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On Saturation Problems for Riemann-Liouville Operators

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The problem of convergence almost everywhere and in weighted Lebesgue norms to the identity for the families of Riemann-Liouville operators is studied.

Key words and phrases: Riemann–Liouville operators, convergence almost everywhere, weighted Lebesgue norms.

1. Introduction

We consider Riemann-Liouville operators of the form

$$\Lambda_{\varphi_\lambda} f(x) := \frac{1}{\Phi_\lambda(x)} \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y, \quad \lambda > 0, \quad \gamma > 0, \quad x > 0,$$

where $\Phi_\lambda(x) := \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) \mathrm{d}y$ and $\Im := \{\varphi_\lambda(y)\}$ is a family of positive functions nondecreasing with respect to y such that $\varphi_\lambda \in L^1(I)$ for any interval $I \subset \mathbb{R}_+ := (0, +\infty)$ and

$$\lim_{\lambda \to \infty} \frac{\varphi_\lambda(ux)}{\Phi_\lambda(x)} = 0 \tag{1}$$

for all x and $u \in (0,1)$.

The paper is devoted to the proof of the convergence $\lim\limits_{\lambda\to\infty} \Lambda_{\varphi_\lambda} f(x) = f(x)$ for almost every (a.e.) $x \in \mathbb{R}_+$ and the similar problem in the weighted Lebesgue norm setting.

2. Main Results

Theorem 1. *Assume that* $\{\varphi_\lambda(y)\} \in \Im$. *Let* f *be a locally integrable function on* $\mathbb{R}_+$. *Then at any Lebesgue point* $x \in \mathbb{R}_+$ *of* f *we have*

$$\lim_{\lambda\to\infty} \frac{1}{\Phi_\lambda(x)} \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y = 0.$$

Proof. Since $\varphi_\lambda(y)$ is nondecreasing then without a loss of generality we may and shall assume that for each $\lambda > 0$, $\varphi_\lambda(y)$ is right-continuous on y. Then

$$\frac{1}{\Phi_\lambda(x)} \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y =$$

$$= \frac{1}{\Phi_\lambda(x)} \int\limits_0^{x-\delta} (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y + \frac{1}{\Phi_\lambda(x)} \int\limits_{x-\delta}^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y.$$

Let $\varepsilon > 0$ be given and x is a Lebesgue point of f. There exsists $\delta_0 > 0$ be such that for $0 < \delta < \delta_0$,

$$\frac{\gamma+1}{\delta^{\gamma+1}} \int_{x-\delta}^{x} (x-y)^{\gamma} |f(y) - f(x)| \, \mathrm{d}y \leqslant \varepsilon.$$

By [1, proposition(12.12)], for $0 < \delta < \delta_0$,

$$\frac{1}{\Phi_\lambda(x)} \int_{x-\delta}^{x} (x-y)^{\gamma} \varphi_\lambda(y) |f(y) - f(x)| \, \mathrm{d}y =$$

$$= \frac{1}{\Phi_\lambda(x)} \int_{x-\delta}^{x} [(x-y)^{\gamma}\varphi_\lambda(x-\delta) - (x-y)^{\gamma}\varphi_\lambda(x-\delta) + (x-y)^{\gamma}\varphi_\lambda(y)] \, |f(y) - f(x)| \, \mathrm{d}y =$$

$$= \frac{1}{\Phi_\lambda(x)} \int_{x-\delta}^{x} \left[(x-y)^{\gamma}\varphi_\lambda(x-\delta) + (x-y)^{\gamma} \int_{(x-\delta,y]} d\varphi_\lambda(t) \right] |f(y) - f(x)| \, \mathrm{d}y =$$

$$= \frac{\varphi_\lambda(x-\delta)}{\Phi_\lambda(x)} \int_{x-\delta}^{x} (x-y)^{\gamma} |f(y) - f(x)| \, \mathrm{d}y + \frac{1}{\Phi_\lambda(x)} \int_{x-\delta}^{x} (x-y)^{\gamma} \int_{(x-\delta,y]} [d\varphi_\lambda(t)] \, |f(y) - f(x)| \, \mathrm{d}y =$$

$$= \frac{\varphi_\lambda(x-\delta)}{\Phi_\lambda(x)} \int_{x-\delta}^{x} (x-y)^{\gamma} |f(y) - f(x)| \, \mathrm{d}y + \frac{1}{\Phi_\lambda(x)} \int_{x-\delta}^{x} \left[\int_{t}^{x} (x-y)^{\gamma} |f(y) - f(x)| \, \mathrm{d}y \right] d\varphi_\lambda(t) \leqslant$$

$$\leqslant \frac{\varphi_\lambda(x-\delta)\varepsilon\delta^{\gamma+1}}{\Phi_\lambda(x)(\gamma+1)} + \frac{1}{\Phi_\lambda(x)} \frac{\varepsilon}{\gamma+1} \int_{(x-\delta,x]} (x-t)^{\gamma+1} d\varphi_\lambda(t) =$$

$$= \frac{\varepsilon}{\Phi_{\lambda(x)}} \left[\frac{\varphi_\lambda(x-\delta)\delta^{\gamma+1}}{\gamma+1} + \frac{1}{\gamma+1} \int_{(x-\delta,x]} (x-t)^{\gamma+1} d\varphi_\lambda(t) \right] = \frac{\varepsilon}{\Phi_\lambda(x)} \int_{(x-\delta,x]} (x-t)^{\gamma}\varphi_\lambda(t)dt \leqslant \varepsilon.$$

For $\gamma > 0$ we have $(x-y)^{\gamma} \approx \delta^{\gamma} + (x-\delta-y)^{\gamma}$, so

$$\limsup_{\lambda\to\infty} \frac{1}{\Phi_\lambda(x)} \int_{0}^{x-\delta} (x-y)^{\gamma}\varphi_\lambda(y) |f(y) - f(x)| \, \mathrm{d}y \approx$$

$$\approx \limsup_{\lambda\to\infty} \frac{\delta^{\gamma}}{\Phi_\lambda(x)} \int_{0}^{x-\delta} \varphi_\lambda(y) |f(y) - f(x)| \, \mathrm{d}y +$$

$$+ \limsup_{\lambda\to\infty} \frac{1}{\Phi_\lambda(x)} \int_{0}^{x-\delta} (x-\delta-y)^{\gamma}\varphi_\lambda(y) |f(y) - f(x)| \, \mathrm{d}y := J_1 + J_2.$$

Since $\varphi_\lambda(x)$ is nondecreasing function, then,

$$J_1 \leqslant \limsup_{\lambda\to\infty} \frac{\delta^{\gamma}\varphi_\lambda(x-\delta)}{\Phi_\lambda(x)} \int_{0}^{x-\delta} |f(y) - f(x)| \, \mathrm{d}y,$$

by (1) $\lim\limits_{\lambda\to\infty} \dfrac{\delta^\gamma \varphi_\lambda(x-\delta)}{\Phi_\lambda(x)} = 0$, so

$$J_1 = \limsup_{\lambda\to\infty} \frac{\delta^\gamma}{\Phi_\lambda(x)} \int\limits_0^{x-\delta} \varphi_\lambda(y)\,|f(y)-f(x)|\,\mathrm{d}y = 0.$$

Also, by (1)

$$J_2 \leqslant \limsup_{\lambda\to\infty} \frac{(x-\delta)^\gamma \varphi_\lambda(x-\delta)}{\Phi_\lambda(x)} \int\limits_0^{x-\delta} |f(y)-f(x)|\,\mathrm{d}y = 0.$$

Thus,

$$\limsup_{\lambda\to\infty} \frac{1}{\Phi_\lambda(x)} \int\limits_0^{x-\delta} (x-y)^\gamma \varphi_\lambda(y)\,|f(y)-f(x)|\,\mathrm{d}y = 0.$$

Since $\varepsilon > 0$ was arbitrary, then

$$\lim_{\lambda\to\infty} \frac{1}{\Phi_\lambda(x)} \int\limits_0^{x} (x-y)^\gamma \varphi_\lambda(y)\,|f(y)-f(x)|\,\mathrm{d}y = 0.$$

Remark 1. The case $\gamma = 0$ was studied in [2].

Example 1. Let f be measurable on $\mathbb{R}_+$. Suppose that exists $\lambda_0 > 0$ such that $y^{\lambda_0} f(y) \in L^1(I)$ for each bounded subinterval $I \subset \mathbb{R}_+$. If $x \in \mathbb{R}_+$ is a Lebesgue point of f, then

$$\lim_{\lambda\to\infty} \Upsilon_\lambda f(x) = f(x),$$

where $\Upsilon_\lambda f(x) = \dfrac{1}{\Theta_\lambda(x)} \displaystyle\int\limits_0^x (x-y)^\gamma y^\lambda f(y)\mathrm{d}y$, and $\Theta_\lambda(x) = \displaystyle\int\limits_0^x (x-y)^\gamma y^\lambda \mathrm{d}y$.

Def 1. Let $E \subseteq \mathbb{R}_+$ be a measurable set. For a measurable function ω such that $\omega(x) > 0$ a.e. on E and all measurable functions f on E for $0 < p < \infty$ we define

$$\|f\|_{L^p_\omega(E)} := \left(\int\limits_E (\omega(x)\,|f(x)|)^p \,\mathrm{d}x \right)^{\frac{1}{p}} \quad \text{and} \quad L^p_\omega(E) := \left\{ f : \|f\|_{L^p_\omega(E)} < \infty \right\}.$$

At first, we consider convergence in $L^p_{x^\gamma}(0,a)$. Since for $0 < p < \infty$, $\gamma > 0$, continuous functions with compact support are dense in $L^p_{x^\gamma}$ [3, Theorem 3.14], then we have the following.

Theorem 2. *If* $f \in L^p_{x^\gamma}, 0 < p < \infty, \gamma > 0$, *then* $\lim\limits_{t\to1} \|f(tx) - f(x)\|_{L^p_{x^\gamma}} = 0$.

Theorem 3. *Let* $a > 0$ *and assume that* $\{\varphi_\lambda(y)\} \in \Im$. *Suppose that there exists* $\Psi_\lambda(u)$ *such that for all* $u \in (0,1)$ *the inequality*

$$\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} \leqslant \Psi_\lambda(u)$$

holds for all $x \in (0,a)$. *Also suppose, that* $\limsup\limits_{\lambda\to\infty} \|\Psi_\lambda\|_{L^1(0,1)} = C < \infty$ *and for all* $\alpha > 0$ *and* $0 < \vartheta < 1$, $\lim\limits_{\lambda\to\infty} \|u^{-\alpha}\Psi_\lambda(u)\|_{L^1(0,\vartheta)} = 0$. *Then for* $f \in L^p_{x^\gamma}(0,a), 1 \leqslant p < \infty, \gamma > 0$,

$$\lim_{\lambda\to\infty} \|\frac{1}{\Phi_\lambda(x)}\int_0^x (x-y)^\gamma \varphi_\lambda(y) f(y)\mathrm{d}y - f(x)\|_{L^p_{x^\gamma}(0,a)} = 0.$$

Proof. Note that for λ sufficiently large and for $r, x \in (0,a)$ we have

$$\int_0^r (x-y)_+^\gamma \varphi_\lambda(y)|f(y)|\mathrm{d}y \leqslant \left(\int_0^r \left((x-y)_+^\gamma y^\gamma |f(y)|\right)^p \mathrm{d}y\right)^{\frac{1}{p}} \left(\int_0^r (y^{-\gamma}\varphi_\lambda(y))^{p'}\mathrm{d}y\right)^{\frac{1}{p'}} \leqslant$$

$$\leqslant x^\gamma \|f\|_{L^p_{y^\gamma}(0,a)} \left(\varphi_\lambda(r)\right)^{\frac{1}{p}} \left(\int_0^r y^{-\gamma p'}\varphi_\lambda(y)\mathrm{d}y\right)^{\frac{1}{p'}}$$

and

$$\left(\int_0^r y^{-\gamma p'}\varphi_\lambda(y)\mathrm{d}y\right)^{\frac{1}{p'}} = r^{-(\gamma+\frac{\gamma}{p'})}\left(\Phi_\lambda(r)\right)^{\frac{1}{p'}} \left(\int_0^1 u^{-\gamma p'}\frac{r^{\gamma+1}\varphi_\lambda(ur)}{\Phi_\lambda(r)}\mathrm{d}u\right)^{\frac{1}{p'}} \leqslant$$

$$\leqslant r^{-(\gamma+\frac{\gamma}{p'})}\left(\Phi_\lambda(r)\right)^{\frac{1}{p'}} \left(\int_0^1 u^{-\gamma p'}\Psi_\lambda(u)\mathrm{d}u\right)^{\frac{1}{p'}} < \infty,$$

since $\gamma > 0$, let $\gamma p' = \beta$, $0 < \vartheta < 1$,

$$I = \int_0^\vartheta u^{-\beta}\Psi_\lambda(u)\mathrm{d}u + \int_\vartheta^1 u^{-\beta}\Psi_\lambda(u)\mathrm{d}u \leqslant C + \vartheta^{-\beta}\int_\vartheta^1 \Psi_\lambda(u)\mathrm{d}u < \infty.$$

Thus, $(x-y)_+^\gamma \varphi_\lambda(y) f(y) \in L^1(0,r)$ for all $r \in (0,a)$. For $0 < \vartheta < 1$ we write

$$\|\frac{1}{\Phi_\lambda(x)}\int_0^x (x-y)_+^\gamma \varphi_\lambda(y) f(y)\mathrm{d}y - f(x)\|_{L^p_{x^\gamma}} \leqslant$$

$$\leqslant \left(\int_0^a \left(\frac{x^\gamma}{\Phi_\lambda(x)}\int_0^x (x-y)_+^\gamma \varphi_\lambda(y)|f(y)-f(x)|\mathrm{d}y\right)^p \mathrm{d}x\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \left(\int_0^a \left(\frac{x^{2\gamma}}{\Phi_\lambda(x)}\int_0^x \varphi_\lambda(y)|f(y)-f(x)|\mathrm{d}y\right)^p \mathrm{d}x\right)^{\frac{1}{p}} =$$

$$= \left(\int_0^a \left(\frac{x^{2\gamma+1}}{\Phi_\lambda(x)}\int_0^1 \varphi_\lambda(ux)|f(ux)-f(x)|\mathrm{d}u\right)^p \mathrm{d}x\right)^{\frac{1}{p}} \leqslant$$

$$\leqslant \int_0^1 \left(\int_0^a \left(\frac{x^{2\gamma+1}}{\Phi_\lambda(x)} \varphi_\lambda(ux) |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u =$$

$$= \int_0^\vartheta \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u +$$

$$+ \int_\vartheta^1 \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u.$$

By Theorem 2, for every $\varepsilon > 0$, there exists $0 < \vartheta_\varepsilon < 1$ such that for $\vartheta_\varepsilon < u < 1$,

$$\left(\int_0^a (x^\gamma |f(ux) - f(x)|)^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \frac{\varepsilon}{C}.$$

So

$$\int_{\vartheta_\varepsilon}^1 \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u \leqslant$$

$$\leqslant \int_{\vartheta_\varepsilon}^1 \Psi_\lambda(u) \left(\int_0^a (x^\gamma |f(ux) - f(x)|)^p \, \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u < \frac{\varepsilon}{C} \int_{\vartheta_\varepsilon}^1 \Psi_\lambda(u) \mathrm{d}u \leqslant \frac{\varepsilon}{C} \int_0^1 \Psi_\lambda(u) \mathrm{d}u$$

and

$$\limsup_{\lambda\to\infty} \int_{\vartheta_\varepsilon}^1 \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u \leqslant \varepsilon.$$

Also

$$\limsup_{\lambda\to\infty} \int_0^{\vartheta_\varepsilon} \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux) - f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u \leqslant$$

$$\leqslant \limsup_{\lambda\to\infty} \int_0^{\vartheta_\varepsilon} \Psi_\lambda(u) \left(\int_0^a (x^\gamma |f(ux) - f(x)|)^p \, \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u \leqslant$$

$$\leqslant \limsup_{\lambda\to\infty} \int_0^{\vartheta_\varepsilon} \Psi_\lambda(u) \left(\left(\int_0^a (x^\gamma |f(ux)|)^p \, \mathrm{d}x \right)^{\frac{1}{p}} + \left(\int_0^a (x^\gamma |f(x)|)^p \, \mathrm{d}x \right)^{\frac{1}{p}} \right) \mathrm{d}u \leqslant$$

$$\leqslant \|f\|_{L^p_{x^\gamma}} \limsup_{\lambda\to\infty} \int_0^\vartheta \Psi_\lambda(u) \left(\frac{1}{u^{\gamma+\frac{1}{p}}} + 1 \right) \mathrm{d}u = 0.$$

Therefore,

$$\limsup_{\lambda\to\infty} \| \frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)_+^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x)\|_{L^p_{x^\gamma}} \leqslant$$

$$\leqslant \limsup_{\lambda\to\infty} \int_0^{\vartheta_\varepsilon} \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux)-f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u +$$

$$+ \limsup_{\lambda\to\infty} \int_{\vartheta_\varepsilon}^1 \left(\int_0^a \left(\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} x^\gamma |f(ux)-f(x)| \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}u \leqslant \varepsilon$$

and so $\lim\limits_{\lambda\to\infty} \left\| \frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)_+^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x) \right\|_{L^p_{x^\gamma}(0,a)} = 0.$ □

Example 2. Similar to Example 1, for $f \in L^p_{x^\gamma}(0,a), 1 \leqslant p < \infty, \gamma > 0$ we have

$$\lim_{\lambda\to\infty} \|(\Upsilon_\lambda - I)f\|_{L^p_{x^\gamma}(0,a)} = 0.$$

Theorem 4. *Assume that,* $\{\varphi_\lambda(y)\} \in \Im$. *Also assume that there exists* $\Psi_\lambda(u)$, *so that for all* $u \in (0,1)$,

$$\frac{x^{\gamma+1}\varphi_\lambda(ux)}{\Phi_\lambda(x)} \leqslant \Psi_\lambda(u) \tag{2}$$

for all $x \in (0,a)$, *and* $\lim\limits_{\lambda\to\infty} \Psi_\lambda(u) = 0$. *Then for any uniformly continuous function* f *on* $(0,a)$, $0 < a < \infty$, *we have*

$$\lim_{\lambda\to\infty} \sup_{0<x<a} \left| \frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x) \right| = 0.$$

Proof. Let $\varepsilon > 0$, since f is uniformly continuous on $(0,a)$, there exists $0 < \delta < a$, such that $|f(u) - f(v)| < \varepsilon$ for $u, v \in (0,a)$, and $|u - v| < \delta$. By (2) there exists λ_0 such that for $\lambda \geqslant \lambda_0$,

$$\Psi_\lambda\left(\frac{a-\delta}{a}\right) < \frac{\varepsilon}{2 \sup\limits_{0<t<a} |f(t)|},$$

$$\left| \frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x) \right| \leqslant \frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y.$$

For $0 < x \leqslant \delta$

$$\frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y \leqslant \frac{\varepsilon}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) \mathrm{d}y = \varepsilon.$$

While for $\delta < x < a$

$$\frac{1}{\Phi_\lambda(x)} \int_0^x (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y =$$

$$= \frac{1}{\Phi_\lambda(x)} \int\limits_0^{x-\delta} (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y + \frac{1}{\Phi_\lambda(x)} \int\limits_{x-\delta}^{x} (x-y)^\gamma \varphi_\lambda(y) \left| f(y) - f(x) \right| \mathrm{d}y \leqslant$$

$$\leqslant \frac{1}{\Phi_\lambda(x)} \left(\left(2 \sup_{0<t<a} |f(t)| \right) (x-\delta) x^\gamma \varphi_\lambda(x-\delta) + \varepsilon \int\limits_{x-\delta}^{x} (x-y)^\gamma \varphi_\lambda(y) \mathrm{d}y \right) \leqslant$$

$$\leqslant \frac{1}{\Phi_\lambda(x)} \left(2 \sup_{0<t<a} |f(t)| \right) x^{\gamma+1} \varphi_\lambda(x-\delta) + \varepsilon \leqslant \left(2 \sup_{0<t<a} |f(t)| \right) \Psi_\lambda \left(\frac{a-\delta}{a} \right) + \varepsilon.$$

Therefore

$$\limsup_{\lambda\to\infty} \sup_{0<x<a} \left| \frac{1}{\Phi_\lambda(x)} \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x) \right| \leqslant 2\varepsilon,$$

and so,

$$\lim_{\lambda\to\infty} \sup_{0<x<a} \left| \frac{1}{\Phi_\lambda(x)} \int\limits_0^x (x-y)^\gamma \varphi_\lambda(y) f(y) \mathrm{d}y - f(x) \right| = 0.$$

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О проблемах насыщаемости для операторов Римана–Лиувилля

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Для семейств операторов Римана–Лиувилля рассматриваются проблемы сходимости почти всюду и по норме весовых пространств Лебега к тождественному оператору.

Ключевые слова: операторы Римана–Лиувилля, сходимость почти всюду, весовые нормы Лебега.