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Critical Dependencies in Three-Layered Josephson Junctions

I. Hristov, S. Dimova, T. Boyadjiev

SU "St. Kl. Ohridski", FMI
5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Sakai-Bodin-Pedersen model is used to investigate numerically the dynamics of the Josephson phases in three stacked long Josephson Junctions. It is shown that the critical currents of the individual junctions depend on the damping and the coupling parameters and that there is a domain in vicinity of zero magnetic field where the junctions switch to nonzero voltage simultaneously, i.e. current locking takes place.

Key words and phrases: stacked Josephson Junctions, current locking, system of perturbed sine-Gordon equations, finite element method, finite difference method.

1. Introduction

The experimental and numerical investigations of multistacked long Josephson Junctions (JJs) lead to invention of new physical effects. One of them is the so-called current locking (CL), experimentally observed in two stacked JJs. The essence of this phenomenon is: there exists a range of the external magnetic field where the junctions of the stack switch to nonzero voltage simultaneously when the external current exceeds some critical value. Later [1] the experimentally found CL for two stacked JJs was simulated numerically in the framework of the inductive coupling model of Sakai-Bodin-Pedersen [2]. In this work we use the same model and find that CL takes place in the case of symmetric three stacked JJs as well. We investigate this phenomenon for different damping and coupling parameters.

2. Mathematical Model

The dynamics of the Josephson phases $\varphi(x,t) = (\varphi_1(x,t), \dots, \varphi_N(x,t))^T$ in geometrically symmetric N stacked JJs is described by the following system of perturbed sine-Gordon equations [2]:

$$\varphi_{tt} + \alpha\varphi_t + J + \Gamma = L^{-1}\varphi_{xx}, \quad -\ell < x < \ell, \quad t > 0, \tag{1}$$

where α is the dissipation coefficient, $J = (\sin\varphi_1, \sin\varphi_2, \dots, \sin\varphi_N)^T$ is the vector of the Josephson current density, $\Gamma = \gamma\,(1,1,\dots,1)^T$ is the vector of the external current density, $L = \mathrm{tridiag}(1, S, 1)$, $(-0.5 < S \leqslant 0$ for arbitrary $N)$ and 2ℓ is the length of the junctions. Here the space x is normalized with respect to the Josephson penetration length and the time t — to the inverse of the plasma frequency.

In this work we consider stacks with overlap geometry placed in external magnetic field h_e, therefore the system (1) should be solved together with the boundary conditions:

$$\varphi_x(-\ell,t) = \varphi_x(\ell,t) = H, \tag{2}$$

where H is the vector $H = h_e\,(1,1,\dots,1)^T$. To close the differential problem appropriate initial conditions must be posed:

$$\varphi(x,0) - \text{given}, \quad \varphi_t(x,0) - \text{given}. \tag{3}$$

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The existence of Josephson current generates a specific magnetic flux. When the external current is less than some critical value, the junctions are in static state. In this case the measured voltages in all junctions are zero. When the external current exceeds this critical value, the system switches to dynamic state and the voltage of at least one of the junctions is nonzero. The voltage in the i-th junction is mathematically given by:

$$V_i = \lim_{T\to\infty} \frac{1}{2\ell T} \int_0^T \int_{-\ell}^{\ell} \varphi_{i,t}(x,t)\mathrm{d}x\mathrm{d}t. \tag{4}$$

We define the critical current of the individual junction as the value at which this junction switches to nonzero voltage.

In order to make a correspondence between the loss of stability of a possible static distribution of the magnetic flux and switching to dynamic state, we solve numerically the static problem, i.e., the system of equations with time independent fluxes. To study the global stability of a possible static solution, the following matrix Sturm-Liouville problem (SLP) is generated:

$$-L^{-1}u_{xx} + Q(x)u = \lambda\, u, \tag{5}$$

$$u_x(\pm\ell) = 0, \quad \int_{-\ell}^{\ell} \langle u, u\rangle \,\mathrm{d}x - 1 = 0, \tag{6}$$

where $Q(x) = J'(\varphi(x))$. This is equivalent to study the positive definiteness of the second variation of the potential energy of the stack. The minimal eigenvalue λ_{min} determines the stability of the distribution under consideration. A minimal eigenvalue equal to zero means a bifurcation caused by change of some parameter, in our case – the external current γ.

3. Numerical Method

The simplest generalizable model of stacked JJs is the case of three stacked JJs because it takes into account the different behavior of the interior and exterior junctions. The numerical results presented here are for the particular case of three stacked JJs.

In order to solve the mentioned above static nonlinear boundary value problem we use an iterative algorithm [3], based on the continuous analog of Newton's method (CANM) [4]. CANM gives a linearized boundary value problem at each iteration step. The linear boundary value problem is solved numerically by means of Galerkin finite element method (FEM) and quadratic finite elements. FEM is used also to reduce the SLP (5), (6) to a linear algebraic problem whose few smallest eigenvalues and the corresponding eigenfunctions are found by the subspace iteration method [5]. To test the accuracy of the above methods we have used the method of Runge by computing the solutions on sequence of embedded meshes. The numerous experiments made show a super-convergence of order four for both the static problem and SLP.

To solve the dynamic problem (1), (2), (3), we use the Finite Difference Method. The main equation (1) is approximated by the "cross-shaped" scheme. To approximate the boundary conditions (2), the three point one-sided finite differences are used. Let h and τ be the steps in space and time respectively, and $\delta = (\tau/h)^2$. In these notations the difference scheme is:

$$\hat{y}_k^l = (1 + 0.5\alpha\tau)^{-1} \left[2y_k^l + (0.5\alpha\tau - 1)\check{y}_k^l - \tau^2(\sin y_k^l + \gamma) + \sum_{m=1}^{3} \delta a_{lm} y_{\bar{x}x,k}^m \right], \tag{7}$$

$$k = 1, \ldots, n-1, \quad l = 1, 2, 3, \quad L^{-1} = (a_{lm})_{l,m=1}^3,$$

$$\hat{y}_0^l = (4\hat{y}_1^l - \hat{y}_2^l - 2hh_e)/3, \quad \hat{y}_n^l = (4\hat{y}_{n-1}^l - \hat{y}_{n-2}^l + 2hh_e)/3. \tag{8}$$

The approximation error of this scheme is $O(\tau^2 + h^2)$.

To check the numerical stability and the real order of accuracy of the difference scheme (7), (8), we have made computations for fixed time levels and embedded meshes in space. The results show second order of convergence in space and time.

The numerical procedure for finding the critical currents of the individual junctions for fixed parameters S, ℓ, α works as follows. We start with $h_e = 0, \gamma = 0, \varphi(x, 0) = 0$, $\varphi_t(x, 0) = 0$. For given magnetic field h_e, increasing the current γ from zero by a small amount $\triangle\gamma$, we approximately calculate the average voltage (4) by using the difference scheme. In our computations, for the next value of the current γ, the phase distributions of the last two time levels achieved are used as initial time levels. We increase the current until all the junctions switch to nonzero voltage. Then the external field h_e is increased by a small amount $\triangle h_e$ and again the current is increased until all the junctions switch to nonzero voltage. As initial data, the phase distributions for $\gamma = 0$ and the previous value of h_e are used.

4. Numerical Results

The results for the critical currents of the individual junctions in a system of three stacked JJs with parameters $S = -0.1, -0.3$; $\alpha = 0.1, 0.05$; $2\ell = 10$ (in total four cases) are graphically shown on Fig. 1, 2.

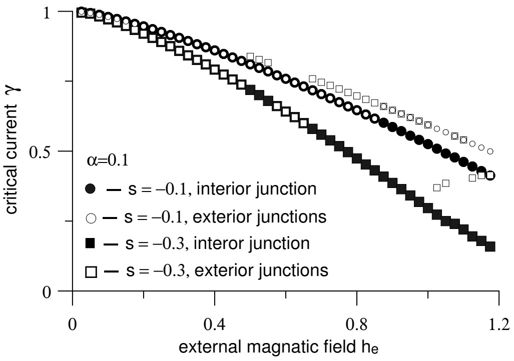

Figure 1. **Critical currents for** $\alpha = 0.1$

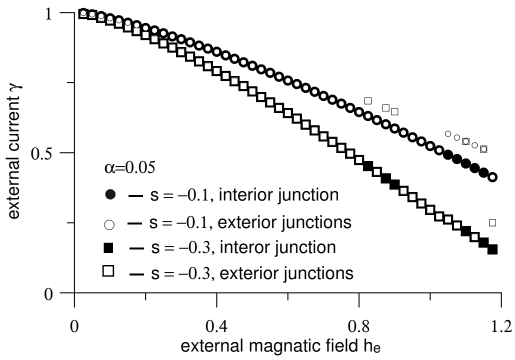

Figure 2. **Critical currents for** $\alpha = 0.05$

The interval $[0, 1.2]$ in h_e is considered. The following conclusions can be made.

In vicinity of zero external magnetic field the junctions switch to nonzero voltage simultaneously, i.e., current locking (CL) takes place. In addition there are smaller domains at which there is CL divided by domains at which the interior junction switches first to resistive (R) state. For fixed damping parameter α, the smaller in absolute value the coupling parameter S, the larger the domain of CL. For fixed coupling parameter, the smaller damping parameter, the larger the domain of CL.

One can see on Fig. 3, 4, 5, 6 that the transient process of switching from static to dynamic state for the two cases — CL and no CL — starts with penetration of fluxons in the interior junction.

In the case of CL (Fig. 3, 4) the switching of the interior junction to R-state triggers the switching of the exterior ones to R state, but in the case of no CL (Fig. 5, 6) this does not happens. In the last case the exterior junctions switch to R-state at higher current, and as the numerical results show, for given magnetic field this critical current weakly depends on the coupling parameter S.

The transient process of switching the exterior junctions to R-state when the interior one is already in R-state is shown on Fig. 7, 8.

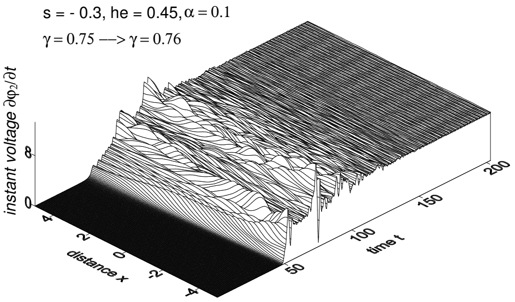

Figure 3. **CL, the interior junction switches to R-state**

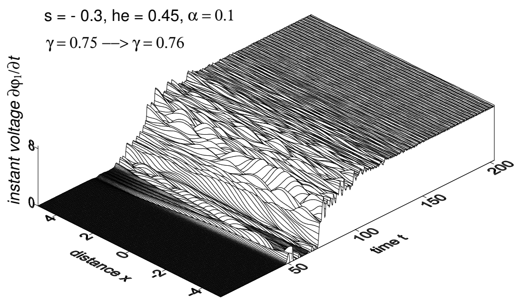

Figure 4. **CL, the exterior junctions switch to R-state**

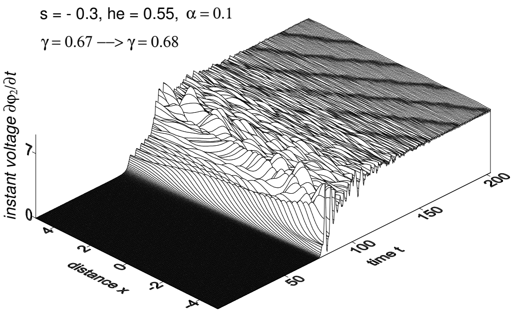

Figure 5. **No CL, the interior junction switches to R-state**

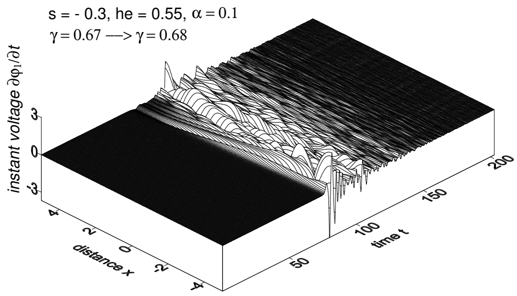

Figure 6. **No CL, the exterior junctions remain at zero voltage**

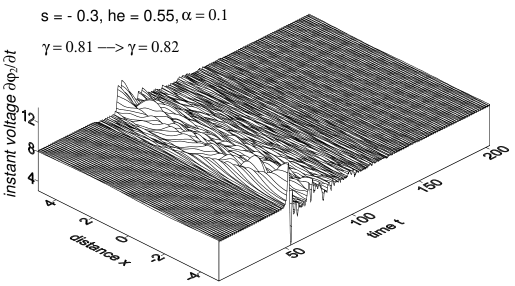

Figure 7. **The interior junction is in R-state**

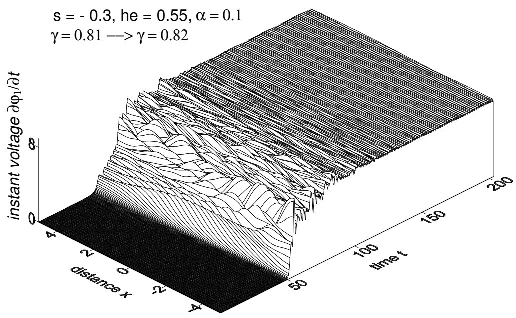

Figure 8. **The exterior junctions switch to R-state**

Let us mention that for every h_e the lower critical current point, found by solving the dynamic problem, lies on some bifurcation curve of a static solution, found by solving the SLP and the static problem.

5. Conclusions

Perfect agreement between the results, found by solving the Sturm-Liouville problem and those, found by solving the dynamic problem, is established. The numerical simulation shows that for symmetric three stacked JJ's current locking takes place.

This is essentially dynamical phenomenon which depends on the coupling and the damping parameters.

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Критические зависимости в трехслойных джозефсоновских контактах

И. Христов, С. Димова, Т. Бояджиев

Факультет математики и информатики
кафедра «Численные методы и алгоритмы»
Софийский университет им. Кл. Охридского
бул. Джеймс Баучер 5, 1164 София, Болгария

Модель Сакаи–Бодина–Педерсена используется для численного исследования динамики джозефсоновских фаз в трехслойных длинных джозефсоновских контактах. Показано, что критические токи отдельных контактов зависят от коэффициента диссипации и параметра связи и что есть область в окрестности нулевого магнитного поля, где контакты переключаются к напряжению отличному от нуля одновременно, т.е. имеет место синхронизация критических токов.

Ключевые слова: многослойный джозефсоновский контакт, синхронизация критических токов, система возмущенных уравнений sine-Гордона, метод конечных элементов, метод конечных разностей.