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Numerical Description of the Fluid Pipe Motion with Multiscale Turbulence Model

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A numerical description of turbulent motion of a pipe localized fluid is given in the paper. The description is based on multiscale turbulent fluid motion model developed by the author. Comparison of numerical results and experimental data is given.

Key words and phrases: mathematical modeling, turbulence, numerical experiment, incorrect task.

The multiscale model of turbulent fluid motion was developed in earlier papers by author [1, 2]. In this mathematical model a fluid was represented as an ensemble of so-called fluid particles of different sizes. In order to describe this ensemble we invoke statistical physics. It helps us to construct an analogue of the Boltzmann equation which describes binary interaction between two fluid particles – first the particles stick together and then break down into two identical fluid particles. Such decay served as a mechanism of turbulent mixing. Boltzmann kinetic equation was solved by the Grad's method, which allowed to obtain closure conditions in the form of the expression of unknown flows. They form part of the corresponding divergences, through the gradients of known quantities.

In accordance with the multiscale model of turbulence, we define the average mass of a fluid particle $\Xi = \Xi(tr)$ and its volume $\Omega = \Omega(t\mathbf{r})$, where t stands for time, $\mathbf{r} = (x_1x_2x_3)$ is radius vector in space. In addition, we introduce the inverse values for the scales of turbulence: $\alpha = 1/\Xi$ and $\beta = 1/\Omega$. We define the density $\rho = \rho(tr)$ and the vector of fluid velocity $\mathbf{w} = \mathbf{w}(t\mathbf{r})$. According to the multiscale model, we can define the internal energy $\rho c T_m$ (c — heat capacity of liquid, T_m — the molecular temperature) in the ordinary sense, and turbulent energy $\frac{3}{2}nT_t$ ($n = \rho/\Xi$ — density of fluid particles) associated with large scale turbulent fluctuations of the liquid.

The resulting system of equations of multiscale turbulence model can be written as:

$$\begin{gathered}
\partial\alpha/\partial t + w\nabla\alpha = -\rho^{-1}\mathrm{div}\, I - \nu^{-1}\alpha, \\
\partial\beta/\partial t + w\nabla\beta = -\rho^{-1}(\beta/\alpha)\mathrm{div}\, I + \rho^{-1}(\beta^2/\alpha)\mathrm{div}\, J - \nu^{-1}\beta, \\
\partial\rho/\partial t + \mathrm{div}\,(\rho w) = 0, \\
\partial(\rho w_i)/\partial t + \partial(\rho w_i w_j + P_{ij}^{(m)} + P_{ij}^{(t)})/\partial x_j = 0, \\
\partial\left(\frac{3}{2}nT_t\right)/\partial t + \mathrm{div}\,\left(\frac{3}{2}nT_t w + q^{(t)}\right) + P_{ij}^{(t)}\partial w_i/\partial x_j + \nu^{-1}\frac{3}{2}nT_t = 0, \\
\partial(\rho c T_m)/\partial t + \mathrm{div}\,(\rho c T_m w + q^{(m)} + q^{(mt)}) + P_{ij}^{(m)}\partial w_i/\partial x_j - \nu^{-1}\frac{3}{2}nT_t = 0.
\end{gathered} \tag{1}$$

Equations (1) include a pair of equations for the scales of turbulence (α, β), and the ordinary laws of conservation of mass, momentum and energy in two forms: molecular and turbulent ones. The system of equations can describe both laminar and turbulent flows. The subindices (m) and (t) denote the molecular and turbulent characteristics respectively. The multiscale model of turbulence [1,2] allows to work out the following closure conditions which permit to express the unknown flows standing under the signs of the divergence, through the gradients of known quantities:

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$$I = k_I \alpha^{-1/2} \beta^{2/3} T_t^{1/2} \nabla \alpha, \quad J = k_j \alpha^{-1/2} \beta^{2/3} T_t^{1/2} \nabla(\alpha \beta^{-1}),$$

$$P_{ij}^{(t)} = P_t \delta_{ij} - \mu_t (\partial w_i / \partial x_j)*, \quad P_t = n T_t, \quad \mu_t = \frac{5}{64\sqrt{\pi}} \left(\frac{4\pi}{3}\right)^{2/3} \alpha^{-1/2} \beta^{2/3} T_t^{1/2}, \tag{2}$$

$$q^{(t)} = \frac{5}{2} k_I \alpha^{-2/3} \beta^{1/2} T_t^{3/2} \nabla \alpha - \frac{75}{256\sqrt{\pi}} \left(\frac{4\pi}{3}\right)^{2/3} \alpha^{-1/2} \beta^{2/3} T_t^{1/2} \nabla(\alpha T_t),$$

$$q^{(mt)} = -k_t \nabla T_m, \quad k_t = c k_q \alpha^{-1/2} \beta^{2/3} T_t^{1/2}.$$

For the final closure of the system of equations (1) it is necessary to add to equations (2) the equation of medium state $P_m = P_m(\rho T_m)$. The value $P_t = nT_t$ in (2) has the meaning of the turbulent “pressure” and is completely analogous to the molecular-kinetic pressure of an ideal gas. In (1) the molecular tensor $P_{ij}^{(m)}$ and energy flux $\mathbf{q}^{(m)}$ were entered. It was made in order to ensure the self-consistence between the desired equations (1) and the Navies–Stokes equations, in the absence of turbulence, i.e. when $T_t = 0$. This self-consistence does occur, because according to (2) at $T_t = 0$ — $\mathbf{I}$, $\mathbf{J}$, $P_{ij}^{(t)}$, $\mathbf{q}^{(t)}$ $\mathbf{q}^{(mt)} = 0$. The linkage between the molecular and turbulent components of the movement is carried out by adding a phenomenological term $\nu^{-1} \frac{3}{2} n T_t$ to the last two equations in (1). The system of equations (1) admits the general law of conservation of the sum of kinetic $(\frac{1}{2}\rho w^2)$, turbulent $(\frac{3}{2} n T_t)$ and internal $(\rho c T_m)$ energies, that is

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho w^2 + \frac{3}{2} n T_t + \rho c T_m \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho w^2 + \frac{3}{2} n T_t + \rho c T_m \right) w + \right.$$
$$\left. + \left(P^{(m)} + P^{(t)} \right) \cdot w + q^{(m)} + q^{(mt)} + q^{(t)} \right] = 0.$$

Let us define the tensor $\sigma_{ij}^{(t)} = P_{ij}^{(t)} - P_t \delta_{ij}$ and relate it and the flow $\mathbf{q}^{(mt)}$ with the Reynolds' turbulent stress tensor $\rho \langle w_i' w_j' \rangle$ ($\rho = const$) and the magnitude $\rho \langle w' T_m' \rangle$. The stress tensor is well-known in the theory of turbulence, and the denotations are used: $\langle ... \rangle$ is the time averaging operation, $\mathbf{w}' = \mathbf{w} - \langle \mathbf{w} \rangle$ — fluctuating velocity component, $T_m' = T_m - \langle T_m \rangle$ is the temperature fluctuating component. So we have: $\sigma_{ij}^{(t)} = \rho \langle w_i' w_j' \rangle$, $q^{(mt)} = \rho c \langle w' T_m' \rangle$. In this case, values μ_t and k_t in (2) can be interpreted as the coefficients of turbulent viscosity and thermal conductivity, which are functions of scale $(\alpha\beta)$ and the “temperature” of turbulence (T_t). This model admits a well-known in the turbulence theory, Boussinesq approximation when the Reynolds' stress tensor is proportional to the strain tensor $(\partial w_i / \partial x_j)^*$ and the turbulent flow of internal energy is proportional to the gradient of the molecular temperature, that is $\rho \langle w_i' w_j' \rangle = -\mu_t (\partial w_i / \partial x_j)^*$, $\rho c \langle w' T_m' \rangle = -k_t \nabla T_m$.

We define a “turbulent” and laminar Reynolds' numbers $\mathrm{Re}_t = \rho V L / \mu_t$, $\mathrm{Re}_m = \rho V L / \mu_m$, where V, L are the characteristic velocity and length. Then the total Reynolds number Re_Σ is the following: $\mathrm{Re}_\Sigma = (\mathrm{Re}_m^{-1} + \mathrm{Re}_t^{-1})^{-1}$. If Re_* is the critical Reynolds number, that is for higher values of Reynolds number a hydrodynamic instability arises, then for $\mathrm{Re}_\Sigma < \mathrm{Re}_*$ such instability does not emerge in the system of equations (1). Equations (1) are balanced so that, for any turbulent motion in which $\mathrm{Re}_m \gg 1$, Re_Σ is a finite and bounded, so that $\mathrm{Re}_\Sigma < \mathrm{Re}_*$. In this case all hydrodynamic functions α, β, ρ, $\mathbf{w}$, P_m, T_m, T_t considered as the solutions of the closed system of equations (1), will be sufficiently smooth functions.

The presence of turbulent viscosity in the motion equation in the system (1) provides automatic adjustment of nonexit beyond the motion equation domain of correctness. This mechanism operates as follows. In the regions where conditions for the development of turbulence arise (the regions of large velocity gradients), turbulence "temperature" T_t increases, that leads to an increase of turbulent viscosity and a decrease in velocity gradient. Turbulent fluid in terms of equations (1) could be imagined as laminar, but with a special kind of viscosity, which depends on traffic conditions.

We'll assume that the fluid is incompressible, that is $\rho = const$, then the first two equations in (1) which describe the behavior of the turbulence scales, conserve the forme, because $\beta = \rho\alpha$ and $\mathbf{J} = 0$. We can write down the equation for the inverse value of the fluid particle volume, that is

$$\partial\beta/\partial t + w\nabla\beta = -\mathrm{div}\,(k_I\rho^{-1/2}\beta^{1/6}T_t^{1/2}\nabla\beta) - 12\pi\left(\frac{3}{4\pi}\right)^{1/3}\rho^{-1}\mu_m\beta^{5/3}. \quad (3)$$

The equation (3) is interesting because the first term in the right side comes with an unusual sign. If the coefficient k_I was negative, then (3) would be the ordinary equation of diffusion type. However, $k_I \geqslant 0$ (as shown in papers [1, 2]), so from here on we shall call it a term of the negative diffusion type or the antidiffusion term.

Unlike ordinary diffusion which leads to smoothing out irregularities in the initial data, antidiffusion, on the contrary, reinforces these inhomogeneities, leading to the development of a complex "picket fence" of the peaks in the form of function β and the peak height rapidly tends to infinity. In other words, antidiffusion term describes the unstable process of progressive turbulence scale Ω refinement. Presence of antidiffusion term in equation (3) is due to the turbulent mechanism of adhesion and further decay of a pair of fluid particles into two identical fluid particles.

The action of turbulent mechanism of pairs agglomeration and decay into two identical fluid particles will lead to a redistribution of the initial fluid mass and volume by equal portions among all fluid particles. Thus, the functions Ξ and Ω decrease (α and β increase) compared with the initial state. If at the beginning the fluid particles are infinitely small and their number is infinitely large, Ξ and Ω will tend to zero (α and β tend to ∞), that is the grinding process of scale will not have the finite lower bound. The second term in the right hand side of (3), associated with molecular viscosity, prevents the growth of the peaks in the distribution of β to infinity. This can be seen from the degree of nonlinearity of the first and second terms. Thus, the molecular viscosity determines the minimum possible size of fluid particles.

Presence of antidiffusion term in equations (1), (3) means that these equations belong to the class of incorrectly formulated problems [3]. In this regard, equation (3) and all the system of equations (1) describing the turbulence become incorrectly formulated problem. To remove the incorrectness of antidiffusion type, we can use one of the algorithms approved in the theory of incorrectly formulated problems [3]. The choice of this algorithm implies the existence of some *a priory* information about the profiles of functions Ξ and $\Omega(\alpha\beta)$. *A priori* information can be very weak, for example, it reduces the total requirement of sufficient smoothness of the functions Ξ and Ω.

Let us calculate the turbulent motion of fluid in the pipe using equation (3). We consider the system of equations (1) for the case of turbulent flow of an incompressible fluid in a pipe. We use the cylindrical coordinate system (r, φ, z), puthing the z-axis at the center of the pipe. Let us assume that the pressure in the fluid decreases uniformly from input to output, that is $P_m + P_t = -(\Delta P z)/L + P$, where $\Delta P = P - P_1$ — differential pressure; P_0, P_1 — pressure at input and output pipes, respectively, L — pipe length.

As the system has axial symmetry and characteristics are uniform with respect to the coordinate z, the unknown values in the problem are the inverse volume of the fluid particles $\beta = \beta(tr)$, axial velocity component $w = w(tr)$ and "temperature" of turbulence $T_t = T_t(tr)$. It is convenient to use in calculations pressure of the turbulence $P_t = \beta T_t$ instead of temperature. So, taking into account the system of equations (1) and the conditions for closure (2), we obtain

$$\frac{\partial\beta}{\partial t}=-\frac{1}{r}\frac{\partial}{\partial r}\left(rk_I\rho^{-1/2}\beta^{-1/3}\frac{\partial\beta}{\partial r}\right)-12\pi\left(\frac{3}{4\pi}\right)^{1/3}\nu_m\beta^{5/3},$$

$$\frac{\partial w}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu_m+\frac{5}{64\sqrt{\pi}}\left(\frac{4\pi}{3}\right)^{2/3}\rho^{-1/2}\beta^{-1/3}P_t^{1/2})\frac{\partial w}{\partial r}\right]+\frac{\Delta P}{\rho L},\qquad(4)$$

$$\frac{\partial P_t}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{25}{128\sqrt{\pi}}\left(\frac{4\pi}{3}\right)^{1/3}\rho^{-1/3}P_t^{1/2}\frac{\partial P_t}{\partial r}\right]-$$
$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{5}{3}k_I\rho^{-1/2}\beta^{-1/3}P_t^{3/2}\frac{\partial\beta}{\partial r}\right)+\frac{5}{64\sqrt{\pi}}\left(\frac{4\pi}{3}\right)^{2/3}\rho^{1/2}\beta^{-1/3}P_t^{1/2}(\frac{\partial w}{\partial r})^2-$$
$$-12\pi\left(\frac{3}{4\pi}\right)^{1/3}\nu_m\beta^{2/3}P_t,$$

where $\nu_m=\mu_m/\rho$ — molecular kinematic viscosity.

The boundary conditions for the system of equations (4) include the condition of symmetry in the center of the stream

$$\partial\beta(t,0)/\partial r,\quad \partial w(t,0)/\partial r,\ \partial P_t(t,0)/\partial r=0\qquad(5)$$

at $r=0$ and the presence of the laminar sublayer near the surface of the pipe

$$\beta(t,r_0),\quad w(t,r_0),\quad P_t(t,r_0)=0,\qquad(6)$$

where r_0 — pipe radius.

The system of equations (4) is incorrect due to the presence of the antidiffusion term in the first equation (4). Due to the ideas of incorrect problems solving [3], use the following method. Let represent the function β in separated variables, that is $\beta(tr)=\gamma(t)\vartheta(r)$, where $\vartheta(r)$ is a known nonnegative function satisfying the boundary conditions (5), (6): $\vartheta'(0)=0$, $\vartheta(r_0)=0$ and the condition of monotonic decreasing from the center of the pipe to the boundary: $\vartheta'(r)<$ at $<r<r_0$.

Substituting the representation $\beta=\gamma\vartheta$ into the first equation (4) and integrating from 0 to r_0, one finds

$$\dot\gamma=a\gamma^{2/3}-b\gamma^{5/3},\qquad(7)$$

where

$$a=\left(-\int_0^{r_0}r^{-1}k_I\rho^{-1/2}P_t^{1/2}\vartheta^{-1/3}\vartheta'dr\right)\left(\int_0^{r_0}\vartheta dr\right)^{-1}>0$$

$$b=12\pi\left(\frac{3}{4\pi}\right)^{1/3}\nu_m\left(\int_0^{r_0}\vartheta^{5/3}dr\right)\left(\int_0^{r_0}\vartheta dr\right)^{-1}>0.$$

After replacing the first equation in system (4) by equation (7) we obtain the correct integro-differential system of equations.

Suppose further that the solution of equation (7) attains a steady state, that is $\dot\gamma=0$ and $\gamma=a/b$. The remaining two equations (4) for the velocity w and pressure turbulence P_t are ordinary nonlinear differential equations of diffusion type with sources and sinks. They were solved numerically with zero initial data with the boundary conditions (5), (6) and limiting steady-state solutions. For the solution of usual for this case parabolic equations we used finite difference scheme [4].

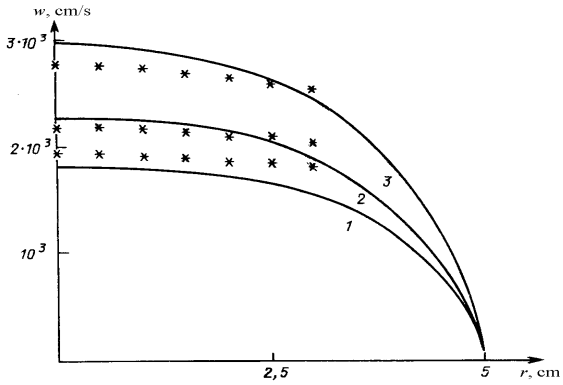

Figure 1. **Comparison of model velocity profiles of turbulent fluid flow in tube with the experimental data (*) from [5] depending on the radius of the pipe**

Figure 1 illustrates the results of fitting the model to the experimental velocity profiles in the tube at different Reynolds numbers Re_m obtained by numerical solving the model equations (4), (7) for the velocity w and pressure P_t, the appropriate assumptions about the function of the inverse volume being made. For variants of calculation 1, 2, 3 — $\mathrm{Re}_m = 1.5\cdot 10^6$, $2.3\cdot 10^6$, $3.2\cdot 10^6$ respectively. An asterisk in Figure 1 denotes the corresponding experimental values of velocity obtained in experiments by I. Nikuradse [5]. Let us choose the following constants: $\rho = 1g/\mathrm{cm}^3$, $r_0 = 5\mathrm{cm}$, $k_I = 2.6$, and for corresponding variants put:
1) $\nu_m = 1.1\cdot 10^{-2}\mathrm{cm}^2/\mathrm{s}$, $\Delta P/L = 1.5\cdot 10^3$ dyn /cm^3;
2) $\nu_m = 8.2\cdot 10^{-3}\mathrm{cm}^2/\mathrm{s}$, $\Delta P/L = 1.8\cdot 10^3$ dyn /cm^3;
3) $\nu_m = 7.5\cdot 10^{-3}\mathrm{cm}^2/\mathrm{s}$, $\Delta P/L = 2.7\cdot 10^3$ dyn /cm^3.

Reynolds number was estimated by the formula $\mathrm{Re}_m = 2r_0u/\nu_m$, u — average velocity and ϑ-function was chosen as $[1-(r/r_0)^2]^{1/2}$. Figure 1 illustrates how to relate to each other modeling and experimental velocity profiles in the core of the turbulent flow at one adjustable coefficient $k_I = 2.6$.

Note a circumstance that was clearly manifested during the solution of equations (4). To remove the incorrectness function describing the inverse scale of turbulence the separated variables were introduced and the equation describing this function was integrated. This procedure having general character, one ascertains the evident non-locality of the model.

Thus incorrectness of the equation for the scale of turbulence leads to non-locality in the description of turbulent flows. From this point of view there appears the similarity to the structure of the non-local spatial model of turbulent exchange given in [6].

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Численное описание движения жидкости в трубе в многомасштабной модели турбулентности

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В работе представлено численное описание движения турбулизованной жидкости в трубе на базе многомасштабной модели турбулентного движения жидкости, разработанной автором ранее. Приводится сравнение численных результатов с экспериментальными данными.

Ключевые слова: математическое моделирование, турбулентность, вычислительный эксперимент, некорректные задачи.