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Method of Singular Sources in Application to Electrovacuum Gravitational Einstein Fields

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By means of the method of singular sources it is possible to construct a generalization of gravitational magnetic dipole.

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1. Introduction

Principally the static axially symmetric problem in general relativity has been formulated and developed in a most elegant manner by Weyl [1]. Among static solutions of the Einstein–Maxwell equations the first and at the time very important result was obtained by Reissner [2] and Nordström [3]. The further works by Majumdar [4] and Papapetrou [5] followed for the problem of the electrostatic field. By means of the method of singular sources it is possible to construct asymptotically flat metrics which reduce to the generalizations of the Schwarzschild metric in the absence of magnetism.

2. Basic Equations

The metric of the static axisymmetric gravitational field can be written in the canonical Weyl coordinates in the form

$$\mathrm{d}s^2 = \frac{1}{f}\left[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2\right] - f\mathrm{d}t^2.$$

The fact that the static Einstein–Maxwell equations allow the existence of either the electric potential, or magnetic one, results from the stationary Einstein–Maxwell equations.

In this case we set and magnetostatic Einstein equations have the form

$$u\Delta u = (\vec{\nabla} u)^2 + \frac{u^4}{\rho^2}(\vec{\nabla} A_3), \quad \vec{\nabla}\left(\frac{u^2}{\rho^2}\vec{\nabla} A_3\right)^2 = 0. \tag{1}$$

Here $u = \sqrt{f}$, $\Delta \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}$, $\vec{\nabla} = \vec{\rho}_0\frac{\partial}{\partial z} + \vec{z}_0\frac{\partial}{\partial z}$ ($\vec{\rho}_0$ and $\vec{z}_0$ are unit vectors) and $A_3(\rho, z)$ is the magnetic component of the electromagnetic 4-potential.

The second equation in (1) can be viewed as the condition for the existence of a new potential A_3' connected with A_3 by relations

$$\frac{\partial A_3'}{\partial\rho} = -\frac{f}{\rho}\cdot\frac{\partial A_3}{\partial z}, \quad \frac{\partial A_3'}{\partial z} = \frac{f}{\rho}\cdot\frac{\partial A_3}{\partial\rho}. \tag{2}$$

In that case Eqs (1) can be rewritten as

$$f\Delta f = (\vec{\nabla} f)^2 + 2f(\vec{\nabla} A_3')^2, \quad f\Delta A_3' = (\vec{\nabla} A_3')\cdot\vec{\nabla} f. \tag{3}$$

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One can easily see that the electrostatic Einstein–Maxwell equations have the same form as Eqs. (3). Therefore, we can put $A_4 = A_3'$, where A_4 is the electric component of the electromagnetic 4-potential $A_i = \big[0, 0, 0, -A_4(\rho, z)\big]$.

While $A_3 = A_4 = 0$ the Eqs (1), (2) turn to the Weyl vacuum static equations:

$$f\Delta f = (\vec{\nabla} f)^2. \tag{4}$$

With the substitution $f = e^{2\psi}$ (4) becomes linear:

$$\Delta\psi = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\cdot\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2}. \tag{5}$$

3. Method of Singular Sources

The right-hand side of (5) contains zero though actually there should be a certain singular unction characterizing the distribution of sources.

Let $\sigma(\rho, z)$ denote the mass density of such sources, and let us rewrite (5) in the form

$$\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} = -4\pi\sigma(\rho, z). \tag{6}$$

This equation has the solution

$$\psi = \frac{1}{4\pi}\int\limits_V \frac{\sigma(\rho', z')}{|\vec{r} - \vec{r}'|}\,\mathrm{d}V'. \tag{7}$$

In the coordinates ρ, φ, z we have

$$\mathrm{d}V' = \rho'\mathrm{d}\rho'\mathrm{d}\varphi'\mathrm{d}z',$$

$$|\vec{r} - \vec{r}'| = \rho^2 + {\rho'}^2 - 2\rho\rho'\cdot\cos(\varphi - \varphi') + (z - z')^2.$$

Since the left-hand side of (6) does not depend on φ, we can set $\varphi = 0$ in the integral.

If we choose

$$\sigma(\rho', z') = \frac{\delta(\rho' - \rho_0)}{\rho'}\sigma(\rho_0, z),$$

where $\rho_0 = \mathrm{const}$, $\delta(\rho' - \rho_0)$ is Dirac's δ-function, we obtain

$$\psi(\rho, z) = \int\limits_{z'=-\infty}^{+\infty}\int\limits_{\varphi'=0}^{2\pi} \frac{\sigma(\rho_0, z')\cdot\mathrm{d}\varphi'\cdot\mathrm{d}z'}{\sqrt{\rho^2 + \rho_0^2 - 2\rho_0\rho\cos\varphi' + (z - z')^2}}\,. \tag{8}$$

Example 1. Let $\rho_0 = 0$, $\sigma_0(z') = \frac{1}{2}\delta(z')$. Integration of (8) then leads to

$$\varphi = -\frac{m}{\sqrt{\rho^2 + z^2}}\,, \tag{9}$$

i.e. to the Chazy–Curzon solution.

Example 2. Let $\sigma_0(z') = \delta_0 = \mathrm{const}$. With this choice we come to the Zipoy solution:

$$\psi = \frac{\delta_0}{2}\ln\left(\frac{z - m_0 + \sqrt{\rho^2 + (z - m_0)^2}}{z + m_0 + \sqrt{\rho^2 + (z - m_0)^2}}\right). \tag{10}$$

If we put $\delta_0 = 1$ in (10), we obtain the Schwarzschild solution

$$f = 1 - \frac{2m_0}{r}, \quad \begin{pmatrix} x = \frac{r}{m_0} - 1, \quad y = \cos\vartheta \\ z = m_0 xy, \quad \rho = m_0\sqrt{(x^2-1)(1-y^2)} \end{pmatrix}$$

Example 3.
a)

$$\psi = \int\limits_0^z \frac{\mathrm{d}z'}{\sqrt{\rho^2 + z'^2}} = \ln\frac{z + \sqrt{\rho^2 + z^2}}{\rho}. \tag{11}$$

It is the soliton solution
b)

$$\sigma_0(z') = \gamma(z') = \begin{cases} \frac{1}{2}, & z' > 0 \\ -\frac{1}{2}, & z' < 0 \end{cases}, \quad \rho_0 = 0.$$

In this case we have

$$\psi = -\frac{1}{2}\int\limits_{-\infty}^{\infty} \frac{\gamma(z')}{\sqrt{\rho^2 + (z-z')^2}}\,\mathrm{d}z' = \ln\frac{z + \sqrt{\rho^2 + z^2}}{\rho}.$$

Example 4. $\sigma(z') = \vartheta(z') \cdot \sigma_0(z')$, $\rho_0 = 0$,

$$\vartheta(z') = \begin{cases} 1, & -m < z' < m \\ 0, & -m > z' > m \end{cases}, \quad \sigma_0(z') = \frac{2}{\pi}\cdot\sigma_0\cdot K\left(\frac{z'}{2m}\cdot\frac{\alpha_0^2}{1+\alpha_0^2}\right).$$

Here K is elliptic integral of the first kind. In this case

$$\psi = \frac{1}{2}\int\limits_{-m}^{m} \frac{\sigma_0(z')}{\sqrt{\rho^2 + (z-z')^2}}\,\mathrm{d}z'. \tag{12}$$

If we put $\alpha_0 = 0$ in (12), then we obtain Schwarzschild solution (10).

Example 5. Let $\sigma(\rho_0, z') = \frac{1}{2}\,\delta(z')\,\vartheta(z')$, $\rho \neq 0$. In this case

$$\psi(\rho, z) = \frac{2m}{\pi}\cdot\frac{K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+z^2}}\right)}{\sqrt{(\rho+\rho_0)^2 + z^2}}. \tag{13}$$

If we put $\rho_0 = 0$, then we obtain the Chazy–Curzon solution.

Example 6. Let $\sigma(\rho_0, z') = \vartheta(z')$. In this case

$$\psi = \frac{1}{\pi}\int\limits_{-m}^{m} \frac{1}{\sqrt{(\rho+\rho_0)^2 + (z-z')^2}}\,K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2 + (z-z')^2}}\right)\mathrm{d}z'. \tag{14}$$

If we put $\rho_0 = 0$, we obtain Schwarzschild solution.

Example 7.

$$\psi(\rho, z) = \frac{2}{\pi}\int\limits_0^z \frac{K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+z'^2}}\right)}{\sqrt{(\rho+\rho_0)^2 + z'^2}}\,\mathrm{d}z'. \tag{15}$$

If we put $\rho_0 = 0$, then we obtain the soliton solution (11).

Example 8.

$$\psi(\rho,z) = -\frac{1}{2}\int\limits_{-\infty}^{+\infty} \frac{\gamma(z')\cdot K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+(z-z')^2}}\right)}{\sqrt{(\rho+\rho_0)^2+(z-z')^2}}\,\mathrm{d}z'. \tag{16}$$

If we put $\rho_0 = 0$, then we obtain the soliton solution (11).

4. The Weyl–Bonnor–Papapetrou–Majumdar Class of Solutions of Einstein–Maxwell Equations

1. Let us consider the subclass of the Weyl electrovacuum solutions of the equations (3)

$$u = \frac{(1-a_0^2)e^{\psi}}{1-a_0^2e^{2\psi}}, \quad A_3' = \frac{a_0\left(1-e^{2\psi}\right)}{1-a_0^2e^{2\psi}}, \quad \Delta\psi = 0. \tag{17}$$

The solution (17) includes also the most famous Weyl–Reissner–Nordström spherical-symmetric solution.

If we put ψ from (14) in (17), we obtain the generalization of the Reissner–Nordström solution.

2. The subclass of Papapetrou–Majumdar solution

$$u = \frac{1}{1+\psi}, \quad A_3' = \frac{\psi}{1+\psi}, \quad \Delta\psi = 0.$$

a) For $\psi = \dfrac{\mu_0 z}{(\rho^2+z^2)^{3/2}}$, we have the gravitational field of magnetic dipole [6]:

$$f = \left[1+\frac{\mu_0 z}{(\rho^2+z^2)^{3/2}}\right]^{-2}, \quad A_3' = \frac{\mu_0 z}{(\rho^2+z^2)^{3/2}}\cdot\left[1+\frac{\mu_0 z}{(\rho^2+z^2)^{3/2}}\right]^{-1}. \tag{18}$$

b) For $\psi = \dfrac{2\mu_0}{\pi}\cdot\dfrac{\partial}{\partial z}\left[\dfrac{K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+z^2}}\right)}{\sqrt{(\rho+\rho_0)^2+z^2}}\right]$, we have the generalization of (18).

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Метод сингулярных источников в задачах электровакуума Эйнштейна

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С помощью метода сингулярных источников возможно построение обобщений известных электровакуумных решений.

Ключевые слова: вакуумные уравнения Эйнштейна–Максвелла, аксиально-симметричные решения, асимптотически плоские метрики.