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Static Configuration with Spherical Symmetry in Conformal Gravitation: Vacuum Solution

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The oscillating model of conformal gravity is considered. The static solution is studied in vacuum-dominated conformal gravitation, which is modeled by scalar field $\chi(x)$. The vacuum solution ($\chi = 0$, $T_{\mu\nu} = 0$) in Einstein's gauge $\varphi(x) = \varphi_0 = \mathrm{const}$ being considered, the Newtonian potential is obtained. The significance of the additional term in the potential is also discussed.

Key words and phrases: oscillating model, conformal cosmology, Einstein's gauge, Newtonian potential.

1. Oscillating Model

The oscillating model of the Universe possesses some attractive properties in comparison with the standard cosmological model based on Einstein's equation [1–3]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

and Friedman's metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right). \tag{2}$$

The properties which give the model its importance among the other models of the Universe are as follows:

1. The periodic dependence of the cosmological scale factor (radius of the Universe) $a(t)$ on cosmic time t allows us to realize the idea of stable (eternal) Universe. In this case it does not contradict the experimental facts witnessing the expansion of the Universe.
2. Oscillating model trivially allows us to surmount the difficulty arising in the interpretation of singularity, in which the scale factor $a(t)$ becomes zero, and density of energy approaches infinity.

It appeares that the first property shows that the closed standard model is correct ($k = 1$, $\rho > \rho_{cr} = 3H_0^2/8\pi G$, where H_0 is the present value of the Hubble constant), but the modern experimental data obtained from measuring the Hubble constant ($H_0 \approx 75$ Km/s·mps) and from mean density of energy, concentrated in galaxies ($\rho_0 \approx 3 \times 10^{-31}$ gm/cm$^3 \ll \rho_{cr} \approx 10^{-29}$ gm/cm^3) does not permit us to affirm that closed standard model is really true. Unfortunately, the presence of the singularity in the history of the early Universe, which is an inevitable property of the standard cosmological model also makes the scenario of the evolution of the Universe within the framework of closed model not so optimistic (all the matter and radiation approache a singularity at the origin of a universal black hole — the "big crunch"). Besides it, R. Tolman remarked in 1931 that the oscillation of closed Universe is not really periodic, since the amplitude of the scale factor $a_{\max}$ and the period T are the functions of the entropy S, and increase monotonically from cycle to cycle with the grow of entropy.

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Nevertheless, the closed standard model is one of the most attractive models in today's cosmology, and some theoreticians try to fit it to experimental data. They hope that it will show proper agreement with astrophysical observations. Small observed mean density of matter $\rho_0 \approx 3 \times 10^{-31}$ gm/cm^3 contradicts estimations of the mean density of substances in metagalactic clusters, obtained from the data of measurements of meansquare peculiar velocities of the motion of peripheral substance in gravitationally related metagalaxies and their analysis within the framework of virial theorem.The most popular method to remove this contradiction is the assumption of the existence of *hidden mass* in the Universe. Massive neutrinos and other exotic objects were considered as the source of this *hidden mass*.

The objective of this article is to find the vacuum static solution with spherical symmetry within the framework of vacuum-dominated conformal gravitation to first order in material excitation, which is modeled by scalar field $\chi(x)$, within the scope of oscillating model of conformal cosmology proposed by V. M. Pyzh [4]. Vacuum equation for stationary metric with spherical symmetry is used.

2. Weyl's Conformal Cosmology

The concept of conformal symmetry was introduced by Herman Weyl in 1918 [5], when he made an attempt to base a new theory of gravitation and electromagnetism on a modified geometry in which there would be room and need for another geometric object besides the metric tensor. This was the first example of theory with local gauge symmetry. Today the principle of gauge invariance lies in the basis of all realistic theories of fundamental interactions and seems to be as natural as the principle of the least action itself. Weyl's term "gauge invariance" and the concept of gauge symmetry opened up before us a new physical content and became a powerful heuristic principle in the time of constructing Lagrangians of unified field theory. In 1973 P. A. M. Dirac in his work returned to the idea of using conformal symmetry in order to construct unified theory of gravitation and electromagnetism on the basis of Weyl's geometry generalizing the Riemannian one.

In Weyl's geometry the concept of Riemannian tensor is generalized till that of co-tensor with the properties of tensor with respect to general local point coordinate transformation process and furthermore showing definite scale degree with respect to Weyl's transformation. Conformal transformation of metric tensor $g_{\mu\nu}(x)$ reads:

$$g_{\mu\nu}(x) \to \overline{g}_{\mu\nu}(x) = e^{2\lambda(x)} g_{\mu\nu}(x), \tag{3}$$

where $\lambda(x)$ is any function of coordinate x. The local quantity $T(x)$ transforming simultaneously with (3) according to the law

$$\bar{T}(x) = e^{n\lambda(x)} T(x) \tag{4}$$

is called the quantity of degree n. If under this condition quantity $\overline{T}(x)$ is the tensor with respect to general point coordinate transformation, then it is named as *co-tensor* of degree n.

Co-tensors of degree $n = 0$ (invariant with respect to conformal gauge transformation (4)) are considered as different class called *in-tensors*.

Affine connection $\Gamma^{\lambda}_{\mu\alpha}(x)$ of Riemannian geometry, in Weyl's space can be transformed in the form

$$g_{\mu\nu,\alpha} - \tilde{\Gamma}^{\lambda}_{\mu\alpha} g_{\lambda\nu} - \tilde{\Gamma}^{\lambda}_{\nu\alpha} g_{\lambda\mu} - 2kB_\alpha g_{\mu\nu} = 0, \tag{5}$$

where k is a constant and $B_\alpha(x)$ is the Weyl's vector field, which controls the change of the length of the vector under its motion along the curve in the Weyl's space. The concept of derivative $T_{;\alpha}$ may be generalized till co-covariant derivative $T_{*\alpha}$, which will be co-tensor.

For co-scalar S with power n we have

$$\overline{S}_\mu = (e^{n\lambda} S)_\mu = e^{n\lambda}(S_\mu + nk\lambda_\mu S) = e^{n\lambda}\left[S_\mu + kn(\overline{B}_\mu - B_\mu)\right]$$

or

$$\overline{S}_\mu - nk\overline{B}_\mu\overline{S} = e^{n\lambda}\left(S_\mu - nkB_\mu S\right). \tag{6}$$

Therefore the quantity

$$S_{*\mu} \equiv S_\mu - nkB_\mu S \tag{7}$$

is the co-vector of power n. Analogous by one can determine co-tensor of power n:

$$V_{\mu*\nu} \equiv V_{\mu\nu} - \tilde{\Gamma}^\alpha_{\mu\nu}V_\alpha - nkB_\nu V_\mu \tag{8}$$

like co-variant derivative with respect to co-vector $V_\mu(x)$ of power n, and the rule of co-variant differentiation may be extended to co-tensor $T(x)$ of power n of any rank and variance.

The above discussion is a brief report on Weyl's geometry.

3. Dirac's Theory and Unified Einstein–Weyl–Dirac Theory

At the time of constructing action in his own theory Dirac used the same method like Weyl [5]. The action in physical theory based on Weyl's geometry

$$I = \int \mathrm{d}^4x\sqrt{|g|}L(x), \tag{9}$$

(where $g \equiv \det g_{\mu\nu}$, $L(x)$ — density of Lagrangian) should be in-invariant. Consequently L must be co-scalar of power (-4).

As was Indicated by Dirac, the method of including linear $\tilde{R}$ in Einsteinian term, in the theory based on Weyl's geometry, permits to assume, that Einstein's GTR and Weyl's theory with quadratic terms with respect to the curvature tensor do not exclude, on the contrary, complement each other. As was shown in the work [6], Einsteinian and Weylian Lagrangians are two different forms of more general unified theory based on Weyl's geometry.

4. Model of Conformal Gravitation with Two Scalar Fields

To study the structure of conformal vacuum and specific properties of the solution in the conformal gravitation, it is better to start with the simpler conformal models of gravitation, which contain the potential with the minimal quantity of field degree in the Lagrangian.

We choose conformal model of gravitation described by the action with two scalar fields $\varphi(x)$ and $\chi(x)$ [7]:

$$I = \int \mathrm{d}^4x\sqrt{|g|}\left\{\frac{1}{12}R(\varphi^2\chi^2) - \frac{1}{2}R(\varphi^\alpha\varphi_\alpha - \chi^\alpha\chi_\alpha) + \frac{\lambda}{4}\varphi^4 - \frac{\beta}{2}\varphi^2\chi^2 - \frac{\alpha}{4}\chi^4\right\}, \tag{10}$$

where $\varphi(x)$ is the dilaton field and $\chi(x)$ is the scalar field, representing matter; α, β, λ are the coupling constants. By varying the action (10) with respect to the metric tensor $g_{\mu\nu}(x)$ as well as fields $\varphi(x)$ and $\chi(x)$ we obtain the following equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\left(\frac{6}{\varphi^2 - \chi^2}\right)(T_{\mu\nu} - \theta_{\mu\nu}) \equiv -\frac{6t_{\mu\nu}}{\varphi^2 - \chi^2}\,, \tag{11}$$

$$\Box\varphi + \frac{1}{6}R\varphi - \beta\chi^2\varphi + \lambda\varphi^3 = 0, \tag{12}$$

$$\Box\chi + \frac{1}{6}R\chi - \beta\varphi^2\chi + \alpha\chi^3 = 0, \tag{13}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter, described by scalar field $\chi(x)$, $\theta_{\mu\nu}$ is the energy-momentum tensor of dilaton field $\varphi(x)$, which is known as observed vacuum degree of freedom, depending on the choice of conformal gauge. Here $\Box$ is the D'Alembertian operator.

Conservation law of complete energy-momentum tensor of matter and vacuum

$$\chi^{\upsilon}(\Box\chi + \frac{1}{6}R\chi + \beta\varphi^2\chi + \alpha\chi^3) - \varphi^{\upsilon}\left(\Box\varphi + \frac{1}{6}R\chi - \beta\chi^2\varphi + \lambda\varphi^3\right) = 0 \tag{14}$$

and the trace equation

$$\chi\left(\Box\chi + \frac{1}{6}R\chi + \beta\varphi^2\chi + \alpha\chi^3\right) - \varphi\left(\Box\varphi - \beta\chi^2\varphi + \lambda\varphi^3\right) = 0 \tag{15}$$

are evident consequences of (12) and (13).

In Einstein's gauge $\varphi_0 = \text{const}$ model, determined by the action (10), is equivalent to GTR with Λ-term. Here the quantity

$$G_0 = \frac{3}{4\pi\varphi_0} \tag{16}$$

should be taken as the gravitational constant and we should assume $\Lambda = (-3\lambda\varphi_0^2/2)$. Such definition of Λ-term leads to the following equation of Einstein:

$$R_{\mu\nu} - \frac{1}{2}\left(R + 2\Lambda\right)g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{17}$$

Vacuum solution of the model (10) for static metric with spherical symmetry has been studied in the work [8], and for Friedman's cosmology in the works [4, 7].

Let us investigate the solution of conformal gravitation, determined by the action (10), for static metric with spherical symmetry, given by the interval

$$\mathrm{d}s^2 = B(r)\mathrm{d}t^2 - A(r)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{18}$$

The solution for $B(r)$ can be represented in the form

$$B(r) = 1 - \left(\frac{2r_g}{r}\right) + \left(\frac{2r^2}{r_0^2}\right), \tag{19}$$

where $r_g = 2MG$ is the radius of Schwarzschild, and $r_0 = 4/\lambda\varphi_0^2$. The quantity r_0 is to be compared with the value of the scale factor of the Universe $r_0 > a(t_0) \sim 10^{28}$ cm, so the last term in the formula (19) can play significant role only in the metagalactic scale.

5. Static Solution with Spherical Symmetry of Conformal Gravitation

Let us investigate the properties of the solution of conformal gravitation, determined by the action (10), for static metric with spherical symmetry, given by the interval

$$\mathrm{d}s^2 = b^2(r)\mathrm{d}t^2 - a^2(r)\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{20}$$

The components of affine connection of Ricci tensor $R_{\mu\nu}(x)$, which are not equal to zero, for metric (20) take the form [1, 7]

$$\begin{gathered}\Gamma^t_{tr} = \frac{b'}{b}, \quad \Gamma^r_{tt} = \frac{bb'}{a^2}, \quad \Gamma^r_{rr} = \frac{a'}{a}, \quad \Gamma^\theta_{\theta r} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \\ \Gamma^r_{\varphi\varphi} = \sin^2\theta, \quad \Gamma^r_{\theta\theta} = -\frac{r^2}{a^2\sin^2\theta}, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta,\end{gathered} \tag{21}$$

$$R_{tt} = -\frac{bb'}{a^2} + \frac{a'bb'}{a^3} - \frac{2bb'}{ra^2}, \quad R_{rr} = -\frac{b''}{b} + \frac{a'b'}{ab} - \frac{2a'}{ra}, \\ R_{\theta\theta} = -1 + \frac{r}{a^2}\left(\frac{b'}{b} - \frac{a'}{a}\right) + \frac{1}{a^2}, \tag{22}$$

where $b'(x) \equiv \mathrm{d}b/\mathrm{d}r$. Scalar curvature $R(x)$ is defined by the expression

$$R(x) = \frac{2}{a^2}\left[-\frac{b''}{b} + \frac{a'b'}{ab} + \frac{2}{r}\left(\frac{a'}{a} - \frac{b'}{b}\right) + \frac{1}{r^2}\left(a^2 - 1\right)\right]. \tag{23}$$

It is convenient to rewrite equation (11) in the equivalent form:

$$R_{\mu\nu} = -\frac{6}{\varphi^2 - \chi}\left[T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} - \theta_{\mu\nu} + \frac{1}{2}\theta g_{\mu\nu}\right], \tag{24}$$

where

$$T \equiv T^{\lambda}_{\lambda} = \chi\Box\chi + 2\beta\chi^2\varphi^2 + \lambda\chi^4, \tag{25}$$

$$\theta \equiv \theta^{\lambda}_{\lambda} = \varphi\Box\varphi + \lambda\varphi^4, \tag{26}$$

$$\Box\chi \equiv \chi^{\lambda}_{\lambda} = -\left(\frac{1}{a^2}\right)\left[\chi'' + \chi'\left(\frac{a'}{a} - \frac{b'}{b} - \frac{2}{r}\right)\right]. \tag{27}$$

At first let us consider vacuum solution ($\chi = 0$, $T_{\mu\nu}$) in Einstein's gauge $\varphi(x) = \varphi_0 = \text{const}$. This type of solution corresponds to Einstein GTR with Λ-term and it is the evident generalization of the Schwarzschild solution [1, 8].

6. Vacuum Solution

At $\chi = 0$, $\varphi = \varphi_{0c} = \text{const}$, equation (24) takes the form

$$R_{tt} = -\frac{bb''}{a^2} + \frac{a'bb'}{a^3} - \frac{2bb'}{ra^2} = -\frac{3}{2}\lambda\varphi_0^2 b^2, \tag{28}$$

$$R_{rr} = \frac{b''}{b} - \frac{a'b'}{ab} - \frac{2a'}{ra} = -\frac{3}{2}\lambda\varphi_0^2 a^2, \tag{29}$$

$$R_{\varphi\varphi} = -1 + \frac{1}{a^2} + \frac{r}{a_2}\left(\frac{b'}{b} - \frac{a'}{a}\right) = \frac{3}{2}\lambda\varphi_0^2 r^2. \tag{30}$$

By multiplying the first of those equations by a^2, and the second by b^2 and by adding them we obtain $-\frac{1}{r}\left(\frac{b'}{b} + \frac{a'}{a}\right) = 0$ and, consequently, $ab = \text{const}$. Here it is convenient to put $ab = 1$, in accordance with the standard solution of Schwarzschild [1]. Then from (30) we obtain

$$2rbb' + b^2 - 1 = \frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(b^2 - 1\right)\right] = \frac{3}{2}\lambda\varphi_0^2 r^2. \tag{31}$$

From (31) we deduce

$$b^2(r) = 1 - \frac{r_g}{r} + \frac{2r^2}{r_0^2}, \tag{32}$$

with the denotations being introduced: $r_g = 2MG$ (for the Schwarzschild radius) and $r_0^2 = 4/\lambda\varphi_0^2 = 16\pi G/3\lambda$. Value of the parameter r_0 is determined by the constants G and λ. It turnes out that r is very big and corresponds to cosmological scale. "Modified" Newtonian potential in conformal gravitation (and GTR with Λ-term) according to (32) has the form:

$$\Phi(r) = -\frac{MG}{r} + \frac{r^2}{r_0^2}. \tag{33}$$

The additional term r^2/r_0^2 is proportional to r^2. It turnes out that this term is negligable for the scale of stellar structures and even for galaxies. However in the analysis of virial theorem results in metagalactic scales, where the contribution of the term $r^2/r_0^2 \sim r^2$ can be significant, it is necessary to use potential $\Phi(r)$ in its proper form (33), in order to escape from the possible contradiction.

7. Conclusion

The article exploits the oscillating model within the scope of Weyl's conformal cosmology and unified theory of Einstein–Weyl–Dirac. It also contains the formulation of conformal model of gravitation, which is modeled by the scalar field $\chi(x)$. Here we consider the vacuum solution $(\chi = 0, T_{\mu\nu} = 0)$ in Einstein's gauge $\varphi(x) = \varphi_0 = \text{const}$. This type of solution corresponds to Einstein's GTR with Λ-term and it is the evident generalization of the Schwarzschild solution. Newtonian potential in conformal gravitation is also obtained. The additional term in it is not significant for the scale of stellar structures and even for galaxies, but in case of metagalactic scales the contribution of this term appears to be significant.

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Статическая сферически-симметричная конфигурация в конформной гравитации. Вакуумное решение

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Рассматривается осцилляционная модель конформной гравитации. Изучается статическое решение в конформной гравитации с вакуумной доминантой, которая моделируется скалярным полем $\chi(x)$. Для вакуумного решения ($\chi = 0$, $T_{\mu\nu} = 0$) в калибровке Эйнштейна ($\varphi(x) = \varphi_0 = \text{const}$) получен ньютоновский потенциал. Обсуждается значение дополнительного слагаемого в потенциале.