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Correlations in Quantum Mechanics as Origin of Modified Newtonian Dynamics

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We present derivation of modified Newtonian force from quantum mechanics with Coulomb potential. We show, that MOND-like theory arises in classical limit of quantum dynamics if nontrivial correlations of canonical variables are taken into account.

Key words and phrases: quantum and classical correlations, effective force, MOND.

1. Introduction

Modified Newtonian Dynamic (MOND) was proposed by Milgrom in 1981 [1, 2]. The most successful relativistic version of MOND, known as 'TeVeS', was introduced by Bekenstein in [3–5]. Although MOND provides impressive experimentally verified predictions [6, 7], there is still lack of fundamental origin of this theory. In this paper we show that nontrivial correlations leads to MOND-like modification of Newtonian force at classical limit of the standard quantum theory. These correlations include dispersions of coordinates, i.e. we propose the nonlocal origin for MOND. Similar approach for relativistic scenario will be considered in forthcoming publication.

2. Classical Theory with Nontrivial Correlations

Let us consider non-relativistic quantum system with one-component wave function $\Psi = \Psi(t,\, x^k)$, where x^k $(k = 1, 2, 3)$ — coordinates of the three-dimensional space. We use exponential parametrization $\Psi = \exp{(iS/\hbar)}$, where the complex phase S is represented in terms of the real functions S_1 and S_2 as

$$S = S_1 + \frac{i\hbar}{2} S_2. \tag{1}$$

In [8] it was shown, that the matrix quantities $A_{1\,kl} = S_{1,\,x_k x_l}$ and $A_{2\,kl} = S_{2,\,x_k x_l}$ define correlations between canonical variables of the theory in the classical limit of $\hbar \to 0$. There it is established, that all these correlations vanish if $A_2^{-1} \to 0$, where A_2 is the matrix with the coefficients $A_{2,\,kl}$.

Now let us study the system of Kepler type with (for example) planet and Sun, with the masses m and M, respectively (where $m \ll M$). Then its Hamiltonian reads:

$$\mathcal{H} = \sum_k \frac{\hat{p}_k^2}{2m} - \frac{GMm}{r}\,, \tag{2}$$

where $\hat{p}_k = i\hbar\partial_{x_k}$, and G — Newton's gravity constant. In the classical limit one obtains the following modified Hamilton's equations [8]:

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} = \frac{p_k}{m} - (A_2^{-1})_{kl}\Gamma_{,\,x_l}, \quad \frac{\mathrm{d}p_k}{\mathrm{d}t} = -\frac{GMm}{r^2}\,\frac{x_k}{r} - (A_1 A_2^{-1})_{kl}\Gamma_{,\,x_l}, \tag{3}$$

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where the function Γ is defined as

$$\Gamma = \frac{1}{m}\Delta S_1, \tag{4}$$

and $\Delta = \sum_l (\partial_l)^2$ is the Laplace operator. We use these dynamical equations to calculate the physical force according the Second Newton's law:

$$F_k = m\frac{\mathrm{d}^2 x_k}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\frac{\mathrm{d}x_k}{\mathrm{d}t}\right); \tag{5}$$

the result reads:

$$F_k = -\frac{GMm}{r^2}\frac{x_k}{r} - \frac{\mathrm{d}[(A_2^{-1})_{kl}\Delta S_{1,\,x_l}]}{\mathrm{d}t} - \frac{1}{m}(A_1 A_2^{-1})_{kl}\Delta S_{1,\,x_l}. \tag{6}$$

Here the first term coincides with the conventional gravitational force, whereas the second and third ones maintain its dispersion corrections. By introduction of the normalized function

$$\Sigma = \frac{S_1}{m} \tag{7}$$

one modifies the correctional terms to make them proportional to the inertial mass m. Thus, the full force is proportional to the inertial mass — this fact proves the equivalence principle in this theory.

Actually, using of the function Σ makes the Hamilton–Jacoby equation $S_{1,\,t}+E = 0$ free of the mass parameter m:

$$\Sigma_{,\,t} + \frac{1}{2}\sum_k \Sigma^2_{,\,x_k} - \frac{GM}{r} = 0, \tag{8}$$

where we mean the classical limit of the theory [8]. Thus, any solution $\Sigma(t,\, x_k)$ of this equation does not depend on m. Then, the function S_2 satisfies the relation

$$S_{2,\,t} + \sum_k \Sigma_{,\,x_k} S_{2,\,x_k} = \Delta\Sigma, \tag{9}$$

which is also independent on the inertial mass. From this it follows that the whole theory under consideration yields the equivalence principle.

3. Modified Newtonian force

To study the total force (6) in the theory, let us represent it in the following form:

$$F_k = F_{0\,k} - m\Phi_k. \tag{10}$$

Here $F_{0\,k} = -GmMx_k/r^3$, whereas the normalized correlation term Φ_k reads:

$$\Phi_k = \left(\hat{\Sigma}A_2^{-1}\right)_{kl}\Delta\Sigma_{,\,x^l} + \Theta_k, \tag{11}$$

where

$$\Theta_k = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(A_2^{-1}\right)_{kl}\Delta\Sigma_{,\,x^l}\right], \tag{12}$$

and $\hat{\Sigma}$ means the matrix with the components $\Sigma_{,\,x^k x^l}$. Taking into account, that the total t-derivative is given by the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \frac{\mathrm{d}x_n}{\mathrm{d}t}\partial_{x_n} = \partial_t + \left[\Sigma_{,\,x^n} - \left(A_2^{-1}\right)_{ns}\Delta\Sigma_{,\,x^s}\right]\partial_{x_n}, \tag{13}$$

and performing the calculations, one obtains the following result:

$$\Theta_k = -\frac{1}{2}\left(A_2^{-1}\right)_{kl} \Delta \left(\Sigma^2_{,x^n}\right)_{,x_l} +$$
$$+ \left[\left(A_2^{-1}\hat{\Sigma} + \hat{\Sigma}A_2^{-1}\right)_{kl} - \left(A_2^{-1}\right)_{kk_1}\left(A_2^{-1}\right)_{lk_2}\left(\Delta\Sigma_{,x_{k_1}x_{k_2}} - \Sigma_{,x^k}S_{2,x_{k_1}x_{k_2}}\right)\right]\Delta\Sigma_{,x_l} +$$
$$+ \left[\Sigma_{,x_n} - \left(A_2^{-1}\right)_{ns}\Delta\Sigma_{,x_s}\right]\times$$
$$\times\left[\left(A_2^{-1}\right)_{kl}\Delta\Sigma_{,x_l x_n} - \left(A_2^{-1}\right)_{kk_1}\left(A_2^{-1}\right)_{lk_2} S_{2,x_{k_1}x_{k_2}x_n}\Delta\Sigma_{,x_l}\right].$$

In the infinitesimal correlation regime, where $A_2^{-1} \to 0$, for the dispersion term one obtains:

$$\Phi_k \approx -\frac{1}{2}\left(A_2^{-1}\right)_{kl} \Delta \left(\Sigma^2_{,x^n}\right)_{,x_l} +$$
$$+ \left(A_2^{-1}\hat{\Sigma} + 2\hat{\Sigma}A_2^{-1}\right)_{kl}\Delta\Sigma_{,x_l} + \Sigma_{,x_n}\left(A_2^{-1}\right)_{kl}\Delta\Sigma_{,x_l x_n}. \quad (14)$$

Note, that Eq. (14) does not include nonlinear correlation terms.

Now let us study one concrete solution of the dynamical system under consideration, which corresponds to the spherically-symmetric potential Σ. Thus, let us consider the anzats with $\Sigma = \Sigma(t,\, r)$, where r means the radial coordinate (i.e., the distance between planet and Sun). In the classical case with separated variables,

$$\Sigma = -\frac{\alpha t}{2} + f(r), \quad (15)$$

where $\alpha = \text{const}$, one deals with

$$f' = \left\{\alpha + \frac{2GM}{r}\right\}^{1/2}. \quad (16)$$

For the isotropic matrix A_2^{-1} (which describes the correlations between the Cartesian coordinates x_k of the planet, see [8])

$$\left(A_2^{-1}\right)_{kl} = \epsilon\delta_{kl}, \quad (17)$$

where $\epsilon \to 0$ in the infinitesimal limit under our study, one obtains:

$$\Phi_k = 2\epsilon\left(\Sigma_{,x_k x_l}\Delta\Sigma_{,x_l} - \Sigma_{,x_k x_l x_n}\Sigma_{,x_l x_n}\right). \quad (18)$$

Note, that Eq. (17) means localization of the planet in the sphere of the $\sim\sqrt{\epsilon}$ radius with the center in its classical position. Using the identity

$$\hat{\Sigma}_{,x_k x_l} = \eta x_k x_l + \xi\delta_{kl}, \quad (19)$$

where $\xi = f'/r$ and $\eta = \xi'/r$, and also the relation

$$\Sigma_{,x_k x_l x_n} = \frac{\eta'}{r}x_k x_l x_n + \eta\left(x_k\delta_{ln} + x_n\delta_{kl} + x_l\delta_{nk}\right), \quad (20)$$

it is possible to calculate the final form of the correlation potential and for the total force as well. This term (and the force) is radial, i.e. $\Phi_k = \Phi\, x_k/r$, where the defining magnitude Φ reads:

$$\Phi = \frac{4\epsilon}{r^3}\,\frac{\left(\alpha + \dfrac{3GM}{r}\right)^2}{\alpha + \dfrac{2GM}{r}}. \quad (21)$$

Note, that $\epsilon > 0$ for the well-localized position of the planet (see [8]), whereas the constant α is of the arbitrary sign. It is seen, that $\Phi \sim r^{-3}$ for $r \to +\infty$, so $F \sim r^{-2}$ for the large distances. Then, the small distances case ($r \to 0$, for the case of $\alpha > 0$) corresponds to $F \sim r^{-4}$. This type of force behavior is actually natural to MOND, see [1, 2].

4. Conclusion

In this article we have derived corrections to the standard gravitational (or Coulomb) force, which can be used for the MOND approach verification. These are given by Eq. (21), which follows from the standard quantum mechanics in the classical limit with nontrivial correlations of the canonical variables. This result can be generalized to the temperature-dependent case, which can be interested in the cosmological framework. In [8] it was shown, that the corresponding consideration needs in introducing of the complex parameter of evolution to the theory.

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Корреляции в квантовой механике как основа модифицированной ньютоновской динамики

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Даётся вывод модифицированной ньютоновской силы из модифицированной квантовой механики с кулоновским потенциалом. Показывается, что теория типа МОНД появляется в классическом пределе квантовой динамики при учёте нетривиальных корреляций канонических переменных теории.

Ключевые слова: квантовые и классические корреляции, эффективная сила, МОНД.