UDC 517.968

Uniqueness and Stability of Solutions for Certain Linear Equations of the First Kind with Two Variables

Avyt Asanov[*], Z. A. Kadenova[†]
[*] *Kyrgyz-Turkish University Manas*
56, Prospect Mira, Bishkek, 720040, Kyrgyzstan
[†] *Ministry of Education and Science of the Kyrgyz Republic*
257, K. Tynystanova str., Bishkek, 720044, Kyrgyzstan

The article is devoted to the study of uniqueness and stability of solutions of linear integral equations of the first kind with two independent variables.

The relevance of the problem is due to the needs in development of new approaches for the regularization and uniqueness of the solution of linear integral equations of the first kind with two independent variables. Integral and operator equations of the first kind with two independent variables arise in theoretical and applied problems.Works of A.N. Tikhonov, M.M. Lavrentyev and B.K. Ivanov, in which a new concept of correctness of setting such targets is given, different from the classical, show tools for research of ill-posed problems, which stimulated the interest to the integral equations that are of great practical importance. At the present time the theory and applications of ill-posed problems have been rapidly developing. One of the classes of such ill-posed problems are integral equations of the first kind with two independent variables. As of approximate solutions of such problems, stable to small variations of the initial data, we use the solutions derived by the method of regularization.

In this article we prove the theorem of uniqueness and obtain estimates of stability for such equations in families of sets of correctnesses. For the tasks solution the methods of functional analysis and method of nonnegative quadratic forms are used. The results of the work are new.

Key words and phrases: linear, inteqral equations, first kind, two variables, solution, uniqueness and stability.

1. Introduction

The integral equations of the first kind were studied in [1–8]. More specifically, fundamental results for Fredholm integral equations of the first kind were obtained in [6], where regularizing operators in the sense of *M.M.Lavrent'ev* were constructed for solutions of linear Fredholm inteqral equations of the first kind. For linear Volterra integral equations of the first kind and third kinds with smooth kernels, the existence of a multiparameter family of solution was proved in [7]. The regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the first kind were investigated in [4,5]. In this work we shall study the problems of uniqueness and stability of solution of the integral equation

$$Ku = f(t,x),\ (t,x) \in G = \left\{(t,x) \in R^2 : t_0 \leqslant t \leqslant T,\ a \leqslant x \leqslant b\right\}, \quad (1)$$

where

$$Ku \equiv \int\limits_a^x P(t,x,y)u(t,y)\mathrm{d}y + \int\limits_{t_0}^t Q(t,x,s)u(s,x)\mathrm{d}s + \int\limits_{t_0}^T \int\limits_a^b C(t,x,s,y)u(s,y)\mathrm{d}y\mathrm{d}s, \quad (2)$$

$P(t,x,y)$ and $Q(t,x,s)$ are given continuous functions, respectively on the domains

$$G_1 = \{(t,x,s) : t_0 \leqslant t \leqslant T,\ a \leqslant y \leqslant x \leqslant b\},$$

$$G_2 = \{(t,x,s) : t_0 \leqslant s \leqslant t \leqslant T,\ a \leqslant x \leqslant b\},$$

$C(t,x,s,y), f(x)$ are given functions, $u(t,x)$ is an unknown function.

2. Uniqueness and Stability of solutions of integral equations

Assume that the following conditions are satisfied:
(i). $P(t,b,a) \geqslant 0 \quad \text{for all} \quad t \in [t_0,T], \quad P(t,b,a) \in C[t_0,T],$

$$P'_y(t,y,a) \leqslant 0 \quad \text{for all} \quad (t,y) \in G, \quad P'_y(t,y,a) \in C(G),$$
$$P'_z(s,b,z) \geqslant 0 \quad \text{for all} \quad (s,z) \in G,$$
$$P'_z(s,b,z) \in C(G), \quad P''_{zy}(s,y,z) \leqslant 0 \quad \text{for all} \quad (s,y,z) \in G_1, \quad P''_{zy}(s,y,z) \in C(G_1).$$

(ii). $Q(T,y,t_0) \geqslant 0 \quad \text{for all} \quad y \in [a,b], \quad Q(T,y,t_0) \in C[a,b],$

$$Q'_s(s,y,t_0) \leqslant 0 \quad \text{for all} \quad (s,y) \in G, \quad Q'_s(s,y,t_0) \in C(G),$$
$$Q'_\tau(T,y,\tau) \geqslant 0 \quad \text{for all} \quad (y,\tau) \in G,$$
$$Q'_\tau(T,y,\tau) \in C(G), \quad Q''_{\tau s}(s,y,\tau) \leqslant 0 \quad \text{for all} \quad (s,y,\tau) \in G_2, \quad Q''_{\tau,s}(s,y,\tau) \in C(G_2).$$

(iii). At least one of the following conditions holds:
(a) $P'_y(s,y,a) < 0$ for almost all $(s,y) \in G$;
(b) $P'_z(s,b,z) > 0$ for almost all $(s,z) \in G$;
(c) $Q'_s(s,y,t_0) < 0$ for almost all $(s,y) \in G$;
(d) $Q'_\tau(T,y,\tau) > 0$ for almost all $(y,\tau) \in G$;
(e) $P''_{zy}(s,y,z) < 0$ for almost all $(s,y,z) \in G_1$;
(f) $Q''_{\tau s}(s,y,\tau) < 0$ for almost all $(s,y,\tau) \in G_2$.

(iv). $C(t,x,s,y) \in L_2(G^2)$ and

$$\frac{1}{2}\left[C(t,x,s,y)+C(s,y,t,x)\right] = \sum_{i=1}^{m} \lambda_i \varphi_i(t,x)\varphi_i(s,y),$$
$$m \leqslant \infty, \;\; \lambda_i \geqslant 0, \;\; i=1,2,...,m, \quad (3)$$

where $\{\varphi_i(t,x)\}$ is an orthonormal sequence of eigenfunctions from $L_2(G)$ and $\{\lambda_i\}$ is the sequence of corresponding nonzero eigenvalues of the Fredholm integral operator C generated by the kernel $\frac{1}{2}\left[C(t,x,s,y)+C(s,y,t,x)\right]$ with the elements $\{\lambda_i\}$ arranged in decreasing order of their absolute values. If $C(t,x,s,y) = 0 \quad \text{for all} \quad (t,x,s,y) \in G^2$, we assume that $\lambda_1 = \lambda_2 = ... = \lambda_m = 0$.

Theorem 1. *Let conditions (i)–(iv) be satisfied. Then the solution of the equation (1) is unique in* $L_2(G)$.

Proof. Taking the multiplication of both sides of the equation (1) with $u(t,x)$, integrating the results on G, we obtain

$$\int_{t_0}^{T}\int_{a}^{b}\int_{a}^{y} P(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}y\mathrm{d}s + \int_{t_0}^{T}\int_{a}^{b}\int_{t_0}^{s} Q(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}y\mathrm{d}s +$$
$$+\int_{a}^{b}\int_{t_0}^{T}\int_{t_0}^{T}\int_{a}^{b} C(s,y,\tau,z)u(\tau,z)u(s,y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y = \int_{a}^{b}\int_{t_0}^{T} f(s,y)u(s,y)\mathrm{d}s\mathrm{d}y. \quad (4)$$

Integrating by parts and using the Dirichlet formula we obtain

$$\int_{t_0}^{T}\int_{a}^{b}\int_{a}^{y} P(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}y\mathrm{d}s = -$$

$$= -\int_{t_0}^{T}\int_{a}^{b}\int_{a}^{y} P(s,y,z)\frac{\partial}{\partial z}\left(\int_{z}^{y} u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}z u(s,y)\mathrm{d}y\mathrm{d}s =$$

$$= \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b} P(s,y,a)\left[\frac{\partial}{\partial y}\left(\int_{a}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\right]\mathrm{d}y\mathrm{d}s +$$

$$+ \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b}\int_{z}^{b} P'_z(s,y,z)\left[\frac{\partial}{\partial y}\left(\int_{z}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\right]\mathrm{d}y\mathrm{d}z\mathrm{d}s =$$

$$= \frac{1}{2}\int_{t_0}^{T}\left[P(s,y,a)\left(\int_{a}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\right]\Bigg|_{y=a}^{y=b}\mathrm{d}s - \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b} P'_y(s,y,a)\left(\int_{a}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}s +$$

$$+ \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b}\left[P'_z(s,y,z)\left(\int_{z}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\right]\Bigg|_{y=z}^{y=b}\mathrm{d}y\mathrm{d}s -$$

$$- \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b}\int_{z}^{b} P''_{zy}(s,y,z)\left(\int_{z}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}z\mathrm{d}s =$$

$$= \frac{1}{2}\int_{t_0}^{T} P(s,b,a)\left(\int_{a}^{b} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}s - \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b} P'_y(s,y,a)\left(\int_{a}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}s +$$

$$+ \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b} P'_z(s,b,z)\left(\int_{z}^{b} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}z\mathrm{d}s -$$

$$- \frac{1}{2}\int_{t_0}^{T}\int_{a}^{b}\int_{a}^{y} P''_{zy}(s,y,z)\left(\int_{z}^{y} u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}z\mathrm{d}y\mathrm{d}s. \quad (5)$$

Similarly integrating by parts and using the Dirichlet formula analogically we have

$$\int_{a}^{b}\int_{t_0}^{T}\int_{t_0}^{s} Q(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y = \frac{1}{2}\int_{a}^{b} Q(T,y,t_0)\left(\int_{t_0}^{T} u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}y -$$

$$- \frac{1}{2}\int_{a}^{b}\int_{t_0}^{T} Q'_s(s,y,t_0)\left(\int_{t_0}^{s} u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y + \frac{1}{2}\int_{a}^{b}\int_{t_0}^{T} Q'_\tau(T,y,\tau)\left(\int_{\tau}^{T} u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}y -$$

$$- \frac{1}{2}\int_{a}^{b}\int_{t_0}^{T}\int_{t_0}^{s} Q''_{\tau s}(s,y,\tau)\left(\int_{\tau}^{s} u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s\mathrm{d}y. \quad (6)$$

Using the Dirichlet formula we have

$$\int_a^b\int_{t_0}^T\int_a^b\int_{t_0}^T C(t,x,s,y)u(s,y)u(t,x)\mathrm{d}s\mathrm{d}y\mathrm{d}t\mathrm{d}x =$$

$$= \int_a^b\int_{t_0}^T\int_a^b\int_{t_0}^t C(t,x,s,y)u(s,y)u(t,x)\mathrm{d}s\mathrm{d}y\mathrm{d}t\mathrm{d}x +$$

$$+ \int_a^b\int_{t_0}^T\int_a^b\int_{t}^T C(t,x,s,y)u(s,y)u(t,x)\mathrm{d}s\mathrm{d}y\mathrm{d}t\mathrm{d}x =$$

$$= \int_a^b\int_{t_0}^T\int_a^b\int_{t_0}^t \left[C(t,x,s,y)+C(s,y,t,x)\right]u(s,y)u(t,x)\mathrm{d}s\mathrm{d}y\mathrm{d}t\mathrm{d}x. \quad (7)$$

Taking into account (5), (6), (7) and (3) from (4) we obtain

$$\frac{1}{2}\int_{t_0}^T P(s,b,a)\left(\int_a^b u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}s - \frac{1}{2}\int_{t_0}^T\int_a^b P'_y(s,y,a)\left(\int_a^y u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}s +$$

$$+\frac{1}{2}\int_{t_0}^T\int_a^b P'_z(s,b,z)\left(\int_z^b u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}z\mathrm{d}s -$$

$$-\frac{1}{2}\int_{t_0}^T\int_a^b\int_a^y P''_{zy}(s,y,z)\left(\int_z^y u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}z\mathrm{d}y\mathrm{d}s +$$

$$+\frac{1}{2}\int_a^b Q(T,y,t_0)\left(\int_{t_0}^T u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}y - \frac{1}{2}\int_a^b\int_{t_0}^T Q'_s(s,y,t_0)\left(\int_{t_0}^s u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y +$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T Q'_\tau(T,y,\tau)\left(\int_\tau^T u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}y -$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s Q''_{\tau s}(s,y,\tau)\left(\int_\tau^s u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s\mathrm{d}y +$$

$$+\sum_{i=1}^m \lambda_i\left(\int_a^b\int_{t_0}^T \varphi_i(s,y)u(s,y)\mathrm{d}s\mathrm{d}y\right)^2 = \int_a^b\int_{t_0}^T f(s,y)u(s,y)\mathrm{d}s\mathrm{d}y. \quad (8)$$

Let $f(t,x)=0$ for all $(t,x)\in G$. Then by virtue of conditions (i)–(iv), from (8) we have $u(t,x)=0$ for almost all $(t,x)\in G$. The theorem 1 is proved. □

Example 1. We consider the equation (1) for

$$P(t,x,y)=\alpha_0(t)\beta_0(x)[\gamma_0(y)+\gamma_1(y)], \quad (t,x,y)\in G_1,$$

$$Q(t,x,s)=\alpha_1(t)\beta_1(x)[\alpha_2(s)+\gamma_2(s)],\quad (t,x,s)\in G_2,$$

$$C(t,x,s,y)=\sum_{i=1}^{m}[c_i(t,x)c_i(s,y)+d_i(t,x)-d_i(s,y)],\quad (t,x,s,y)\in G^2,$$

where

$$\alpha_0(t),\alpha_1(t),\alpha_1'(t),\alpha_2(t),\alpha_2'(t),\gamma_2(t),\gamma_2'(t)\in C[t_0,T],\beta_0(x),\beta_0'(x),\gamma_0(x),$$
$$\gamma_0'(x),\gamma_1(x),\gamma_1'(x),\beta_1(x)\in C[a,b],c_i(t,x),d_i(t,x)\in C(G)(i=1,2,...,m),\alpha_1'(t)\leqslant 0$$

and $\gamma_2'(t)+\alpha_2'(t)\geqslant 0$ for all $t\in[t_0,T],\beta_0'(x)<0$ for almost all $x\in[a,b]$, $\gamma_0'(x)+\gamma_1'(x)\geqslant 0$ for all $x\in[a,b],\gamma_0(a)+\gamma_1(a)>0,\alpha_0(t)>0$ for almost all $t\in[t_0,T]$, $\beta_1(x)>0$ for almost all $x\in[a,b],\gamma_2(t_0)+\alpha_2(t_0)\geqslant 0$.

In this case the conditions (i)-(iv) be satisfied.

The following condition is assumed to hold in what follows.

(v). The Fredholm operator C generated by the kernel $\frac{1}{2}\left[C(t,x,s,y)+C(s,y,t,x)\right]$ defined by (3) is positive, i.e. all the eigenvalues λ_i of $\frac{1}{2}\left[C(t,x,s,y)+C(s,y,t,x)\right]$ are positive $(i=1,2,...,m,\ m=\infty)$.

The family of well-posedness depending on the parameter α is defined as

$$M_\alpha=\left\{u(t,x)\in L_2(G):\sum_{\nu=1}^{\infty}\lambda_\nu^{-\alpha}\left|u^{(\nu)}\right|^2\leqslant c_0\right\},$$

where $c_0>0,\alpha\in(0,\infty)$, and $u^{(\nu)}=\int\limits_{t_0}^{T}\int\limits_{a}^{b}u(t,x)\varphi_\nu(t,x)\mathrm{d}x\mathrm{d}t,\ \nu=1,2,...,\infty.$

Theorem 2. *Let conditions (i)-(ii) and (v) be satisfied.*

Then the solution $u(t,x)$ of the equation (1) is unique in $L_2(G)$. Moreover, on the set $K(M_\alpha)$ (where $K(M_\alpha)$ is the image of M_α under the action of the operator K defined by formula (2)), the inverse K^{-1} of operator K is uniformly continuous with the Holder exponent $\dfrac{\alpha}{\alpha+2}$; i.e

$$\|u(t,x)\|_{L_2}\leqslant c_0^{\frac{1}{2+\alpha}}\,\|f(t,x)\|_{L_2}^{\frac{\alpha}{2+\alpha}},\quad 0<\alpha<\infty, \tag{9}$$

where

$$u(t,x)\in M_\alpha,f(t,x)\in K(M_\alpha),\|u(t,x)\|_{L_2}=\left(\int\limits_a^b\int\limits_{t_0}^{T}|u(t,x)|^2\mathrm{d}t\mathrm{d}x\right)^{1/2}.$$

Proof. a) In this case, the orthonormal sequence of eigenfunctions $\{\varphi_i(t,x)\}$ is complete in $L_2(G)$. Therefore (8) implies the uniqueness of the solution to equation (1) in $L_2(G)$.

Let $f(t,x)\in K(M_\alpha)$. Then the equation (1) has a solution $u(t,x)\in M_\alpha$ and it follows from (8) that $\sum\limits_{\nu=1}^{\infty}\lambda_\nu\left|u^\nu\right|^2\leqslant\|f(t,x)\|_{L_2}\cdot\|u(t,x)\|_{L_2}$. On the other hand,

$$\|u(t,x)\|_{L_2}^2=\sum_{\nu=1}^{\infty}\frac{|u^{(\nu)}|^{\frac{2}{1+\alpha}}}{\lambda_\nu^{\frac{\alpha}{1+\alpha}}}\cdot\lambda_\nu^{\frac{\alpha}{1+\alpha}}\cdot|u^{(\nu)}|^{\frac{2\alpha}{1+\alpha}}\leqslant$$

$$\leqslant \left[\sum_{\nu=1}^{\infty} \frac{|u^{(\nu)}|^2}{\lambda_\nu^\alpha}\right]^{\frac{1}{1+\alpha}} \left[\sum_{\nu=1}^{\infty} \lambda_\nu |u^{(\nu)}|^2\right]^{\frac{\alpha}{1+\alpha}}. \quad (10)$$

Combining the last two inequalities gives estimate (9). The theorem 2 is proved. □

References

1. *Aparstyn A. S.* Nonclassical Linear Volterra Equations of the First Kind. — Utrecht: VSP, 2003.
2. *Asanov A.* Regularization, Uniqueness and Existence of Solutions of Volterra Equations of the First Kind. — Utrecht: VSP, 1998.
3. *Bukhgeim A. L.* Volterra Equations and Inverse Problems. — Utrecht: VSP, 1999.
4. *Imanaliev M. I., Asanov A.* On Solutions of Systems of Volterra Nonlinear Integral Equations of the First Kind // Doklady Akademii Nauk. — 1989. — Vol. 309, No 5. — Pp. 1052–1055.
5. *Imanaliev M. I., Asanov A.* On Solutions of Systems of Nonlinear Two Dimensional Volterra Integral Equations of the First Kind // Doklady Akademii Nauk. — 1991. — Vol. 317, No 2. — Pp. 330–333.
6. *Lavrent'ev M. M., Romanov V. G., Shishatskii S. P.* ILL-posed Problems of Mathematical Physics and Analysis. — American Mathematical Society: Providence, R.I, 1986.
7. *Magnitskii N. A.* Linear VolterrA Integral Equations of the First and Third Kind // Doklady Akademii Nauk. — 1991. — Vol. 317, No 2. — Pp. 330–333.
8. *Shishatskii S. P., Asanov A., Atamanov E. R.* Uniqueness Problems for Degenerating Equations and Nonclassical Problems. — Utrecht: VSP, 2001.

УДК 517.968

Единственность и устойчивость решений для некоторых интегральных уравнений первого рода с двумя независимыми переменными

А. Асанов*, З. А. Каденова†

* *Кыргызско-Турецкий университет «Манас»*
Проспект Мира, 56, Бишкек, 720040, Кыргызстан
† *Министерство образования и науки Кыргызской республики*
ул. К. Тыныстанова, 257, Бишкек, 720044, Кыргызстан

Статья посвящена исследованию единственности и устойчивости решений линейных интегральных уравнений первого рода с двумя независимыми переменными.

Актуальность проблемы обусловлена потребностями в разработке новых подходов для регуляризации и единственности решения линейных интегральных уравнений первого рода с двумя независимыми переменными. Интегральные и операторные уравнения первого рода с двумя независимыми переменными возникают в теоретических и прикладных задачах. В работах А.Н. Тихонова, М.М. Лаврентьева и В.К. Иванова, в которых дано новое понятие корректности постановки таких задач, отличное от классического, показано средство для исследования некорректных задач, что стимулировало интерес к интегральным уравнениям, имеющим большое прикладное значение. В настоящее время бурно развивается теория и приложения некорректных задач. Один из классов таких некорректных задач составляют интегральные уравнения первого рода с двумя независимыми переменными.

В статье доказано теорема единственности и получены оценки устойчивости для таких уравнений в семействах множеств корректностей. Для решения задачи использованы методы функционального анализа и метод неотрицательных квадратичных форм. Полученные результаты работы являются новыми.

Ключевые слова: линейный, интегральные уравнения, первого рода, двух переменных, решение, единственность и устойчивость.