_D\dot{\Theta} + K_P\Theta - \xi\right), \qquad (30)$$

where ξ defined in Eq. (5).

Equation (30) is said to be a Modified Lagrange's Equation. By comparing Eq. (1) and (30) we can see that the approximate value for the Lagrangian multiplier is given by:

$$\lambda \equiv A\left(\ddot{\Theta} + K_D\dot{\Theta} + K_P\Theta\right). \qquad (31)$$

Note that [6], the term $\Theta_{\mathbf{q}}^T A(K_D\dot{\Theta} + K_P\Theta - \xi)$ in Eq. (30) represents the projection in the direction of the coordinates $\mathbf{q}$ of all the internal forces that are generated by the dynamic system when the constraints $\ddot{\Theta}, \dot{\Theta}$ and Θ are violated.

Equation (30) forms a modified Lagrange's equations given in Eq. (1). The advantages of formulation (30) are due to the inherent features of the leading matrix $M + \Theta_{\mathbf{q}}^T A \Theta_{\mathbf{q}}$. For any possibly varying number of constraints on the system, including the case of redundant constraints, the dimension of $(M + \Theta_{\mathbf{q}}^T A \Theta_{\mathbf{q}})$ is $n \times n$. In other words, this sum is always positive definite matrix.

Remark 4.
1. In the modified Lagrange's equation given by equation (30) the problem of choosing the entries of A, K_D and K_P still persists.
2. The modified Lagrange's equation developed in this paper is similar to penalty method of constraint stabilization developed in [5, 6]. The basic difference is, in this paper we use symmetric positive definite matrices A, K_D and K_P. This is helps us to use different correction parameters in contrary to the same correction parameters say, used in penalty method. The same difference exists between Generalized Baumgarte's method developed in this paper and Baumgarte's method in [4].

Example 1. Consider a two-link manipulator, $L_1 = l$, $L_2 = l/2$, $m_2 = m$, $m_1 = 2m_2 = 2m$ as shown in Figure 1. Let us assume that point P follows the horizontal line $y = l/2$ with a constant velocity along the x-axis. We can expect that for $q_1 = \pi/2$ and $q_2 = \pi$ the system Jacobian matrix becomes singular. We apply the two methods constructed above to simulate the system. In this example it is verified that, simulating by the Modified Lagrange's method developed in this paper solved the problem of singularity at the indicated points, whereas, the Generalized Baumgarte's method is unable to run the simulation in this example at the singularity points.

The generalized coordinates are taken to be $q = \{q_1, q_2\}$ as indicated above.

Notations: The following notations are used in this example.

$$C1 = \cos(q_1), \quad C2 = \cos(q_2), \quad S1 = \sin(q_1), \quad S2 = \sin(q_2),$$

$$C12 = \cos(q_1 + q_2), \quad S12 = \sin(q_1 + q_2),$$

$$K_D = \begin{pmatrix} h & 0 \\ 0 & r \end{pmatrix}, \quad K_P = \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}.$$

The constraint equations are $y = l/2$ for point P, and the velocity constraint to keep point P at a constant velocity on the x-axis starting from the position $x = x_0$ at $t = 0$.

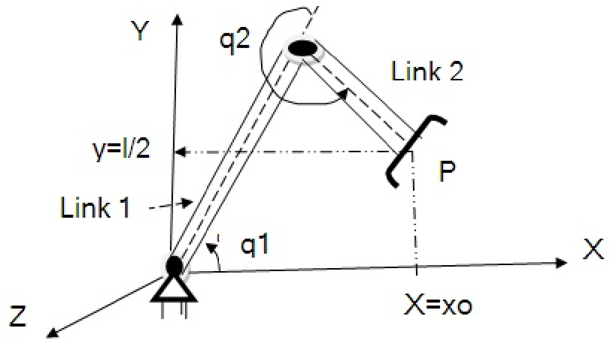

Figure 1. Two-link manipulator

Thus the constraints are: $\Theta = (\Theta_1, \Theta_2)^T$ given by:

$$\Theta_1(q_1, q_2, t) = lS1 + l/2S12 - l/2 = 0,$$

$$\Theta_2(q_1, q_2, t) = lC1 + l/2C12 + vt - x_0 = 0.$$

The constraint equation at the acceleration level becomes:

$$\begin{pmatrix} \ddot{\Theta}_1 \\ \ddot{\Theta}_2 \end{pmatrix} = \begin{pmatrix} lC1 + l/2C12 & l/2C12 \\ -lS1 - l/2S12 & -l/2S12 \end{pmatrix} \begin{pmatrix} \ddot{q_1} \\ \ddot{q_2} \end{pmatrix} - \begin{pmatrix} \dot{q_1}^2 lS1 + l/2S12(\dot{q_1} + \dot{q_2})^2 \\ \dot{q_1}^2 lC1 + l/2C12(\dot{q_1} + \dot{q_2})^2 \end{pmatrix} \tag{32}$$

Expression (16) is given

$$\begin{aligned} &\ddot{\Theta} + K_D \dot{\theta} + K_P \Theta = \\ &= \begin{pmatrix} lC1 + l/2C12 & l/2C12 \\ -lS1 - l/2S12 & -l/2S12 \end{pmatrix} \begin{pmatrix} \ddot{q_1} \\ \ddot{q_2} \end{pmatrix} - \begin{pmatrix} \dot{q_1}^2 lS1 + l/2S12(\dot{q_1} + \dot{q_2})^2 \\ \dot{q_1}^2 lC1 + l/2C12(\dot{q_1} + \dot{q_2})^2 \end{pmatrix} + \\ &+ K_D \begin{pmatrix} \dot{q_1} lC1 + l/2C12(\dot{q_1} + \dot{q_2}) \\ -l\dot{q_1}S1 - l/2S12(\dot{q_1} + \dot{q_2}) + v \end{pmatrix} + \\ &+ K_P \begin{pmatrix} lS1 + l/2S12 - l/2 \\ lC1 + l/2C12 + vt - x_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Ξ defined in Eq. (20) for this example becomes:

$$\begin{aligned} \Xi = &\begin{pmatrix} \dot{q_1} lS1 + l/2S12(\dot{q_1} + \dot{q_2}) & l/2S12(\dot{q_1} + \dot{q_2}) \\ \dot{q_1} lC1 + l/2C12(\dot{q_1} + \dot{q_2}) & l/2C12(\dot{q_1} + \dot{q_2}) \end{pmatrix} \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \end{pmatrix} - \\ &- \begin{pmatrix} h(\dot{q_1} lC1 + l/2C12(\dot{q_1} + \dot{q_2})) \\ r(-l\dot{q_1}S1 - l/2S12(\dot{q_1} + \dot{q_2}) + v) \end{pmatrix} - \begin{pmatrix} p(lS1 + l/2S12 - l/2) \\ s(lC1 + l/2C12 + vt - x_0) \end{pmatrix}, \end{aligned}$$

$$M = \begin{pmatrix} 1/2ml^2 + I_1 + I_2 & 1/4ml^2C2 + I_2 \\ 1/4ml^2C2 + I_2 & 1/16ml^2 + I_2 \end{pmatrix},$$

$$\Theta_q = \begin{pmatrix} lC1 + l/2C12 & l/2C12 \\ -lS1 - l/2S12 & -l/2S12 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1/4ml^2\dot{q_2}(\dot{q_1} + \dot{q_2}\sin(q_2) - 2mgl\sin(q_1) \\ -1/4ml^2\sin(q_2) - 1/4mgl\sin(q_1 + q_2) \end{pmatrix}.$$

I. Simulation by Generalized Baumgarte's Method.

In the case of Generalized Baumgarte's method, we use Eq. (21) and simulate the state-space form of system of first order differential equation given in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{q}}\\ M^{-1}[Q-(\Theta_{\mathbf{q}}^T(\Theta_{\mathbf{q}}M^{-1}\Theta_{\mathbf{q}}^T)^{-1}(\Theta_{\mathbf{q}}M^{-1}Q-\Xi))].\end{bmatrix} \quad (33)$$

Let us use the following values for numerical investigation $m_1 = 2$ kg, $m = m_2 = 1$ kg, $I_1 = 1/12m_1l^2 = 0.167$ kgm^2 $I_2 = 1/12m_2l^2 = 0.208$ kgm^2. The initial angular position for link 1 and 2 are respectively 1.396263 rad. and 3.563268 rad. From these we can find that $x_0 = 0.295953m$. The initial angular velocities are chosen to be 0.358620 and -0.867745 rad/s to set the initial velocity of the velocity constraint close to zero. Note that singularity point occurs in the Jacobian matrix $\Theta_{\mathbf{q}}$ at $q_1 = \pi/2$ and $q_2 = \pi$.

We consider the following cases.

Case 1: Simulation graph (Fig. 2) with stabilization constants $h = 20$; $r = 10$; $p = 100$; $s = 25$. At the time when the simulation of the system reaches the singular point of $q_1 = \pi/2$ and $q_2 = \pi$. In Fig. 2 there is a failure of the simulation $t = 4.067105e+0$. That verifies, the generalized Baumgart's method can't run simulations near the singular points. This problem is circumvented by Modified Lagrange's Equation method as shown in the subsequent parts of this example.

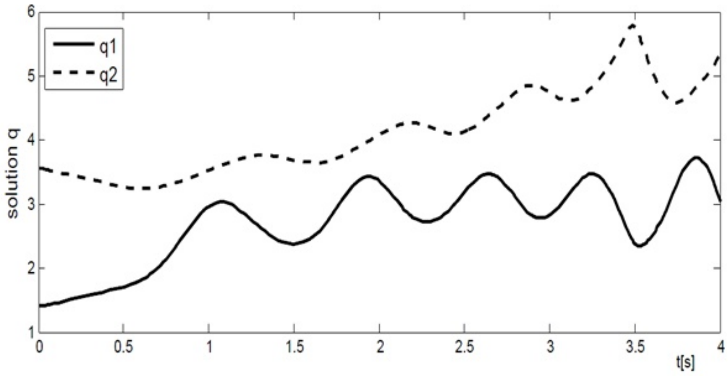

Figure 2. Simulation graph with stabilization constants $h = 20$; $r = 10$; $p = 100$; $s = 25$

II. Simulation by Modified Lagrange's Equation method of Eq. (30)**.**

In this case we simulate the state-space form of system first order differential equation given in by: for simplicity we use $A = \alpha I$ where I is an identity matrix of appropriate size.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{q}}\\ (M+\Theta_{\mathbf{q}}^T\alpha\Theta_{\mathbf{q}})^{-1}[Q-\Theta_{\mathbf{q}}^T\alpha(K_D\dot{\Theta}+K_P\Theta-\xi)].\end{bmatrix} \quad (34)$$

Note that: the difficult of choosing appropriate penalty numbers is observed in the following graphs and the simulation for different numbers is shown. The overall result is that the system is asymptotically stable and the problem of running the simulation near and at the singular points is totally removed in this method.

Case I. Modified Lagrange's Equation graph (Fig. 3) for $h = 20$; $r = 10$; $p = 100$; $s = 25$; $\alpha = 10$.

Case 2: Modified Lagrange's Equation graph (Fig. 4) for $h = 20$, $r = 10$, $p = 100$, $s = 250$, $\alpha = 3$

Case 3: Modified Lagrange's Equation graph (Fig. 5) for $h = 20$; $r = 10$; $p = 10$; $s = 2500$; $\alpha = 50$

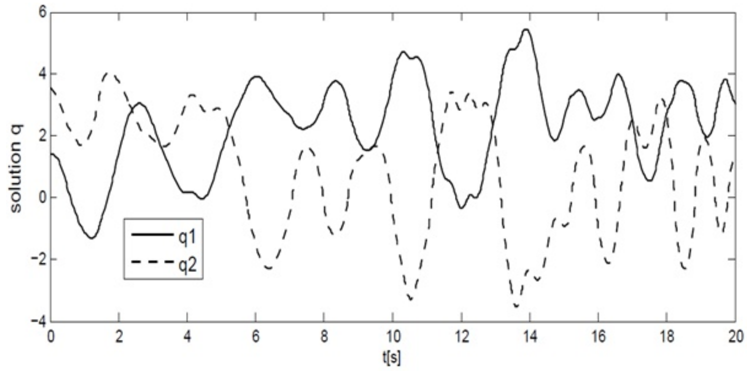

Figure 3. Modified Lagrange's Equation graph for $h = 20$; $r = 10$; $p = 100$; $s = 25$; $\alpha = 10$

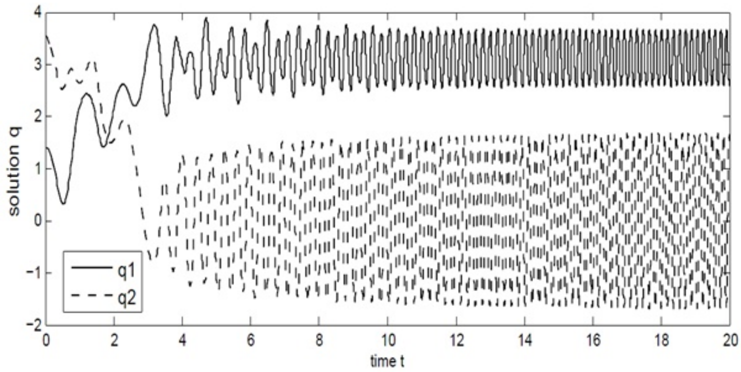

Figure 4. Modified Lagrange's Equation graph for $h = 20,\ r = 10,\ p = 100,\ s = 250,\ \alpha = 3$

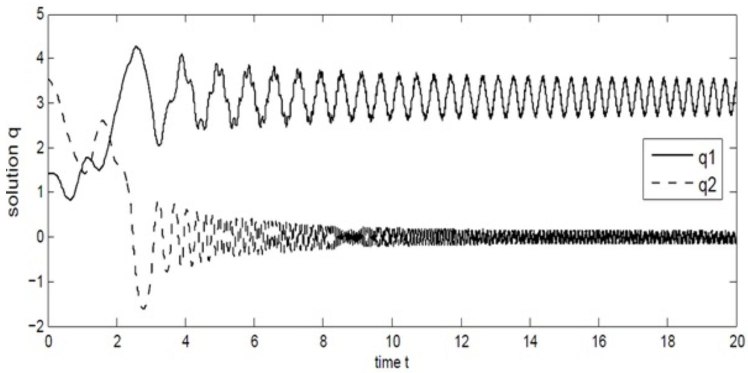

Figure 5. Modified Lagrange's Equation graph for $h = 20$; $r = 10$; $p = 10$; $s = 2500$; $\alpha = 50$

It was observed during simulation experimentation that, choosing alpha too high makes the system unstable. More appropriate scalars can be chosen by running the simulation for different values of the constants.

4. Conclusion

The generalized penalty method developed in this paper circumvents the issue of redundant constraints and singular mass matrices where as, the generalized Baumgarte's method fails to run near singular points as shown in the example. But it can be concluded that the two methods developed have equivalent effect on a dynamic systems with non-singular mass matrices. The method developed in this paper improves the usual Baumgarte's method and the penalty method in that it made the possibility of using different correction parameters.

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Стабилизация избыточно ограниченной динамической системы

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В данной статье рассматривается вопрос стабилизации связей динамической системы. Широко использовано уравнение движения Лагранжа второго порядка для моделирования динамики механических систем, рассматриваемых в этой статье. Известно, что метод Баумгарта по ограничению стабилизации не позволяет избежать проблемы сингулярности массовых матриц, которая может возникнуть в результате избыточности ограничений, и не сможет запускать симуляции вблизи и на точках сингулярности. Разработан обобщённый метод Баумгарта и определены условия стабилизации на основе метода Ляпунова. Разработанный метод позволяет определить коррекцию параметров ограничений, накладываемых на фазовые переменные. Известный метод Баумгарта, использующий коррекцию уравнений связей, следует из методов, предлагаемых в работе. Модифицированные уравнения Лагранжа построены в соответствии с условиями стабилизации связей и охватывают также случай сингулярной матрицы коэффициентов

кинетической энергии. Как и в случае метода Баумгарта, обычное уравнение Лагранжа является частным случаем более совершенного метода, описанного в данной статье. Численный пример иллюстрирует эффективность разработанных методов. Предлагаемый метод моделирования обеспечивает асимптотическую устойчивость решения уравнений динамики по отношению к уравнениям связей также в сингулярном случае.

Ключевые слова: стабилизация, обобщённый метод Баумгарта, модифицированные уравнения Лагранжа, сингулярная массовая матрица, избыточные ограничения, прямой метод Ляпунова.

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