Физика

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On Cosmological Solutions with Sigma-Model Source

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A multidimensional model of gravity with a sigma-model action for scalar fields is considered. The gravitational model is defined on the manifold, which contains n Einstein factor spaces. General cosmological-type solutions to the field equations are obtained when $n-1$ factor-spaces are Ricci-flat. The solutions are defined up to solutions of geodesic equations corresponding to a sigma-model target space. Several examples of sigma-models are considered. A subclass of non-singular solutions is singled-out for the case when all factor-spaces are Ricci-flat.

Key words and phrases: cosmological solutions, sigma-model, acceleration.

1. Introduction

Scalar-tensor theories are well-known and important as alternatives to Einstein's general relativity. They are widely used, in particular, for explaining present-day accelerated expansion of the Universe [1] and in many other applications.

Here we consider a gravitational model governed (in essence) by a Lagrangian

$$\mathcal{L} = R[g] - h_{ab}(\varphi) g^{MN} \partial_M \varphi^a \partial_N \varphi^b, \tag{1}$$

where g is a metric and non-linear "scalar fields" φ^α come to the Lagrangian in a sigma-model form with a target space metric h assumed. For a review of sigma-models see [2] and refs. therein.

The Lagrangian (1) with $h_{ab} = const$ describes the truncated $NS - NS$ sector of various $D = 10$ and $D = 11$ supergravity theories in the Einstein frame [3]. Usually these theories contain form-fields (fluxes) in addition to massless scalar fields and Chern-Simons (CS) terms. In this sense, the Lagrangian (1) matches zero flux (and CS) limit. For $D = 3$ the Lagrangians of such types are generic ones when dimensional reductions of (bosonic sectors of) supergravity models are considered, see [4, 5] and refs therein.

Here we deal with cosmological-type solutions defined on the product of n Einstein spaces (e.g. Ricci-flat ones). The integrable cosmological configurations were studied in numerous papers, see [6, 7] (without scalar fields), [8–11] (with one scalar field), [12] etc. The authors of these papers restricted their attention to a linear sigma-model for which components h_{ab} are constant. Here we study the solutions for $h_{ab}(\varphi)$ with arbitrary dependence on φ^a (e.g. for a non-linear sigma-model).

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2. The Model

We start by considering an action of the form

$$S = \frac{1}{2\kappa^2} \int\limits_M \mathrm{d}^D x \sqrt{|g|} \Big\{ R[g] - h_{ab}(\varphi) g^{MN} \partial_M \varphi^a \partial_N \varphi^b \Big\} + S_{YGH}, \tag{2}$$

where κ^2 is a D-dimensional gravitational coupling, $g = g_{MN} \mathrm{d}x^M \otimes \mathrm{d}x^N$ is a metric defined on a manifold M, $\varphi : M \to M_\varphi$ is a smooth sigma-model map and M_φ is a l-dimensional manifold (target space) equipped with the metric $h = h_{ab}(\varphi) \mathrm{d}\varphi^a \otimes \mathrm{d}\varphi^b$ (φ^a are coordinates on M_φ). Here S_{YGH} is the standard York-Gibbons-Hawking boundary term [13, 14].

The field equations for the action (2) read as follows

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN}, \tag{3}$$

$$\frac{1}{\sqrt{|g|}} \partial_M (g^{MN} \sqrt{|g|} h_{ab}(\varphi) \partial_N \varphi^b) - \frac{1}{2} \frac{\partial h_{cb}(\varphi)}{\partial \varphi^a} \partial_K \varphi^c \partial_L \varphi^b g^{KL} = 0, \tag{4}$$

where

$$T_{MN} = h_{ab}(\varphi) \partial_M \varphi^a \partial_N \varphi^b - \frac{1}{2} h_{ab}(\varphi) g_{MN} \partial_K \varphi^a \partial_L \varphi^b g^{KL}. \tag{5}$$

is the stress-energy tensor.

Here we consider a cosmological-type ansatz for the metric and “scalar fields”

$$g = w e^{2\gamma(u)} \mathrm{d}u \otimes \mathrm{d}u + \sum_{i=1}^{n} e^{2\beta^i(u)} g^i, \tag{6}$$

$$\varphi^a = \varphi^a(u), \tag{7}$$

where $a = 1, \ldots, l$.

The metric is defined on the manifold

$$M = \mathbb{R}_* \times M_1 \times \ldots \times M_n, \tag{8}$$

where $\mathbb{R}_* = (u_-, u_+)$ and any factor-space M_i is a d_i-dimensional Einstein manifold with the metric g^i obeying

$$R_{m_i n_i}[g^i] = \xi_i g^i_{m_i n_i}, \tag{9}$$

where $i = 1, \ldots, n$.

To find solutions for the equations (3)–(4) seems to be complicated due to the non-linear structure of the Einstein equations and intricacy having scalar fields. However it may be shown that the field equation for the model (2) with the metric and “scalar fields” from (6), (7) are equivalent to the Lagrange equations corresponding to the Lagrangian of the one-dimensional $(n + l)$-component σ-model

$$L = \frac{1}{2} \mathcal{N}^{-1} [G_{ij} \dot\beta^i \dot\beta^j + h_{ab}(\varphi) \dot\varphi^a \dot\varphi^b] - \mathcal{N} V_\xi. \tag{10}$$

Here $\mathcal{N} = \exp(\gamma - \gamma_0) > 0$ is a modified lapse function, $\gamma_0 = \sum\limits_{i=1}^{n} d_i \beta^i$,

$$G_{ij} = d_i \delta_{ij} - d_i d_j, \tag{11}$$

where $i,j = 1,\dots,n$, are components of the gravitational part of the minisuperspace metric and

$$V_{\xi} = \frac{w}{2}\sum_{i=1}^{n}\xi_i d_i e^{-2\beta^i+2\gamma_0} \tag{12}$$

is the potential. For the constant $h_{ab}(\varphi) = h_{ab}$ the reduction to the sigma-model was proved (for more general set up) in [15]. We note that $h_{ab}(\varphi)$ can be interpreted as a scalar part of the target space metric. Here and in what follows $\dot{A} = \frac{\mathrm{d}A}{\mathrm{d}u}$.

When all M_i have finite volumes the substitution of (6) and (7) into the action (2) gives us the following relation

$$S = \mu\int L\mathrm{d}t \tag{13}$$

where L is defined in (10), $\mu = -\frac{w}{\kappa^2}\prod_{i=1}^{n}V_i$, and $V_i = \int_{M_i}\mathrm{d}^{d_i}y\left(\sqrt{\det\left(g^i_{m_in_i}\right)}\right)$ is the volume of M_i, $i = 1,\dots,n$.

The relation (13) can be derived using the following expression for the scalar curvature

$$R = -we^{-2\gamma}\Big(2\ddot{\gamma}_0 - 2\dot{\gamma}_0\dot{\gamma} + \dot{\gamma}_0^2 + \sum_{i=1}^{n}d_i\left(\dot{\beta}^i\right)^2\Big) + \sum_{i=1}^{n}e^{-2\beta^i}R[g^i], \tag{14}$$

where $R[g^i] = \xi_i d_i$ is the scalar curvature corresponding to the M_i-manifold. To obtain (13) one should extract the total derivative term in (14) which is canceled by the York-Gibbons-Hawking boundary term.

We write the Lagrange equations for (10) and then put $\mathcal{N} = 1$, or equivalently $\gamma = \gamma_0$, i.e. when u is a harmonic variable. We get

$$G_{ij}\ddot{\beta}^j + w\sum_{j=1}^{n}\xi_j d_j(-\delta^j_i + d_i)e^{-2\beta^j+2\gamma_0} = 0, \tag{15}$$

where $i = 1,\dots,n$,

$$\frac{\mathrm{d}(h_{cb}(\varphi)\dot{\varphi}^b)}{\mathrm{d}u} - \frac{1}{2}\frac{\partial h_{ab}(\varphi)}{\partial\varphi^c}\dot{\varphi}^a\dot{\varphi}^b = 0, \tag{16}$$

$c = 1,\dots,l$, and

$$\frac{1}{2}G_{ij}\dot{\beta}^i\dot{\beta}^j + \frac{1}{2}h_{ab}(\varphi)\dot{\varphi}^a\dot{\varphi}^b + V_{\xi} = 0. \tag{17}$$

In fact, equations (15) are nothing else but Lagrange equations corresponding to the Lagrangian

$$L_{\beta} = \frac{1}{2}G_{ij}\dot{\beta}^i\dot{\beta}^j - V_{\xi} \tag{18}$$

with the energy integral of motion

$$E_{\beta} = \frac{1}{2}G_{ij}\dot{\beta}^i\dot{\beta}^j + V_{\xi}. \tag{19}$$

Likewise (15), the equations (16) are Lagrange equations corresponding to the Lagrangian

$$L_{\varphi} = \frac{1}{2}h_{ab}(\varphi)\dot{\varphi}^a\dot{\varphi}^b \tag{20}$$

with the energy integral of motion

$$E_{\varphi} = \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b. \tag{21}$$

Equations (16) are equivalent to geodesic equations corresponding to the metric h.

The relation (17) is the energy constraint

$$E = E_{\beta} + E_{\varphi} = 0, \tag{22}$$

coming from $\partial L / \partial \mathcal{N} = 0$ (for $\mathcal{N} = 1$).

Equations (15) may be rewritten in an equivalent form

$$\ddot{\beta}^i - w \xi_i e^{-2\beta^i + 2\gamma_0} = 0, \tag{23}$$

where $i = 1, \ldots, n$. These expressions may be obtained from (15) by using the inverse matrix $(G^{ij}) = (G_{ij})^{-1}$:

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D} \tag{24}$$

and the following relations for $u^{(k)}$-vectors:

$$u_i^{(k)} = -\delta_i^k + d_i, \qquad u^{(k)i} = G^{ij} u_j^{(k)} = -\frac{\delta^{ki}}{d_i}. \tag{25}$$

where $i, j, k = 1, \ldots, n$.

In what follows we will use the following relation

$$(u^{(k)}, u^{(k)}) = G^{ij} u_i^{(k)} u_j^{(k)} = \frac{1}{d_k} - 1, \tag{26}$$

where $k = 1, \ldots, n$.

Hence, the problem of finding the cosmological-type solutions for the model (2) (with u being harmonic variable) is reduced to solving the equations of motion for the Lagrangians L_{β} and L_{φ} with the energy constraint (22) imposed.

Geodesics for a flat metric h.

For the constant $h_{ab}(\varphi) = h_{ab}$ eqs (16) read

$$\ddot{\varphi}^a = 0, \tag{27}$$

or, equivalently,

$$\varphi^a = \upsilon_{\varphi}^a u + \varphi_0^a, \tag{28}$$

where $\upsilon_{\varphi}^a, \varphi_0^a$ are integration constants, $a = 1, \ldots, l$.

The energy for scalar fields (21) takes the form

$$E_{\varphi} = \frac{1}{2} h_{ab} \upsilon_{\varphi}^a \upsilon_{\varphi}^b. \tag{29}$$

More examples of geodesic solutions will be given in Section 4.

3. Cosmological-Type Solutions

In this section we deal with certain examples of cosmological-type solutions with the metric and "scalar fields" from (6) and (7), respectively.

3.1. Solutions with n Ricci-flat Spaces

In this subsection, we focus on the solutions for the case when all factor-spaces M_i are Ricci-flat:

$$\mathrm{Ric}[g^i] = 0, \tag{30}$$

where $i = 1, \dots, n$.

Due to (30) the potential V_ξ is equal to zero and the equations of motion (23) for β^i now become

$$\ddot{\beta}^i = 0, \tag{31}$$

where $i = 1, \dots, n$.

Integration of the equations (31) yields

$$\beta^i = v^i u + \beta^i_0, \quad \gamma_0 = \sum_{i=1}^{n} d_i \left(v^i t + \beta^i_0 \right), \tag{32}$$

where the parameters v^i and β^i_0 are integration constants and the energy (19) takes the form

$$E_\beta = \frac{1}{2} G_{ij} v^i v^j, \tag{33}$$

where the minisuperspace metric G_{ij} is given by (11).

The metric reads

$$g = w \exp\left[2 \sum_{i=1}^{n} d_i (v^i u + \beta^i_0) \right] \mathrm{d}u \otimes \mathrm{d}u + \sum_{i=1}^{n} \exp\left[2 \left(v^i u + \beta^i_0 \right) \right] g^i. \tag{34}$$

The “scalar fields” obey eqs. (16) with the energy constraint

$$E_\varphi = \frac{1}{2} h_{ab}(\varphi) \dot{\varphi}^a \dot{\varphi}^b = -\frac{1}{2} G_{ij} v^i v^j. \tag{35}$$

In a special case of one (non-fantom) scalar field ($h_{11} = 1$) and $w = -1$ this solution was obtained in [8].

The scalar curvature for the metric (34) reads (see (14))

$$R[g] = -w \left(G_{ij} v^i v^j \right) e^{-2\gamma_0}. \tag{36}$$

In what follows we use a parameter

$$\Sigma = \Sigma(v) \equiv \sum_{i=1}^{n} d_i v^i \tag{37}$$

to classify the solutions.

Non-special Kasner-like solutions.

First we shall consider the non-special case when $\Sigma(v) \neq 0$.

Let us define a “synchronous” variable

$$\tau = \frac{1}{|\Sigma(v)|} \exp\left[\sum_{j=1}^{n} (v^j u + \beta^j_0) d_j \right] \tag{38}$$

obeying $e^{2\gamma_0(\beta)} \mathrm{d}u^2 = \mathrm{d}\tau^2$.

We introduce new parameters:

$$\alpha^i = \upsilon^i/\Sigma(\upsilon), \tag{39}$$

where $i = 1, \ldots, n$, and

$$\mathcal{E}_\varphi = E_\varphi/(\Sigma(\upsilon))^2. \tag{40}$$

Then the metric reads

$$g = w \mathrm{d}\tau \otimes \tau + \sum_{i=1}^{n} c_i^2 \tau^{2\alpha_i} g^i, \tag{41}$$

where $\tau > 0$. "Scalar fields" are solutions to equations of motion (see (16))

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\tau h_{cb}(\varphi)\frac{\mathrm{d}\varphi^b}{\mathrm{d}\tau}\right] - \frac{1}{2}\tau\frac{\partial h_{ab}(\varphi)}{\partial\varphi^c}\frac{\mathrm{d}\varphi^a}{\mathrm{d}\tau}\frac{\mathrm{d}\varphi^b}{\mathrm{d}\tau} = 0, \tag{42}$$

where $a = 1, ..., l$. The parameters (39) obey the Kasner-like conditions

$$\sum_{i=1}^{n} d_i \alpha^i = 1, \tag{43}$$

$$\sum_{i=1}^{n} d_i (\alpha^i)^2 = 1 - 2\mathcal{E}_\varphi, \tag{44}$$

where

$$2\mathcal{E}_\varphi = \tau^2 h_{ab}(\varphi)\frac{\mathrm{d}\varphi^a}{\mathrm{d}\tau}\frac{\mathrm{d}\varphi^b}{\mathrm{d}\tau}, \tag{45}$$

is the integral of motion for eqs. (42).

In (41) c_i are constants

$$c_i = |\Sigma|^{\alpha^i} \exp\left[\beta_0^i - \alpha^i \sum_{j=1}^{n} \beta_0^j d_j\right], \tag{46}$$

where $i = 1, \ldots, n$, obeying $\prod_{i=1}^{n} c_i^{d_i} = |\Sigma(\upsilon)|$.

Flat h**.** For the special case of the flat target space metric $h_{ab}(\varphi) = h_{ab}$ we get

$$\varphi^a = \alpha_\varphi^a \ln\tau + \bar{\varphi}_0^a, \tag{47}$$

where $\bar{\varphi}_0^a$ are constants, $a = 1, \ldots, l$, and

$$\mathcal{E}_\varphi = \frac{1}{2} h_{ab}\alpha_\varphi^a \alpha_\varphi^b. \tag{48}$$

The scalar curvature (36) reads

$$R[g] = 2w\mathcal{E}_\varphi \tau^{-2}. \tag{49}$$

It diverges for $\tau \to +0$ if $\mathcal{E}_\varphi \neq 0$. Hence all solutions with $\mathcal{E}_\varphi \neq 0$ are singular.

For $\mathcal{E}_\varphi \neq 0$ the solutions with non-Milne-type sets of the Kasner parameters are singular when all g^i have Euclidean signatures since the Riemann tensor squared is divergent at $\tau \to +0$ [16]. For Milne-type sets of parameters, i.e. when $d_i = 1$

and $\alpha^i = 1$ for some i ($\alpha^j = 0$ for all $j \neq i$) the metric is regular, when either i) $g^{(i)} = -w dy^i \otimes dy^i$, $M_i = \mathbb{R}$ ($-\infty < y^i < +\infty$), or ii) $g^{(i)} = w dy^i \otimes dy^i$, M_i is a circle of length L_i ($0 < y^i < L_i$) and $c_i L_i = 2\pi$ (i.e. when the cone singularity is absent).

Special (steady state) solutions. Now we consider the special case when $\Sigma(v) = 0$. Due to (35) we obtain

$$E_\varphi = -\frac{1}{2}\sum_{i=1}^{n} d_i (v^i)^2 \leqslant 0. \tag{50}$$

We get in this case $\gamma_0 = \sum_{i=1}^{n} d_i \beta_0^i = \mathrm{const}$ and hence the scalar curvature

$$R[g] = 2wE_\varphi e^{-2\gamma_0} \tag{51}$$

and the volume scale factor $v = e^{\gamma_0}$ are constants.

The "synchronous" variable is proportional to u ($\tau = e^{\gamma_0} u$).

Hence, we obtained a restriction for the energy $E_\varphi \leqslant 0$. For $E_\varphi = 0$, all $v^i = 0$, and we are led to a static Ricci-flat solution.

For $E_\varphi < 0$ we get $R[g] \neq 0$. This possibility occurs if the target space metric h is not positive-definite (e.g. there are phantom scalar fields for flat h). For solutions with one (phantom) scalar field see [8].

Solutions with acceleration. Let $d_1 = 3$ and $M_1 = \mathbb{R}^3$. The factor-space M_1 may be considered as describing our space. In both cases there exist subclasses of solutions describing accelerated expansion of our space.

Indeed, for Kasner-like solutions with $w = -1$ one could make a replacement $\tau \mapsto \tau_0 - \tau$ where τ_0 is a constant (corresponding to the so-called "big rip"). For such replacement the scale factor of M_1 reads

$$a_1(\tau) = c_1(\tau_0 - \tau)^{\alpha_1}, \tag{52}$$

where $c_1 > 0$.

If $\alpha_1 < 0$ we get accelerated expansion of 3-dimensional factor space M_1. For the Hubble parameter we get

$$H = \dot{a_1}/a_1 = (-\alpha_1)/(\tau_0 - \tau), \tag{53}$$

while the variation of the effective gravitational constant reads

$$\dot{G}/G = (1 - 3\alpha_1)/(\tau_0 - \tau). \tag{54}$$

Here we used the relation $G = \mathrm{const}\prod_{i=2}^{n}(\tau_0 - \tau)^{-d_i\alpha_i} = \mathrm{const}(\tau_0 - \tau)^{3\alpha_1 - 1}$ [17] (see (43)). This implies the relation

$$\delta = \dot{G}/(GH) = (3\alpha_1 - 1)/\alpha_1. \tag{55}$$

The condition $\alpha_1 < 0$ yields the huge value $|\delta| > 3$ which does not obey the observational limits [17]: $|\delta| < 0.1$. Thus, accelerated expansion of M_1 factor-space is incompatible with tests on $G - dot$.

Analogous consideration may be carried out for special (steady state) solutions. For $w = -1$, $d_1 = 3$ and $v^1 > 0$ we get accelerated expansion of 3-dimensional factor-space M_1. In this case due to $\Sigma = 0$ one get $\delta = 3$ which also does not pass the $G - dot$ test.

3.2. Solutions with One Curved Einstein Space and $n-1$ Ricci Flat Spaces

Here we put

$$\mathrm{Ric}[g^1] = \xi_1 g^1, \quad \xi_1 \neq 0, \qquad \mathrm{Ric}[g^i] = 0, \quad i > 1, \tag{56}$$

i.e. the first space (M_1, g^1) is an Einstein space of non-zero scalar curvature and other spaces (M_i, g^i) are Ricci-flat.

The Lagrangian (18) reads in this case

$$L_\beta = \frac{1}{2} G_{ij} \dot{\beta}^i \dot{\beta}^j - \frac{w}{2} \xi_1 d_1 \exp\left(-2\beta^1 + 2\gamma_0\right), \tag{57}$$

where $-\beta^1 + \gamma_0 = u_i^{(1)} \beta^i$ and $u_i^{(1)} = -\delta_i^1 + d_i$.

The Lagrange equations corresponding to the Lagrangian (57) are integrated in Appendix. The solution reads

$$\beta^1 = \ln |f|^{\frac{1}{1-d_1}} + v^1 u + \beta_0^1, \tag{58}$$

$$\beta^i = v^i t + \beta_0^i, \quad i > 1, \tag{59}$$

where β_0^i, v^i are constants obeying

$$v^1 = \sum_{i=1}^n v^i d_i, \quad \beta_0^1 = \sum_{i=1}^n \beta_0^i d_i. \tag{60}$$

The function f is following

$$f = \begin{cases} R \sinh(\sqrt{C}(u - u_0)), & C > 0, \quad w\xi_1 > 0; \\ |\xi_1 (d_1 - 1)|^{1/2} (u - u_0), & C = 0, \quad w\xi_1 > 0; \\ R \sin(\sqrt{-C}(u - u_0)), & C < 0, \quad w\xi_1 > 0; \\ R \cosh(\sqrt{C}(u - u_0)), & C > 0, \quad w\xi_1 < 0, \end{cases} \tag{61}$$

where u_0 and C are constants and

$$R = \sqrt{\frac{|\xi_1 (d_1 - 1)|}{|C|}}. \tag{62}$$

For γ_0 we get

$$\gamma_0 = \beta^1 - \ln |f|. \tag{63}$$

The energy integral of motion E_β corresponding to L_β reads (see Appendix)

$$E_\beta = \frac{C d_1}{2(1 - d_1)} + \frac{1}{2} G_{ij} v^i v^j. \tag{64}$$

Using (58), (59) and (63) we are led to the relation for the metric

$$g = |f|^{\frac{2d_1}{1-d_1}} \exp\left[2(v^1 u + \beta_0^1)\right] \left(w du \otimes du + f^2 g^1\right) + \sum_{i=2}^n \exp\left[2(v^i u + \beta_0^i)\right] g^i. \tag{65}$$

The “scalar fields” obey eqs (16) with the energy constraint

$$E_\varphi = \frac{1}{2} h_{ab}(\varphi)\dot{\varphi}^a \dot{\varphi}^b = \frac{C d_1}{2(d_1 - 1)} - \frac{1}{2} G_{ij} v^i v^j. \tag{66}$$

Here the constraints (60) on β_0^i, v^i should be kept in mind, and the function f is defined in (61).

In a special case of one (non-fantom) scalar field ($h_{11} = 1$) and $w = -1$ this solution was obtained earlier in [9, 10], see also [11].

4. More Examples of Geodesic Solutions

In this section we consider three examples of solutions to geodesic equations corresponding to the metric h that may be used for the cosmological-type solutions above.

4.1. Metric on S^2

Let h be a metric on a two-dimensional sphere S^2

$$h = \mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2 \vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi. \tag{67}$$

The simplest solution to geodesic equations (16) for the metric reads

$$\varphi = \omega u, \quad \vartheta = \pi/2, \tag{68}$$

where ω is constant. Here $E_\varphi = \frac{1}{2}\omega^2$. The general solution to geodesic equations may be obtained by a proper isometry $SO(3)$- transformation of the solution from (68).

4.2. Metric on $\mathrm{d}S^2$

Now we put h to be a metric on a two-dimensional de Sitter space $\mathrm{d}S^2$

$$h = -\mathrm{d}\chi \otimes \mathrm{d}\chi + \cosh^2 \chi \mathrm{d}\varphi \otimes \mathrm{d}\varphi. \tag{69}$$

There are three basic solutions to geodesic equations (16) in this case

$$\varphi = \omega u, \qquad \chi = 0, \tag{70}$$

$$\chi = v u, \qquad \varphi = 0, \tag{71}$$

$$\tan \varphi = \sinh \chi = m u, \tag{72}$$

where ω, v and m are constants. For the energy we have $E_\varphi = \frac{1}{2}\omega^2$, $-\frac{1}{2}v^2$ and 0, for space-like, time-like,and null geodesics, respectively. The general solution to geodesic equations may be obtained by a proper isometry $SO(1,2)$ — transformation of the solutions from (70)–(72).

4.3. A Diagonal Metric h

Here we deal with a diagonal metric

$$h = \varepsilon_0 d\varphi \otimes \mathrm{d}\varphi + \sum_{k=1}^{l-1} \varepsilon_k A_k^2(\varphi) \mathrm{d}\psi^k \otimes \mathrm{d}\psi^k, \tag{73}$$

where $\varepsilon_0 = \pm 1$, $\varepsilon_k = \pm 1$ ($k > 0$) and all $A_k(\varphi) > 0$, are smooth functions.

The Lagrange function for the non-linear sigma-model is given by

$$L_\varphi = \frac{1}{2}\left[\varepsilon_0\dot{\varphi}^2 + \sum_{k=1}^{l-1}\varepsilon_k A_k^2(\varphi)(\dot{\psi}^k)^2\right]. \tag{74}$$

Equations of motion for cyclic variables ψ^k

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\varepsilon_k A_k^2(\varphi)\dot{\psi}^k\right) = 0 \tag{75}$$

yield the following of integrals of motion

$$\varepsilon_k A_k^2(\varphi)\dot{\psi}^k = M_k, \tag{76}$$

where $k = 1, \ldots, l-1$.

Another integration constant is energy E_φ

$$E_\varphi = \frac{1}{2}\left[\varepsilon_0\dot{\varphi}^2 + \sum_{k=1}^{l-1}\varepsilon_k A_k^2(\varphi)(\dot{\psi}^k)^2\right] \tag{77}$$

which due to (76) reads

$$E_\varphi = \frac{1}{2}\left[\varepsilon_0\dot{\varphi}^2 + \sum_{k=1}^{l-1}\varepsilon_k M_k^2 A_k^{-2}(\varphi)\right]. \tag{78}$$

This relation implies the following quadrature

$$\int\limits_{\varphi_0}^{\varphi}\frac{\mathrm{d}\bar{\varphi}}{\sqrt{2\varepsilon_0 E_\varphi - \varepsilon_0\sum_{k=1}^{l-1}\varepsilon_k M_k^2 A_k^{-2}(\bar{\varphi})}} = u - u_0, \tag{79}$$

which implicitly defines the function $\varphi = \varphi(u)$.

Another quadratures just following from (76)

$$\psi^k - \psi_0^k = \int\limits_{u_0}^{u}\mathrm{d}\bar{u}\varepsilon_k M_k A_k^{-2}(\bar{u}), \tag{80}$$

complete the integration of the geodesic equations for the metric (73).

For $A_k(\varphi) = \exp(\lambda\varphi)$, $\lambda \neq 0$, the metric (73) may describe either a part of de-Sitter space (if $\varepsilon_0 = -1$, $\varepsilon_k = 1$, $k > 0$) or a part of anti-de-Sitter space (if $\varepsilon_1 = -1$, $\varepsilon_r = 1$, $r \neq 1$). The case $l = 3$ is of interest in connection with the so-called the AWE hypothesis [18].

5. Conclusions

Here we have considered a multidimensional model of gravity with a sigma-model source (for scalar fields). The model is defined on the manifold M, which contains n Einstein spaces.

We have obtained exact cosmological-type solutions to the field equations in two cases: i) when either all factor-spaces are Ricci-flat or ii) when only one factor-space space has nonzero scalar curvature.

In the first case i) the solutions have either Kasner-like form or describe steady-state solutions, generalizing those from [8]. The Kasner-like solutions are mostly singular with certain exceptions (of Milne-type).

For the case when all factor-spaces are Ricci-flat we have singled-out a subclass of solutions describing accelerated expansion of 3-dimensional manifold. We have shown that these solutions do not obey the tests on variation of G.

The second subclass of solutions ii) (e.g. for spherically symmetric configurations) will be considered in a separate publication (e.g. a possible fitting of acceleration with bounds on G-dot, see [19] and ref. therein).

6. Appendix: Solutions Governed by Liouville Equation

Here we consider a Toda-like system with the following Lagrangian

$$L = \frac{1}{2}\langle \dot{\beta}, \dot{\beta} \rangle - A \exp\left(2\langle b, \beta \rangle\right), \tag{81}$$

where $\beta \in \mathbb{R}^n$, $A \neq 0$, $b \in \mathbb{R}^n$. The scalar product for vectors belonging to $\mathbb{R}^n$ is defined by

$$\langle \beta_1, \beta_2 \rangle = G_{ij} \beta_1^i \beta_2^j, \tag{82}$$

where G_{ij} is a non-degenerate symmetric matrix (e.g. given by (11)).

The Lagrange equations corresponding to the model (81) read (in a condensed vector form)

$$\ddot{\beta} + 2Ab \exp\left(2\langle b, \beta \rangle\right) = 0. \tag{83}$$

Let $\langle b, b \rangle \neq 0$.

Eqs (83) is exactly integrable and the solution has the following form

$$\beta = \frac{b}{\langle b, b \rangle} q + vt + \beta_0, \tag{84}$$

where $\langle b, b \rangle \neq 0$ and $v, \beta_0 \in \mathbb{R}^n$ are constant vectors obeying

$$\langle v, b \rangle = \langle \beta_0, b \rangle = 0. \tag{85}$$

The function $q = q(t)$ obeys the Liouville equation

$$\ddot{q} + 2A\langle b, b \rangle e^{2q} = 0, \tag{86}$$

The solution to Liouville equation reads

$$q = -\ln|f|, \tag{87}$$

where

$$f = \begin{cases} R \sinh(\sqrt{C}(t - t_0)), \quad C > 0, \quad \bar{A} < 0; \\ |2\bar{A}|^{1/2}(t - t_0), \quad C = 0, \quad \bar{A} < 0; \\ R \sin(\sqrt{-C}(t - t_0)), \quad C < 0, \quad \bar{A} < 0; \\ R \cosh(\sqrt{C}(t - t_0)), \quad C > 0, \quad \bar{A} > 0; \end{cases} \tag{88}$$

here we put $\bar{A} = A\langle b, b \rangle$ and

$$R = \sqrt{\frac{2|A\langle b, b \rangle|}{|C|}}. \tag{89}$$

The energy corresponding to the model (81) reads

$$E = \frac{1}{2}\langle \dot{\beta}, \dot{\beta} \rangle + A e^{2\langle b, \beta \rangle}. \tag{90}$$

After substitution of (84) to (90) we obtain

$$E = E_T + \frac{1}{2}\langle v, v \rangle, \tag{91}$$

where

$$E_T = \frac{1}{2\langle b, b \rangle}\dot{q}^2 + A e^{2q}. \tag{92}$$

Due to (88) we get

$$E_T = \frac{C}{2\langle b, b \rangle}. \tag{93}$$

Proposition. For $\langle b, b \rangle \neq 0$ all solutions to Lagrange equations (83) are covered by the relations (84), (85), (87) and (88).

Proof. $\Rightarrow$. It is obvious that the solutions (84), (85) with q from (87), (88) obey the equations of motion (83).

$\Leftarrow$. Let us show that the relations (84), (85), (87) and (88) follow from (83).

Let $q = \langle b, \beta \rangle$ and $y = \beta - (bq)/\langle b, b \rangle$. It is obvious that $\langle b, y \rangle = 0$. It follows from (83) that that the equation (86) and

$$\ddot{y} = 0, \quad \Rightarrow \quad y = vt + \beta_0, \tag{94}$$

where constant vectors v and β_0 obey (due to $\langle b, y \rangle = 0$)

$$\langle b, y \rangle = 0 \quad \Rightarrow \langle b, v \rangle = \langle b, \beta_0 \rangle = 0. \tag{95}$$

Hence

$$\beta = \frac{bq}{\langle b, b \rangle} + y = \frac{bq}{\langle b, b \rangle} + vt + \beta_0 \tag{96}$$

where $q = q(t)$ obeys (86) and hence it is given by relations (87) and (88).

The Proposition is proved. $\square$

Let us introduce a dual vector $u = (u_i)$: $u_i = G_{ij} b^j$. Then we get $u(\beta) = u_i \beta^i = \langle b, \beta \rangle$, $(u, u) = G^{ij} u_i u_j = \langle b, b \rangle$ $((G^{ij}) = (G_{ij})^{-1})$ and the solution (84) reads

$$\beta^i = -\frac{u^i}{(u, u)} \ln |f| + v^i t + \beta_0^i, \tag{97}$$

where $i = 1, \ldots, n$, where $(u, u) \neq 0$,

$$u(v) = u_i v^i = 0, \qquad u(\beta_0) = u_i \beta_0^i = 0, \tag{98}$$

and function f is defined in (88) with

$$R = \sqrt{\frac{2|A(u, u)|}{|C|}}, \quad \bar{A} = A(u, u). \tag{99}$$

For the energy (92) we obtain from (91), (93)

$$E = \frac{C}{2(u,u)} + \frac{1}{2} G_{ij} v^i v^j. \qquad (100)$$

Example. Let us consider the Lagrange system from Section 3 with parameters: $A = \frac{w}{2} \xi_1 d_1$, $u_i = u_i^{(1)} = -\delta_i^1 + d_i$, $d_1 > 1$. Then due to (25) and (26) we get $u^i = -\frac{\delta^{1i}}{d_1}$ and $(u,u) = \frac{1}{d_1} - 1 < 0$. The solution reads

$$\beta^i = \frac{\delta_1^i}{1 - d_1} \ln |f| + v^i t + \beta_0^i, \qquad (101)$$

where $i = 1, \ldots, n$, with constraints

$$v^1 = \sum_{i=1}^n v^i d_i, \qquad \beta_0^1 = \sum_{i=1}^n \beta_0^i d_i, \qquad (102)$$

imposed. In (88) we should put $\bar{A} = \frac{w}{2} \xi_1 (1 - d_1)$ and $R = \sqrt{\frac{|\xi_1|(d_1 - 1)}{|C|}}$.

For the energy we get

$$E = \frac{C d_1}{2(1 - d_1)} + \frac{1}{2} G_{ij} v^i v^j. \qquad (103)$$

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О космологических решениях с сигма-модельным источником

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Рассматривается многомерная модель скалярно-тензорной гравитации с сигма-модельным действием для скалярного сектора. Гравитационная модель определена на многообразии, которое содержит n фактор-пространств Эйнштейна. Получены общие решения космологического типа для полевых уравнений, когда все фактор-пространства, за исключением одного, риччи-плоские. Решения определены с точностью до решения уравнений геодезических на пространстве мишеней. В случае, когда все фактор-пространства риччи-плоские, выделен подкласс несингулярных решений.

Ключевые слова: космологические решения, сигма-модель, ускорение.