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Equations of Motion of Rapidly Driven Systems

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A general theory of a large class of classical dynamical systems with an external rapidly oscillating driving action is proposed. The main results are the effective equations for the mean motion. The scope of the present work is to generalize the approach to the case when the fast perturbation contains odd and even terms and amplitudes may depend on velocities.

Key words and phrases: high-frequency oscillations, Kapitza pendulum, vibrational mechanics, high-frequency stabilization.

Dynamical systems periodically influenced by external pulses recently have been attracting more and more attention [1]. The visible reason for such interest is that a driven system totally changes its typical characteristics. The bright example of such changes is inverted pendulum. Fast periodical fluctuations of the point of suspension stabilize the overturned position of the pendulum, which is unstable under normal conditions. It is necessary to remind that it only happens when the frequency of external pulses Ω greatly exceeds the pendulum characteristic frequency ω_0. In other words, the stabilization effect is possible only under the influence of fast external pulses. Hence there is a special class of driven systems — fast driven systems, those systems for which the frequency of external pulses is much more significant than the system's characteristic frequency.

In our article [2] we proposed an approach to studying the wide class of such systems based on the Hamilton formalism. The method is based on the search of solution of dynamical equations with fast oscillating parts, like:

$$q_i = Q_i + \mu\, \chi_i; \quad p_i = P_i + \mu\, \rho_i, \tag{1}$$

where Q_i, P_i are "slow" and "fast" parts of coordinates and impulses, which characteristic times are $2\pi/\omega_0$ and $2\pi/\Omega$ respectively and $\mu = \omega_0/\Omega \ll 1$.

In other words, dynamical trajectories of the system can be looked upon as super positions of fast pulsations and slow modulations. The physical base of relations (1) lies in the fact that due to inertia the system responds weekly to fast external pulses. In [2] we obtained the effective equations for mean slow trajectories. We used the traditional method of decomposing the averaging over "short" and "long" periods [1,3]. Due to external pulses these equations contain some additional terms as compared with unperturbed equations. For example, in the simple case of a pendulum we have to take into consideration the additional restoring force, which stabilizes the pendulum in its inverted position.

However, the dynamical equations considered in [2] are not of the general type, since they contain even terms over time only. The scope of the present work is to generalize the approach to the case of the following dynamical equations:

$$\dot{q}_i = \frac{\partial H\,(q,p,t)}{\partial p_i} + g_i^{(0)}\,(q,p) + \sum_{k\geqslant 1} g_i^{(k)}\,(q,p)\cos\,(k\,\Omega\,t) + \sum_{k\geqslant 1} f_i^{(k)}\,(q,p)\sin\,(k\,\Omega\,t), \tag{2}$$

$$\dot{p}_i = -\frac{\partial H\,(q,p,t)}{\partial p_i} + h_i^{(0)}\,(q,p) + \sum_{k\geqslant 1} h_i^{(k)}\,(q,p)\cos\,(k\,\Omega\,t) + \sum_{k\geqslant 1} d_i^{(k)}\,(q,p)\sin\,(k\,\Omega\,t), \tag{3}$$

where g, h, f and d are arbitrary functions of canonical variables.

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The backbone of our approach [2] is the search for solutions of equations (2), (3) in the form (1). Substituting (1) into (2), (3) and expanding all functions in power series of μ and retaining terms up to the first order (*e.g.* $g_i^{(k)}(q,p) \approx g_i^{(k)}(Q,P) + \mu\ (\chi_j \cdot \nabla_{Q_j})\ g_i^{(k)}(Q,P) + \mu\ (\rho_j \cdot \nabla_{P_j})\ g_i^{(k)}(Q,P)$ etc.), we find

$$\begin{aligned}
\dot{Q}_i + \mu\,\dot{\chi}_i = {} & \frac{\partial H}{\partial P_i} + \mu\,(\chi_j \cdot \nabla_{Q_j})\,\frac{\partial H}{\partial P_i} + \mu\ (\rho_j \cdot \nabla_{P_j})\,\frac{\partial H}{\partial P_i} + \\
& + g_i^{(0)}(Q,P) + \mu\ (\chi_j \cdot \nabla_{Q_j})\ g_i^{(0)}(Q,P) + \mu\ (\rho_j \cdot \nabla_{P_j})\ g_i^{(0)}(Q,P) + \\
& + \sum_{k\geqslant 1} g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1} f_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \\
& + \sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \\
& + \sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ f_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ f_i^{(k)}(Q,P)\sin(k\,\Omega\,t)\,, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\dot{P}_i + \mu\,\dot{\rho}_i = {} & -\frac{\partial H}{\partial Q_i} - \mu\ (\chi_j \cdot \nabla_{Q_j})\,\frac{\partial H}{\partial Q_i} - \mu\ (\rho_j \cdot \nabla_{P_j})\,\frac{\partial H}{\partial Q_i} + \\
& + h_i^{(0)}(Q,P) + \mu\ (\chi_j \cdot \nabla_{Q_j})\ h_i^{(0)}(Q,P) + \mu\ (\rho_j \cdot \nabla_{P_j})\ h_i^{(0)}(Q,P) + \\
& + \sum_{k\geqslant 1} h_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1} d_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \\
& + \sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ h_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ h_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \\
& + \sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ d_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ d_i^{(k)}(Q,P)\sin(k\,\Omega\,t). \qquad (5)
\end{aligned}$$

Here summation is made over repeated indices. The equations contain slow and fast terms, which can be separated by averaging over the period$2\pi/\Omega$. The fast part of equations (4), (5) thus read:

$$\begin{aligned}
\mu\,\dot{\chi}_i = {} & \mu\ (\chi_j \cdot \nabla_{Q_j})\,\frac{\partial H}{\partial P_i} + \mu\ (\rho_j \cdot \nabla_{P_j})\,\frac{\partial H}{\partial P_i} + \\
& + \mu\ (\chi_j \cdot \nabla_{Q_j})\ g_i^{(0)}(Q,P) + \mu\ (\rho_j \cdot \nabla_{P_j})\ g_i^{(0)}(Q,P) + \\
& + \sum_{k\geqslant 1} g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1} f_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \\
& + \mu\sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \mu\sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ g_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \\
& + \mu\sum_{k\geqslant 1}(\chi_j \cdot \nabla_{Q_j})\ f_i^{(k)}(Q,P)\sin(k\,\Omega\,t) + \mu\sum_{k\geqslant 1}(\rho_j \cdot \nabla_{P_j})\ f_i^{(k)}(Q,P)\sin(k\,\Omega\,t), \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\mu\,\dot{\rho}_i = {} & -\mu\ (\chi_j \cdot \nabla_{Q_j})\,\frac{\partial H}{\partial Q_i} - \mu\ (\rho_j \cdot \nabla_{P_j})\,\frac{\partial H}{\partial Q_i} + \\
& + \mu\ (\chi_j \cdot \nabla_{Q_j})\ h_i^{(0)}(Q,P) + \mu\ (\rho_j \cdot \nabla_{P_j})\ h_i^{(0)}(Q,P) + \\
& + \sum_{k\geqslant 1} h_i^{(k)}(Q,P)\cos(k\,\Omega\,t) + \sum_{k\geqslant 1} d_i^{(k)}(Q,P)\sin(k\,\Omega\,t) +
\end{aligned}$$

$$+\,\mu\sum_{k\geqslant 1}\left(\chi_j\cdot\nabla_{Q_j}\right)\,h_i^{(k)}\,(Q,P)\cos\,(k\,\Omega\,t)+\mu\sum_{k\geqslant 1}\left(\rho_j\cdot\nabla_{P_j}\right)\,h_i^{(k)}\,(Q,P)\cos\,(k\,\Omega\,t)+$$

$$+\,\mu\sum_{k\geqslant 1}\left(\chi_j\cdot\nabla_{Q_j}\right)\,d_i^{(k)}\,(Q,P)\sin\,(k\,\Omega\,t)+\mu\sum_{k\geqslant 1}\left(\rho_j\cdot\nabla_{P_j}\right)\,d_i^{(k)}\,(Q,P)\sin\,(k\,\Omega\,t). \quad (7)$$

The terms in (6), (7) are not all of the same order. While the terms$\mu\,\dot\chi\cong\mu\,\Omega\,\chi\cong\omega_0\chi$, $\mu\,\dot\rho\cong\mu\,\Omega\,\rho\cong\omega_0\rho$ are not small, the terms with $\mu\,\chi$, $\mu\,\rho$ are much smaller. So, we find for the fast equations accurate up to zero-order terms in μ

$$\begin{aligned}\mu\,\dot\chi_i&=\sum_{k\geqslant 1}g_i^{(k)}\cos\,(k\,\Omega\,t)+\sum_{k\geqslant 1}f_i^{(k)}\sin\,(k\,\Omega\,t),\\ \mu\,\dot\rho_i&=\sum_{k\geqslant 1}h_i^{(k)}\cos\,(k\,\Omega\,t)+\sum_{k\geqslant 1}d_i^{(k)}\sin\,(k\,\Omega\,t).\end{aligned} \quad (8)$$

Integrating these equations, we obtain

$$\begin{aligned}\chi_i&\approx\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{g_i^{(k)}}{k}\sin\,(k\,\Omega\,t)-\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{f_i^{(k)}}{k}\cos\,(k\,\Omega\,t),\\ \rho_i&\approx\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{h_i^{(k)}}{k}\sin\,(k\,\Omega\,t)-\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{d_i^{(k)}}{k}\cos\,(k\,\Omega\,t).\end{aligned} \quad (9)$$

Subsequently, we solve the fast equations taking into account the terms of first-order in μ too, substituting (9) into (6), (7) and again integrating with respect to time (recall that Q_i, P_i, $H\,(Q,P,t)$ are held constant)

$$\begin{aligned}\chi_i\approx{}&\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(g_i^{(k)}\sin\,(k\,\Omega\,t)-\sum_{k\geqslant 1}f_i^{(k)}\cos\,(k\,\Omega\,t)\right)-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{k\geqslant 1}\frac{\cos\,(k\,\Omega\,t)}{k^2}\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\left(\frac{\partial H}{\partial P_i}+g_i^{(0)}\right)-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{k\geqslant 1}\frac{\sin\,(k\,\Omega\,t)}{k^2}\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\left(\frac{\partial H}{\partial P_i}+g_i^{(0)}\right)-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{\substack{l,\,k\geqslant 1\\ l\neq k}}\frac{\cos\,((l-k)\ \Omega\,t)}{2\,l\ (l-k)}\left[\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)g_i^{(k)}+\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)f_i^{(k)}\right]-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{l,\,k\geqslant 1}\frac{\cos\,((l+k)\ \Omega\,t)}{2\,l\ (l+k)}\left[\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)g_i^{(k)}-\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)f_i^{(k)}\right]-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{\substack{l,\,k\geqslant 1\\ l\neq k}}\frac{\sin\,((l-k)\ \Omega\,t)}{2\,l\ (l-k)}\left[\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)g_i^{(k)}-\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)f_i^{(k)}\right]-\\ &-\frac{1}{\mu\,\Omega^2}\sum_{l,\,k\geqslant 1}\frac{\sin\,((l+k)\ \Omega\,t)}{2\,l\ (l+k)}\left[\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)g_i^{(k)}+\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)f_i^{(k)}\right],\end{aligned} \quad (10)$$

$$\begin{aligned}\rho_i\approx{}&\frac{1}{\mu\,\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(h_i^{(k)}\sin\,(k\,\Omega\,t)-\sum_{k\geqslant 1}d_i^{(k)}\cos\,(k\,\Omega\,t)\right)+\\ &+\frac{1}{\mu\,\Omega^2}\sum_{k\geqslant 1}\frac{\cos\,(k\,\Omega\,t)}{k^2}\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\left(\frac{\partial H}{\partial Q_i}-h_i^{(0)}\right)+\end{aligned}$$

$$
+\frac{1}{\mu\,\Omega^2}\sum_{k\geqslant 1}\frac{\sin(k\,\Omega\,t)}{k^2}\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\left(\frac{\partial H}{\partial Q_i}-h_i^{(0)}\right)-
$$

$$
-\frac{1}{\mu\,\Omega^2}\sum_{\substack{l,\,k\geqslant 1\\ l\neq k}}\frac{\cos((l-k)\,\Omega\,t)}{2\,l\,(l-k)}\left[\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)h_i^{(k)}+\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)d_i^{(k)}\right]-
$$

$$
\frac{1}{\mu\,\Omega^2}\sum_{l,\,k\geqslant 1}\frac{\cos((l+k)\,\Omega\,t)}{2\,l\,(l+k)}\left[\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)h_i^{(k)}-\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)d_i^{(k)}\right]-
$$

$$
\frac{1}{\mu\,\Omega^2}\sum_{\substack{l,\,k\geqslant 1\\ l\neq k}}\frac{\sin((l-k)\,\Omega\,t)}{2\,l\,(l-k)}\left[\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)h_i^{(k)}-\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)d_i^{(k)}\right]-
$$

$$
\frac{1}{\mu\,\Omega^2}\sum_{l,\,k\geqslant 1}\frac{\sin((l+k)\,\Omega\,t)}{2\,l\,(l+k)}\left[\left(f_j^{(l)}\cdot\nabla_{Q_j}+d_j^{(l)}\cdot\nabla_{P_j}\right)h_i^{(k)}+\left(g_j^{(l)}\cdot\nabla_{Q_j}+h_j^{(l)}\cdot\nabla_{P_j}\right)d_i^{(k)}\right]. \quad (11)
$$

Expressions (10), (11) represent a rapidly oscillating response of coordinates and moments of the system to the fast external actions. But it is not the only reaction of the system. The mean trajectories $(\mathbf{Q};\mathbf{P})$ are altered too. Inserting (10), (11) into (4), (5) and averaging over the period $2\pi/\Omega$ we find the following equations of motion for the mean canonical coordinates:

$$
\dot{Q}_i=\left(\frac{\partial H}{\partial P_i}+g_i^{(0)}\right)-
$$

$$
-\frac{1}{2\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)g_i^{(k)}+\frac{1}{2\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)f_i^{(k)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{k\geqslant 1}\frac{1}{k^2}\left\{\left(g_{j/}^{(k)}\cdot\nabla_{Q_{j/}}+h_{j/}^{(k)}\cdot\nabla_{P_{j/}}\right)\left[\left(\frac{\partial H}{\partial P_j}+g_j^{(0)}\right)\cdot\nabla_{Q_j}-\left(\frac{\partial H}{\partial Q_j}-h_j^{(0)}\right)\cdot\nabla_{P_j}\right]\right\}g_i^{(k)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{k\geqslant 1}\frac{1}{k^2}\left\{\left(f_{j/}^{(k)}\cdot\nabla_{Q_{j/}}+d_{j/}^{(k)}\cdot\nabla_{P_{j/}}\right)\left[\left(\frac{\partial H}{\partial P_j}+g_j^{(0)}\right)\cdot\nabla_{Q_j}-\left(\frac{\partial H}{\partial Q_j}-h_j^{(0)}\right)\cdot\nabla_{P_j}\right]\right\}f_i^{(k)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(g_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+h_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\right\}g_i^{(|l-k|)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(f_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+d_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\right\}g_i^{(|l-k|)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(g_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+h_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\right\}g_i^{(l+k)}+
$$

$$
+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(f_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+d_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\right\}g_i^{(l+k)}-
$$

$$
-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(f_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+d_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\right\}f_i^{(|l-k|)}+
$$

$$
+\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(g_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+h_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\right\}f_i^{(|l-k|)}-
$$

$$
+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(f_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+d_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)\right\}f_i^{(l+k)}+
$$

$$
+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(g_{j/}^{(l)}\cdot\nabla_{Q_{j/}}+h_{j/}^{(l)}\cdot\nabla_{P_{j/}}\right)\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)\right\}f_i^{(l+k)}, \quad (12)
$$

$$
\dot{P}_i=-\left(\frac{\partial H}{\partial Q_i}-h_i^{(0)}\right)-
$$

$$
-\frac{1}{2\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(f_j^{(k)}\cdot\nabla_{Q_j}+d_j^{(k)}\cdot\nabla_{P_j}\right)h_i^{(k)}+\frac{1}{2\Omega}\sum_{k\geqslant 1}\frac{1}{k}\left(g_j^{(k)}\cdot\nabla_{Q_j}+h_j^{(k)}\cdot\nabla_{P_j}\right)d_i^{(k)}-
$$

$$-\frac{1}{2\Omega^2}\sum_{k\geqslant 1}\frac{1}{k^2}\left\{\left(g^{(k)}_{j/}\cdot\nabla_{Q_{j/}}+h^{(k)}_{j/}\cdot\nabla_{P_{j/}}\right)\left[\left(\frac{\partial H}{\partial P_j}+g^{(0)}_j\right)\cdot\nabla_{Q_j}-\left(\frac{\partial H}{\partial Q_j}-h^{(0)}_j\right)\cdot\nabla_{P_j}\right]\right\}h^{(k)}_i-$$

$$-\frac{1}{2\Omega^2}\sum_{k\geqslant 1}\frac{1}{k^2}\left\{\left(f^{(k)}_{j/}\cdot\nabla_{Q_{j/}}+d^{(k)}_{j/}\cdot\nabla_{P_{j/}}\right)\left[\left(\frac{\partial H}{\partial P_j}+g^{(0)}_j\right)\cdot\nabla_{Q_j}-\left(\frac{\partial H}{\partial Q_j}-h^{(0)}_j\right)\cdot\nabla_{P_j}\right]\right\}d^{(k)}_i-$$

$$-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(g^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+h^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(g^{(k)}_j\cdot\nabla_{Q_j}+h^{(k)}_j\cdot\nabla_{P_j}\right)\right\}h^{(|l-k|)}_i-$$

$$-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(f^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+d^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(f^{(k)}_j\cdot\nabla_{Q_j}+d^{(k)}_j\cdot\nabla_{P_j}\right)\right\}h^{(|l-k|)}_i-$$

$$-\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(g^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+h^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(g^{(k)}_j\cdot\nabla_{Q_j}+h^{(k)}_j\cdot\nabla_{P_j}\right)\right\}h^{(l+k)}_i+$$

$$+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(f^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+d^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(f^{(k)}_j\cdot\nabla_{Q_j}+d^{(k)}_j\cdot\nabla_{P_j}\right)\right\}h^{(l+k)}_i-$$

$$-\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(f^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+d^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(g^{(k)}_j\cdot\nabla_{Q_j}+h^{(k)}_j\cdot\nabla_{P_j}\right)\right\}d^{(|l-k|)}_i+$$

$$+\frac{1}{2\Omega^2}\sum_{l\neq k\geqslant 1}\frac{1}{2k\,(l-k)}\left\{\left(g^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+h^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(f^{(k)}_j\cdot\nabla_{Q_j}+d^{(k)}_j\cdot\nabla_{P_j}\right)\right\}d^{(|l-k|)}_i-$$

$$+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(f^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+d^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(g^{(k)}_j\cdot\nabla_{Q_j}+h^{(k)}_j\cdot\nabla_{P_j}\right)\right\}d^{(l+k)}_i+$$

$$+\frac{1}{2\Omega^2}\sum_{l,k\geqslant 1}\frac{1}{2k\,(l+k)}\left\{\left(g^{(l)}_{j/}\cdot\nabla_{Q_{j/}}+h^{(l)}_{j/}\cdot\nabla_{P_{j/}}\right)\left(f^{(k)}_j\cdot\nabla_{Q_j}+d^{(k)}_j\cdot\nabla_{P_j}\right)\right\}d^{(l+k)}_i. \quad (13)$$

Notice that in (12)–(13) the gradients in the left positions act only on the functions within curly brackets.

Relations (12), (13) constitute the effective microscopic equations of motion of a system subject to high-frequency external actions. The true trajectories of particles consist of a superposition of the rapidly oscillating motion (10), (11) and the “slow” motion over the mean trajectories described by equations (12), (13). The additional terms in these equations are real forces and velocities, which do exist and affect the behavior of the system.

In the simplest case of one-dimensional non-linear oscillator driven by a perturbation containing odd and even terms with amplitudes depending on coordinates and velocities:

$$\ddot{x}+f\,(x,\dot{x})=H\,(x,\dot{x})\ \cos{(\Omega t)}+G\,(x,\dot{x})\ \sin{(\Omega t)} \quad (14)$$

it follows from equations (12), (13) the equation of motion in the mean coordinate:

$$\ddot{X}+f\,(x)+\frac{1}{2\Omega^2}H\left(\frac{\partial H}{\partial x}\right)_X+\frac{1}{2\Omega^2}G\left(\frac{\partial G}{\partial x}\right)_X+\frac{1}{2\Omega}G\left(\frac{\partial H}{\partial \dot{x}}\right)_X-\frac{1}{2\Omega}H\left(\frac{\partial G}{\partial \dot{x}}\right)_X=0. \quad (15)$$

The first additional term in (15) coincides with the result first obtained by Kapitza [3] and it is responsible for the stabilization of inverted pendulums. However, the most remarkable terms in the right-hand side of (12), (13) and respectively (15), are those proportional to $1/\Omega$, while the remaining additional terms are second-order in $1/\Omega$, i.e. much smaller. As in the quantum case, they arise under very special conditions, namely when the perturbation contains odd and even terms and at least one of the amplitudes depends on the velocity.

This brings up the question whether such a situation may occur in some realistic classical systems (let us note that some kind of automatic control systems are of no interest in this connection, reminding that inverted pendulum is a pure *dynamic* system). In the book [1, p. 120], mention is made of systems with dry friction which may represent such an example.

So, the generalization of equations to the case when the fast perturbation contains odd and even terms and amplitudes may depend on velocities, is in no way trivial and new additional terms may be responsible for many interesting effects.

References

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Уравнения движения для систем с быстрым внешним воздействием

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Предложена общая теория классических систем, находящихся под быстрым внешним воздействием. Основной результат представляют собой эффективные уравнения для усреднённого движения. Целью настоящей работы является обобщение разработанного ранее подхода на случай, когда внешнее быстрое возмущение содержит как чётные, так и нечётные по времени члены с амплитудами, зависящими от скоростей.