UDC 531.3

Constructing Dynamic Equations of Constrained Mechanical Systems

Chernet Tuge Deressa

Department of Mathematics
Jimma University
Jimma, Ethiopia

In this paper constructing equation of mechanical systems based on their kinetic energy, potential energy and dissipative force is discussed. Both the holonomic and non-holonomic constraints are considered. Equations of constraint forces resulting from ideal and non-ideal nature of the constraints are developed.It is shown that, the constraint force is a sum of two forces resulting from the ideal and non-ideal nature of the constraints. An explicit equation of the acceleration of the system is developed basing on the constraint forces from the nature of the constraints. For investigating the deviation of the system from the trajectory of the constraint equations, excess variables are included in the equations of the constraints. The stability of the system is based on determining the sign of constants emerging from developing the Lagrange's equation of motion for the constraints. The determination of the sign of the constants is made based on Routh-Hurwitz Criterion for Stability.

An example is used to demonstrate each of the equations developed in the paper and constructing state-space equation of the system.

Key words and phrases: dissipative force, excess variables, ideal constraints, Lagrange equation, non-ideal constraints, stability, Routh-Hurwitz criterion for stability, state-space equation.

1. Introduction

In an article by R.G. Mukharlyamov [1], constructing equation of mechanical system when the kinetic energy, potential energy and dissipative forces are known is detailed. Both holonomic and non-holonomic constraints are considered. In order to investigate the stability of constraints, excess variables are used.All the discussions made by R.G. Mukharlyamov in [1] were for Ideal constraints. In an article by Udwadia F. [2, 3] constructing equation of mechanical system involving both the ideal and non-ideal constraints is discussed. The method detailed is so general that it makes the ideal constraints only a particular case.The issue of constraint stabilization is not included in Udwadia F. The idea for developing this paper is mainly an insight from investigating the above two articles. In this paper, firstly, a new idea of developing equation of mechanical system simultaneously from Lagrange's equation of motion for a constrained mechanical system and Lagrange's equation of motion for the constraints is discussed. In other words dynamic equation for the constraints is developed and is used for investigation of asymptotic stability of the system.Secondly, the equation of a constraint force generated as a result of constraining a system is developed. In constructing an equation for the constraint forces, a general case that involves both the ideal and non-ideal situation of the constraints is considered. It is discussed that, the constraint force is a sum of two forces resulting from the ideal and non-ideal nature of the constraints which agrees with what is discussed in [2, 4]. It is also shown that, from the general equation of the constraint forces,it is possible to develop an equation for only ideal constraints by taking into account the D'Alembert-Lagrange principle of the work done by virtual displacement.It should be noted that this paper brings about new idea in that, the excess variables are included in all the equations developed, as a result of which we have new constants that need to be determined in order to make the system asymptotically stable. Thirdly, an explicit equation of the acceleration of the system is developed basing on the constraint forces from the ideal and non-ideal nature of the constraints.

The fourth point involved in this paper is determining the constants, obtained as a result of equations of motion of the constraint equations, so that the system becomes asymptotically stable. With this regard, we use *Routh–Hurwitz criterion for stability* in determining the region of stability [4, 5].

A further discussion in this paper is made using an example provided. In the example each of the methods developed are demonstrated.Moreover it is discussed that the method developed in this paper can easily be extended to develop **state-space equation of the system** which is a power full approach in dynamic system for multiple input and multiple out put situation of linear and non-linear system analysis [4, 5].

In summery, some of the advantages of the approach developed in this paper are:

1. An explicit equation of the acceleration of a system can be constructed with relatively minimum steps. This is because, we don't need to calculate Lagrange's multipliers in the method developed in this paper.
2. Problems that result from the redundant constraints,such as rank deficiency of the Jacobian matrix, can be managed.
3. The stability of the system can easily be made using*Routh-Hurwitz criterion for stability* even at singular points.
4. Since the equation resulting from the method can be developed into **state-space equation**, it can be used for further investigations, such as controllability and control design of complex systems.

2. Equations of Mechanical System.

In this section discussion is made on the construction of equation of motion for mechanical systems.

Let $T^0 = T^0(q, \dot{q})$, $V^0 = V^0(q)$, $D^0 = D^0(q, \dot{q})$ be respectively, the kinetic energy,potential energy and dissipative function of an unconstrained mechanical system, where $q = (q^1, q^2., ., ., q^n)$ is n generalized coordinate. Moreover, we shall assume that this system is subjected to a set of $m = h + s$ consistent equality constraints of the form:

$$\varPhi(q, t) = 0, \tag{1}$$

$$\varPsi(q, \dot{q}, t) = 0, \tag{2}$$

where $\varPhi$ is an h vector and $\varPsi$ is an s vector.

Construction of dynamic equations of mechanical systems, are usually based on the assumption that, if constraint equations at the position and velocity level are satisfied at $t = t_0$ then are satisfied for all $t > t_0$. But in reality this is not the case. If a constraint equation at an acceleration level is used, for instance, the position and velocity level acceleration equations suffers drift phenomena. Ones these equations deviate from the exact value at some time $t \geqslant t_0$ the error will keep on accumulating. This creates problem in stabilization and control design of mechanical systems [1, 6].

For instance let us consider a holonomic case given in equation (1) such that $\varPhi = \varepsilon, \dot{\varPhi} = \delta$ which deviates from the exact value $\varPhi(q, t) = 0$, then from the differential equation $\ddot{\varPhi} = 0$ we obtain $\varPhi = \delta t + \varepsilon$.This indicates that deviation of the constraint equation from the exact value at time $t \geqslant t_0$ accumulates in linear manner for all time. We note that, this is when the constraints at the acceleration level is used to construct model of mechanical system to get Ordinary Differential Equations.

In order to stabilize the constraints in (1) and (2) it is necessary to take account of the deviation from equations (1) , (2) and introduce a corresponding correction to the dynamic equation of the system [1, 6]. Let the deviation of the constraints be denoted by y,$\dot{y}$ and $\dot{p}$ called excess variables [1] such that:

$$\varPhi(q, t) = y, \quad \dot{\varPhi}_q \dot{q} + \varPhi_t = \dot{y}, \quad \varPsi(q, \dot{q}, t) = \dot{p}, \tag{3}$$

where: $\varPhi = (\varPhi^1, \varPhi^2, \ldots, \varPhi^h)$, $y = (y^1, y^2, \ldots, y^h)$, $\varPsi = (\varPsi^{h+1}, \varPsi^{h+2}, \ldots, \varPsi^{h+s})$, $\dot{p} = (\dot{y}^{h+1}, \dot{y}^{h+2}, \ldots, \dot{y}^{h+s})$.

The mechanical system is determined by the generalized coordinates, q, y and the generalized velocities $\dot{q}, \dot{\mathbf{r}}$ where

$$\mathbf{r} = (y^1, y^2, ., ., ., y^h, y^{h+1}, ., ., ., y^m)^T,$$

$$\dot{\mathbf{r}} = (\dot{y}^1, \dot{y}^2, \ldots, \dot{y}^h, \dot{y}^{h+1}, \dot{y}^{h+2}, \ldots, \dot{y}^{h+s})^T = (\dot{y}, \dot{p})^T.$$

Let L^1 represent Lagrangian for the constraints such that:

$$L^1 = L^1(y, \dot{\mathbf{r}}), \quad T^1 = T^1(y, \dot{\mathbf{r}}), \quad V^1 = V^1(y), \quad D^1 = D^1(y, \dot{\mathbf{r}}),$$

where L, T, D and V respectively represents the Lagrangian, the Kinetic Energy, the Dissipative Function and the Potential Energy of the constrained mechanical system such that:$L = L(q, y, \dot{q}, \dot{\mathbf{r}})$, $T = T(q, y, \dot{q}, \dot{\mathbf{r}})$,$D = D(q, y, \dot{q}, \dot{\mathbf{r}})$, $V = V(q, y)$, $L(q, \dot{q}, 0, 0) = L^0(q, \dot{q})$, $L = L^0 + L^1$, $L^0 = T^0 - V^0$, $L^1 = T^1 - V^1$, $D(q, \dot{q}, 0, 0) = D^0(q, \dot{q})$. Here L^0 is the Lagrangian of the an unconstrained system. We now expand the constraint equations by Taylor expansion method, around $y = 0, \dot{\mathbf{r}} = 0$. That is for sufficiently small values of the variables y and $\dot{\mathbf{r}}$ we obtain:

$$2T^1 = \sum_{\eta,k=1}^{m} (a_{\eta k})\dot{y}^\eta \dot{y}^k, \quad 2V^1 = \sum_{\eta,k=1}^{h} (v_{\eta k})y^\eta y^k,$$

and the dissipative force $2R^1 = -\sum\limits_{\eta,k=1}^{m} (c_{\eta k})\dot{y}^\eta \dot{y}^k$.

Now let us obtain the Lagrange's equation, for the constraints, based on T^1, V^1, R^1 and $L^1 = T^1 - V^1$

$$\frac{\partial L^1}{\partial \dot{y}^i} = \frac{1}{2}\sum_{\eta,k=1}^{m} (a_{\eta k})\frac{\partial \dot{y}^\eta}{\partial \dot{y}^i}\dot{y}^k + \frac{1}{2}\sum_{\eta,k=1}^{m} (a_{\eta k})\frac{\partial \dot{y}^k}{\partial \dot{y}^i}\dot{y}^\eta =$$

$$= \frac{1}{2}\sum_{\eta,k=1}^{m} (a_{\eta k})\delta_{\eta i}\dot{y}^k + \frac{1}{2}\sum_{\eta,k=1}^{m} (a_{\eta k})\delta_{ki}\dot{y}^\eta, \quad i = 1, 2, \ldots, m,$$

where $\delta_{ij} = 1$, when $i = j$, and is 0, when $i \neq j$:

$$\frac{\partial L^1}{\partial \dot{y}^i} = \frac{1}{2}\sum_{k=1}^{m}(a_{ik})\dot{y}^k + \frac{1}{2}\sum_{\eta=1}^{m}(a_{\eta i})\dot{y}^\eta.$$

Since $a_{ik} = a_{ki}$ and the summation in dices are dummy, the above expression can be simplified as

$$\frac{\partial L^1}{\partial \dot{y}^i} = \frac{1}{2}\sum_{k=1}^{m}(a_{ik})\dot{y}^k + \frac{1}{2}\sum_{\eta=1}^{m}(a_{\eta i})\dot{y}^\eta = \frac{1}{2}\sum_{\eta=1}^{m}(a_{\eta i})\dot{y}^\eta + \frac{1}{2}\sum_{\eta=1}^{m}(a_{i\eta})\dot{y}^\eta = \frac{1}{2}\sum_{\eta=1}^{m}(a_{i\eta})\dot{y}^\eta$$

Hence:

$$\frac{d}{dt}\frac{\partial L^1}{\partial \dot{y}^i} = \frac{1}{2}\sum_{\eta=1}^{m}(a_{i\eta})\ddot{y}^\eta \tag{4}$$

A similar analysis for the potential energy and dissipative force gives:

$$\frac{\partial L^1}{\partial y^i} = \frac{1}{2}\sum_{\eta=1}^{h}(v_{i\eta})y^\eta, \quad \frac{\partial R^1}{\partial \dot{y}^i} = \frac{1}{2}\sum_{\eta=1}^{m}(c_{i\eta})\dot{y}^\eta. \tag{5}$$

Now from the equation: $\dfrac{d}{dt}\dfrac{\partial L^1}{\partial \dot{y}^i} - \dfrac{\partial L^1}{\partial y^i} = -\dfrac{\partial R^1}{\partial \dot{y}^i}$ we obtain:

$$\sum_{\eta=1}^{m}(a_{i\eta})\ddot{y}^\eta - \sum_{\eta=1}^{h}(v_{i\eta})y^\eta = -\sum_{\eta=1}^{m}(c_{i\eta})\dot{y}^\eta. \tag{6}$$

Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}, \quad C = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,m} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m,1} & c_{m,2} & \cdots & c_{m,m} \end{pmatrix},$$

$$W = -\begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,h} & 0 & \cdots & 0_{1,m} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,h} & 0 & \cdots & 0_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ v_{h,1} & v_{h,2} & \cdots & v_{h,h} & 0 & \cdots & 0_{h,m} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \ddots & \vdots \\ 0_{m,1} & 0_{m,2} & \cdots & 0_{m,h} & 0 & \cdots & 0_{m,m} \end{pmatrix}.$$

Now equation (6) can be given as:

$$A\ddot{\mathbf{r}} = -C\dot{\mathbf{r}} - W\mathbf{r}, \tag{7}$$

where $\dot{\mathbf{r}}$ and $\mathbf{r}$ are defined above.

Equation (7) can be reduced to:

$$\ddot{\mathbf{r}} = -B\dot{\mathbf{r}} - K\mathbf{r}, \tag{8}$$

where $B = A^{-1}C$ and $K = A^{-1}W$.

Remark 1. The matrices A, C and W contain the elements a_{ij}, c_{ij} and h_{ij} for the different constraint equations, do not account for the coupling between the different constraints and hence the entries off the diagonals can be taken to be zero and therefore the matrices in this case are each diagonal matrices. For example A can be expressed as [7,8]:

$$\begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{m,m} \end{pmatrix} = a\begin{pmatrix} d_{1,1} & 0 & \cdots & 0 \\ 0 & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{m,m} \end{pmatrix} = aP.$$

So that the different coefficients can be seen as a single factor a multiplied by a proportionality factor d_{ii} for each constraint equation. The matrices C and Wcan be treated in the same way.Moreover, from the above discussion it can be seen that each of the values a_{ii}, c_{ii} and h_{ii} may be different for each constraint equations. However, for a wide range of purposes, the use of single scalar values for a_{ii}, c_{ii} and h_{ii}in every constraint equation is an acceptable solution, and it is frequently found in the literature. It can then be concluded that the matrices A, C and W can be taken to be a single factor each, say a, c and h respectively [7, 8].

Equation (8) can also be written as a system of first order differential equations shown below.

$$\begin{cases} \dfrac{dr}{dt} = \dot{\mathbf{r}}, \\ \dfrac{d\dot{\mathbf{r}}}{dt} = -B\dot{\mathbf{r}} - K\mathbf{r}. \end{cases} \tag{9}$$

From Lagrange's equation for the constrained mechanical system and equation (8) we obtain the following system of Differential Equations given in (10) below.

That is from $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = Q_{ex} + Q^C$ and $\dfrac{d}{dt}\dfrac{\partial L^1}{\partial \dot{y}^i} - \dfrac{\partial L^1}{\partial y^i} = -\dfrac{\partial R^1}{\partial \dot{y}^i}$ it results:

$$\begin{cases} M\ddot{q} = L_q + Q_{ex} + Q^c + \dot{M}\dot{q}, \\ \ddot{\mathbf{r}} = -B\dot{\mathbf{r}} - K\mathbf{r}, \end{cases} \tag{10}$$

where M is the mass matrix of the unconstrained system, $L_q = \dfrac{\partial L}{\partial q}$, Q_{ex} is an external force, $Q = L_q + Q_{ex} - \dot{M}\dot{q}$, Q^c is a constraint force applied on the system as a result of the constraints. B and K are constant matrices of size m by m. From Remark 1 we can assume B and K to be constants.

3. Construction of the Constraint Force Q^c

In this section we shall discuss the construction of equation for the constraint force Q^c. It is shown that Q^c is a sum of two terms resulting from the ideal and non-ideal nature of the constraints. By considering the D'Alembert-Lagrange principle of virtual displacement, it is shown that the force due to the non-ideal nature of the constraints will add up to zero.

Differentiating (1) twice, (2) once , replace $\ddot{\boldsymbol{r}}$ by:

$$\ddot{\mathbf{r}} = (\dot{\Phi}_q\dot{q} + \Phi_q\ddot{q} + \dot{\Phi}_t) + (\Psi_q\dot{q} + \Psi_{\dot{q}}\ddot{q} + \Psi_t).$$

Then equation (10) with $Q = L_q + Q_{ex} - \dot{M}\dot{q}$ becomes

$$\begin{cases} \ddot{q} = M^{-1}Q^c + M^{-1}Q, \\ (\Phi_q + \Psi_{\dot{q}})\ddot{q} = -B\dot{\mathbf{r}} - K\mathbf{r} - (\dot{\Phi}_q\dot{q} + \dot{\Phi}_t + \Psi_q\dot{q} + \Psi_t). \end{cases} \tag{11}$$

Multiplying the first part of (11) by $\Phi_q + \Psi_{\dot{q}}$ and solving for Q^c from the two equations leads to:

$$(\Phi_q + \Psi_{\dot{q}})M^{-1}Q^c = -B\dot{\mathbf{r}} - K\mathbf{r} - (\Psi_t + \Psi_q\dot{q} + \dot{\Phi}_q\dot{q} + \dot{\Phi}_t) - (\Phi_q + \Psi_{\dot{q}})M^{-1}Q. \tag{12}$$

Let $A = (\Phi_q + \Psi_{\dot{q}})M^{-1}$, $\mathbf{b} = -B\dot{\mathbf{r}} - K\mathbf{r} - (\Psi_t + \Psi_q\dot{q} + \dot{\Phi}_q\dot{q} + \dot{\Phi}_t)$, $(\Phi_q + \Psi_{\dot{q}})M^{-1}Q = AQ$.

Note also that:

$$\Phi_q + \Psi_{\dot{q}} = (\Phi_q^1, \Phi_q^2, \ldots, \Phi_q^h, 0, 0, \ldots, 0)^T + (0, 0, \ldots, 0, \Psi_{\dot{q}}^{h+1}, \Psi_{\dot{q}}^{h+2}, \ldots, \Psi_{\dot{q}}^{h+s})^T.$$

Then we can write equation (12) as:

$$AQ^c = \mathbf{b} - AQ. \tag{13}$$

The general solution of equation (13) is given by:

$$Q^c = A^+(\mathbf{b} - AQ) + (I - A^+A)w, \tag{14}$$

where w is a non-zero n-vector, A^+ is a generalized inverse of matrix A and I is an identity matrix of appropriate size.

From(14) we can infer that: The constraint force is given as a sum of two components. The first component is the extent to which the acceleration of the unconstrained system deviates from the acceleration of the constrained system with constant of proportionality A^+. In other words,If $AQ = \mathbf{b}$ then,there is no deviation and the constraint equation is satisfied. Otherwise, there is a deviation from the trajectory of the constraint equation.The second component of (14) is proportional to a non-zero n-vector w with constant of proportionality $(I - A^+A)$.

The constraint force given in (14) is general in that D'Alembert-Lagrange principle is not taken into account. That means (14) is applicable irrespective of the whether the constraint is ideal or non-ideal. To further strengthen this observation let us see what form (14) would have if D'Alembert-Lagrange principle is taken into account.

D'Alembert-Lagrange principle states that, at each instant of time t, and for all virtual displacement δr, at that time t, the work done by the force of constraint, Q^c, under this virtual displacements,δr, must be zero; that is, $(\delta r)^T Q^c = 0$ at each instant of time t [2, 3].

The constraint equation, the second part of (11), is considered to define a virtual displacement. We can write the constraint equation in (11) in the form:

$$B\ddot{q} = \mathbf{b}, \tag{15}$$

where $B = (\Phi_q + \Psi_{\dot{q}})$ is m by n matrix and $\mathbf{b}$ is defined in (12) is an m vector. A non-zero vector δr such that $B\delta r = 0$ at time t is said to be a virtual displacement (δr is a non-zero vector in the null space of B) [2, 3]. Suppose that at a particular time t, $(\delta r)^T Q^c = 0$ (that is D'Alembert-Lagrange principle is satisfied). We want to show that the term,in equation (14), $(I - A^+A)w = 0$.

To define δr,the virtual displacement, in terms of A, put $\delta r = M^{-1}\delta\mu$ then $B\delta r = (\Phi_q + \Psi_{\dot{q}})M^{-1}\delta\mu$ and then $A\delta\mu = 0$. (A is defined above in (12)). From $A\delta\mu = 0$ we obtain $\delta\mu = (I - A^+A)u$ for a non- zero n-vector u. Thus in (12) D'Alembert–Lagrange principle requires that $\{\delta\mu : A\delta\mu = 0\}$

Now:

$$\begin{aligned}(\delta r)^T Q^c = M^{-1}(I - A^+A)uA^+(\mathbf{b} - AQ) + M^{-1}(I - A^+A)u(I - A^+A)w = \\ = M^{-1}u^T(I - A^+A)A^+(\mathbf{b} - AQ) + M^{-1}u^T(I - A^+A)(I - A^+A)w = \\ = 0 + M^{-1}u^T(I - A^+A)w\end{aligned}$$

and then we obtain:

$$(I - A^+A)w = 0. \tag{16}$$

Hence, we can make the following conclusions:

a) The constraint force, taking into account the D' Alembert-Lagrange principle is given by:

$$Q^i \triangleq A^+(\mathbf{b} - AQ). \tag{17}$$

This force,Q^i, is the control force that provides feedback control based on the error signal $\mathbf{b} - AQ$ which measures the extent to which the unconstrained acceleration does not satisfy the trajectory requirement of the constraint equation (15) with the gain matrix A^+ [4, 5].

b) Thus, (17) is a force resulting from idea nature of the constraints and

$$Q^{ni} \triangleq (I - A^+A)w \tag{18}$$

is a force from the non-ideal nature of the constraints.

c) It is also possible to say that, the work done by the constraint force by a non-zero virtual displacement vector δr, since the ideal constraint do no work at any time t, is given by:

$$W^{ni} \triangleq (\delta r)^T (I - A^+A)w. \tag{19}$$

d) An explicit equation of acceleration of the constrained mechanical system from the first equation of (11) and equation (14) is given by:

$$\ddot{q} = M^{-1}[A^+(\mathbf{b} - AQ) + (I - A^+A)w] + M^{-1}Q \tag{20}$$

which reduces,based on (16), to:

$$\ddot{q} = M^{-1}[A^+(\mathbf{b} - AQ)] + M^{-1}Q. \tag{21}$$

Regarding the characterization of vector w which is used starting from equation (14) one can set $w = M^{-1}C(t)$ in order that equations (20) and (22) are identical [3].

$$\ddot{q} = M^{-1}[A^+(\mathbf{b} - AQ)] + (I - A^+A)M^{-1}C(t) + M^{-1}Q. \tag{22}$$

In other words the non-ideal constraint force is given by:

$$(I - A^+A)M^{-1}C(t), \tag{23}$$

where [3] the n-vector $C(t)$ needs to be specified at each instant of time t, and it depends on the nature of the non-ideal constraints in the specified mechanical system under consideration. More information can be obtained in [3]. If the constraint forces are ideal then $C(t) = 0$.

4. Stability

We Use *Routh-Hurwitz* criterion for stability in determining the constants K and B [4, 5]. The *Routh-Hurwitz* stability method provides an answer to the question of stability by considering the characteristic equation of the system. The characteristic equation in the Laplace variable is written as

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0. \tag{24}$$

The *Routh–Hurwitz* criterion states that the number of roots of $q(s)$ with positive real parts is equal to the number of changes in sign of the first column of the Routh array. This criterion requires that there be no changes in sign in the first column for a stable system. This requirement is *both necessary and sufficient* [4, 5]. This criterion will be applied in the next example.

Example. A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in figure 1. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, find the acceleration and each of the constraint forces before the hoop falls off the cylinder.

Solution. The two constraint equations, the distance of the center of mass of the hoop from the center of the cylinder and the no slipping of the hoop as long as it is

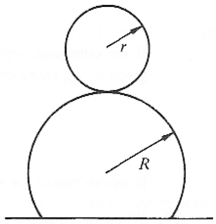

Figure 1. A hoop rolls on a cylinder.

touching the cylinder are respectively given by:

$$\rho = r + R, \quad r(\phi - \theta) = R\theta,$$

where ϕ, the angle r makes with the vertical and θ, the angle ρ makes with the vertical are the generalized coordinates. The kinetic energy is the sum of the kinetic energy of the center of mass of the hoop and the kinetic energy of the hoop about the cylinder given by:

$$T = \frac{1}{2}m((\rho\dot{\theta})^2 + (r\dot{\phi})^2).$$

The potential energy is the height above the center of the cylinder and is given by:

$$V = mg\rho\cos(\theta).$$

The Lagrangian $L = T - V$.

The first part of equation (10) becomes:

$$\begin{bmatrix} m\rho^2 & 0 \\ 0 & mr^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{pmatrix} -mg\rho\sin(\theta) \\ 0 \end{pmatrix} + Q^c. \tag{25}$$

Based on equation (3) we have the excess variables $\mathbf{r} = (y^1, y^2)$ for the two constraints and Since, $\ddot{\mathbf{r}} = (\ddot{y}^1, \ddot{y}^2)$ and $\ddot{y}^1 = 0, \ddot{y}^2 = r\ddot{\phi} - \rho\ddot{\theta}$ then the second part of equation (10) becomes:

$$\ddot{\mathbf{r}} = -B\dot{\mathbf{r}} - K\mathbf{r}, \tag{26}$$

where K and B are appropriate constants to be chosen for asymptotic stability of the equilibrium point of system. First let us find $Q^c = Q^i + Q^{ni}$, where, based on the substitutions used in (12) we obtain:

$$M = \begin{pmatrix} m\rho^2 & 0 \\ 0 & mr^2 \end{pmatrix}, A = [\Phi_q + \Psi_{\dot{q}}]M^{-1} = \begin{pmatrix} 0 & 0 \\ \frac{-1}{m\rho} & \frac{1}{mr} \end{pmatrix},$$

where $\Phi_1(q,t) = \rho - (r + R)$ and $\Phi_2(q,t) = r(\phi - \theta) - R\theta$ and $\Phi = (\Phi_1, \Phi_2)$.

It need to be noted that both the constraints in this example are holonomic constraints. $Q = \begin{pmatrix} -mg\rho\sin(\theta) \\ 0 \end{pmatrix}$, $\mathbf{b} = -B\dot{\mathbf{r}} - K\mathbf{r} = \begin{pmatrix} -Ky^1 \\ -Ky^2 - B\dot{y}^2 \end{pmatrix}$,

$$A^{nc} = AQ = \begin{pmatrix} 0 & 0 \\ \frac{-1}{m\rho} & \frac{1}{mr} \end{pmatrix} \begin{pmatrix} -mg\rho\sin(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ g\sin(\theta) \end{pmatrix},$$

$$A^+ = \frac{1}{(\frac{1}{m\rho})^2 + (\frac{1}{mr})^2} A^T = \begin{pmatrix} 0 & \frac{-m\rho r^2}{r^2+\rho^2} \\ 0 & \frac{mr\rho^2}{r^2+\rho^2} \end{pmatrix}.$$

Hence,

$$Q^i = A^+(\mathbf{b} - AQ) = \begin{pmatrix} \frac{-m\rho r^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \\ \frac{mr\rho^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \end{pmatrix}$$

If we assume the constraint between the hoop and the cylinder at some time to be non-ideal then the constraint force resulting from such a constraint is given by equation (23) as:

$$Q^{ni} = (I - A^+A)M^{-1}C(t) = \begin{pmatrix} \frac{1}{m(r^2+\rho^2)} & \frac{\rho}{mr(r^2+\rho^2)} \\ \frac{r}{m\rho(r^2+\rho^2)} & \frac{1}{m(r^2+\rho^2)} \end{pmatrix} C(t),$$

where [3] the 2-vector $C(t)$ needs to be specified at each instant of time t,and it depends on the nature of the non-ideal constraints in the specified mechanical system under consideration.

The equation of motion of the hoop becomes:

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{-m\rho r^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \\ \frac{mr\rho^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \end{pmatrix} + \\ + M^{-1}\begin{pmatrix} -mg\rho\sin(\theta) \\ 0 \end{pmatrix} + M^{-1} \begin{pmatrix} \frac{1}{m(r^2+\rho^2)} & \frac{\rho}{mr(r^2+\rho^2)} \\ \frac{r}{m\rho(r^2+\rho^2)} & \frac{1}{m(r^2+\rho^2)} \end{pmatrix} C(t).$$

Next point of discussion is experimental determination of the constants K and B so that the system becomes Asymptotically stable. Since the determination of $C(t)$ is beyond the scope of this paper let us assume the case of ideal constraints $C(t) = 0$.

Hence the equation of the system reduces to:

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{-m\rho r^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \\ \frac{mr\rho^2}{r^2+\rho^2}(-Ky^2 - B\dot{y}^2 - g\sin(\theta)) \end{pmatrix} + M^{-1}\begin{pmatrix} -mg\rho\sin(\theta) \\ 0 \end{pmatrix}.$$

Again for Numerical investigation of K and B, let $m = 2$kg, $r = 0.2$ meter, $R = 1$m, $\rho = R + r$, $g = 9.8\frac{kgm}{s^2}$.

Then we obtain:

$$M = \begin{pmatrix} 0.3472 & 0 \\ 0 & 12.5 \end{pmatrix}, \quad \frac{-m\rho r^2}{r^2+\rho^2} = 0.06487, \quad \frac{mr\rho^2}{r^2+\rho^2} = 0.3891, \quad mg\rho = 23.52$$

and with this numerical values the equation of the system becomes:

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} -0.0225(-Ky^2 - B\dot{y}^2 - 23.2995\sin(\theta)) \\ 4.86389(-Ky^2 - B\dot{y}^2 - 47.665\sin(\theta)) \end{pmatrix}.$$

Next we write the system equation as a system First order Differential Equations. In so doing let $x_1 = \theta, x_2 = \dot{\theta}, \dot{x}_2 = \ddot{\theta}, x_3 = \phi, x_4 = \dot{\phi}, \dot{x}_4 = \ddot{\phi}$. Now the system becomes:

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x}_2 = -0.0225[K(-0.2x_3 + 1.2x_1) + B(-0.2x_4 + 1.2x_2)] - 23.2995\sin(x_1), \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = 4.8638[K(-0.2x_3 + 1.2x_1) + B(-0.2x_4 + 1.2x_2)] - 47.665\sin(x_1). \end{cases}$$

The state variables are then x_1, x_2, x_3 and x_4. Define the output variables $x_1 = \theta$ and $x_3 = \phi$ define u, input variable as step in put.

Define the state derivatives as:

$$\begin{cases} \dot{x_1} = f_1(x,u) = x_2, \\ \dot{x}_2 = f_2(x,u) = -0.0225[K(-0.2x_3 + 1.2x_1) + B(-0.2x_4 + 1.2x_2)] - 23.2995\sin(x_1), \\ \dot{x}_3 = f_3(x,u) = x_4, \\ \dot{x}_4 = f_4(x,u) = 4.8638[K(-0.2x_3 + 1.2x_1) + B(-0.2x_4 + 1.2x_2)] - 47.665\sin(x_1). \end{cases}$$

Choosing the output variables $x_1 = \theta$ and $x_3 = \phi$ define:

$$\begin{cases} y_1 = g_1(x,u) = x_1, \\ y_2 = g_2(x,u) = x_3. \end{cases}$$

Suppose the initial condition is such that $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0) = (0,0,0,0)$ and the initial input $u^0 = 0$.

We want to write the system in a linearized form:

$$\begin{cases} \dot{x} = Ax + EU, \\ y = Cx + DU. \end{cases}$$

Which are the state and out put equations respectively and:

$$A = \frac{\partial f}{\partial x}(x^0, u^0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.027K - 23.2995 & -0.027B & 0.0045K & 0.0045B \\ 0 & 0 & 0 & 1 \\ 5.8366K - 47.665 & 5.8366B & -0.9728K & -0.9728B \end{pmatrix},$$

$$E = \frac{\partial f}{\partial u}(x^0, u^0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; C = (\frac{\partial g}{\partial x}(x^o, u^0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; D = \frac{\partial g}{\partial u}(x^0, u^0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

With the above Linearization the state equation becomes:

$$\begin{pmatrix} \triangle\dot{x}_1 \\ \triangle\dot{x}_2 \\ \triangle\dot{x}_3 \\ \triangle\dot{x}_4 \end{pmatrix} = \begin{pmatrix} \triangle\dot{\theta} \\ \triangle\ddot{\theta} \\ \triangle\dot{\phi} \\ \triangle\ddot{\phi} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -0.027K - 23.2995 & -0.027B & 0.0045K & 0.0045B \\ 0 & 0 & 0 & 1 \\ 5.8366K - 47.665 & 5.8366B & -0.9728K & -0.9728B \end{pmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \triangle U.$$

The out put equation is given by:

$$y = \begin{pmatrix} y1 \\ y2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \theta \\ \phi \end{pmatrix}.$$

Next we shall determine the constants K and B in order to make the system asymptotically stable. We use Routh-Hurwitz Criterion for stability in determining the constants K and B [4]. This method requires the characteristic equation of the system in the Laplace variable s.

$$det(sI - A) = \begin{vmatrix} s & -1 & 0 & 0 \\ 0.027K + 23.2995 & s + 0.027B & -0.0045K & -0.0045B \\ 0 & 0 & s & -1 \\ -5.8366K + 47.665 & -5.8366B & 0.9728K & s + 0.9728B \end{vmatrix} =$$
$$= s^4 + 0.9998Bs^3 + (0.0000009B^2 + 0.9998K + 23.2995)s^2 +$$
$$+ (0.0000018BK + 22.88024611B)s + 0.0000009K^2 + 22.8802461K.$$

For construct Routh table let $a = 0.9998B, b = 0.0000009B^2 + 0.9998K + 23.2995, c = 0.0000018BK + 22.8802461B, d = 0.0000009K^2 + 22.8802461K$

Table 1

Routh-table

column of S	column#1	column#2	column#3
s^4	1	b	d
s^3	a	c	0
S^2	$\frac{ab-c}{a}$	d	0
s	$\frac{abc-c^2-a^2d}{ab-c}$	0	0
s^0	d	0	0

According to Routh–Hurwitz criterion the system is stable if the first column of Routh table does not change sign. Accordingly, we need to have

$$a > 0, \quad ab - c > 0, \quad abc - c^2 - a^2d > 0, \quad d > 0. \tag{27}$$

since $1 > 0$.

Upon investigating the graph (figure 2) of $abc - c^2 - a^2d > 0$ using MATLAB,it has a maximum point $(x, y) = (0, 0)$. The MATLAB command used and the graph are shown below.(B=x and K=y is used for this case.)

$$fsolve(0.9998 * x * (0.0000009 * x(22.8802461 * x + 0.0000018 * x * y)^2 +$$
$$+ 0.9998 * y(22.8802461 * x + 0.0000018 * x * y) + 23.2995) -$$
$$- (22.8802461 * x + 0.0000018 * x * y)^2 - 0.99960004 * x^2 = 0, y),$$

$$plot(0.9998 * x * (0.0000009 * x(22.8802461 * x + 0.0000018 * x * y)^2 +$$
$$+ 0.9998 * y(22.8802461 * x + 0.0000018 * x * y) + 23.2995) -$$
$$- (22.8802461 * x + 0.0000018 * x * y)^2 - 0.99960004 * x^2, y).$$

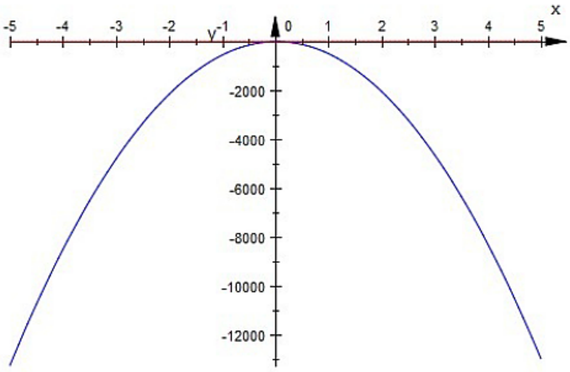

Figure 2. Graph with maximum point (0,0)

This implies that the first column of Routh table (Table 1) changes sign since at least 1 and $abc - c^2 - a^2d > 0$ have opposite sign. Hence it can be concluded from the Routh table indicated above that the system is unstable.But for $B = K = 0$ we obtain the fourth (row of s) to be a zero row which indicates that the eigenvalues (or poles) of the system are on the imaginary axis , axis y.It can be seen for the case of $B = K = 0$ the system is marginally stable.The graph of the system for the case $B = K = 100$ and non-zero initial condition $X0 = [0.0001; 0; 0.0001; 0]$ is indicated below in figure 3.

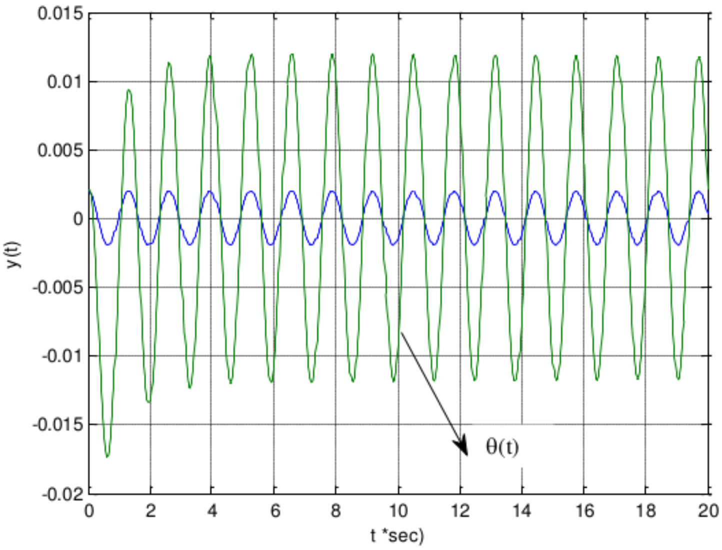

Figure 3. Initial Condition Response

5. Conclusion

In this paper a method of constructing equations of mechanical system that includes both the ideal and non-ideal nature of constraints is developed. With the method developed, problems that may arise from redundant constraints, like rank deficient Jacobian matrix of the constraints and other related problems can be managed.The method can also be used to develop state-space equation of a system and then after, can be used for further study on controllability, observability and other related issues in complex mechanical systems. The state space of a system being constructed, a wide range use of different software like MATLAB can be employed for investigation of dynamic system.This paper can also be used as a bench mark for further research in the area.

References

1. *Mukharlyamov R. G.* Stabilization of the Motions of Mechanical Systems in Prescribed Phase-Space Manifolds // Applied Mathematics and Mechanics. — 2006. — Vol. 70. — Pp. 210–222.
2. *Udwadia F. E.* Recent Advances in Multi-body Dynamics and Nonlinear Control // J. of the Braz. Soc. of Mech. Sci. & Eng. — 2006. — Vol. XXVIII, No 3 / 311. — Pp. 311–314.
3. *Udwadia F. E., Kalaba R. E.* On the Foundation of Analytical Dynamics // International Journal of Nonlinear Mechanics. — 2002. — Vol. 37. — Pp. 1079–1090.
4. *Dorf R. C., Bishop R. H.* Modern Control Systems. — 12th edition. edition. — Prentice Hall, 2011.
5. *Norman S. N.* Control Systems Engineering. — 6th edition edition. — John Wiley & Sons, Inc, 2011.
6. *Amirouche F.* Fundamentals of Multibody Dynamics: Theory and Applications. — Boston: Birkhäuser, 2006.
7. *González F., Kövecses J.* Use of Penalty Formulations in Dynamic Simulation and Analysis of Redundantly Constrained Multibody Systems. // Multibody System Dynamics. — 2012.
8. *Blajer W.* Advanced Design of Mechanical Systems: From Analysis to Optimization. — New York: Springer Wien, 2009.

УДК 531.3

Построение уравнений динамики связанных механических систем

Чернет Туге Дересса

Кафедра математики
Университет г. Джимма
г. Джимма, Эфиопия

В статье предлагается новый метод решения задачи построения уравнений динамики механической системы, обеспечивающий стабилизацию связей при численном решении. Исходными данными для составления уравнений динамики являются функция Лагранжа, диссипативные и непотенциальные силы и ограничения, выраженные уравнениями голономных и неголономных связей. Рассматриваются случаи идеальных и неидеальных связей. Определение правых частей систем дифференциальных уравнений используется обобщенная обратная матрица.

Для исследования поведения отклонений решения системы от уравнений связей вводятся добавочные переменные. Устойчивость по отношению к уравнениям связей определяется по уравнениям возмущений связей, составленных по расширенным функциям Лагранжа и диссипативной функции. Ограничиваясь добавлением в лагранжиан и диссипативную функцию квадратичных форм с постоянными коэффициентами, получены дифференциальные уравнения возмущений связей линейными с постоянными коэффициентами. Это позволяет обеспечить условия асимптотической устойчивости на основе критерия Рауса–Гурвица. Метод иллюстрируется на примере решения задачи скатывания цилиндра с поверхности закреплённого цилиндра без проскальзывания.

Ключевые слова: избыточные переменные, идеальные связи, неидеальные связи, устойчивость, критерий Рауса–Гурвица, пространство состояний, функция Лагранжа, диссипативная функция.