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Uniqueness of Solutions for One Class of Linear Equations of the First Kind with Two Variables

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This article is devoted to the study of the uniqueness of solutions of linear integral equations of the first kind with two independent variables in which the operator generated by the kernel, is not compact operator.

The relevance of the problem is due to the needs in development of new approaches for the regularization and uniqueness of the solution of linear integral equations of the first kind with two independent variables. For approximate solutions of such tasks, stable to small variations of the initial data, we use the solutions derived by the method of regularization and belonging to the class of incorrectly formulated tasks. One of the classes of such ill-posed problems are integral equations of the first kind with two independent variables.

The aim of the work is to prove the theorems of uniqueness for solving linear integral equations of the first kind with two independent variables.

In the paper a theorem of the uniqueness of the solution of integral equations of the first kind with two independent variables is proved. To obtain the results formulated in the article the methods of functional analysis and method of nonnegative quadratic forms are used. The obtained results are new. The reliability of the result is set by prooves and illustrated by examples. The work has a theoretical character.

The obtained theoretical results can be used in various fields of science and technology.

Key words and phrases: linear, inteqral equations, first kind, two variables, solution and uniqueness.

1. Introduction

Various issues concerning integral equations of the first kind were studied in [1–8]. More specifically, fundamental results for Fredholm integral equations of the first kind were obtained in [6], where regularizing operators in the sense of M.M.Lavrent'ev vere constructed for solutions of linear Fredholm integral equations of the first kind. For linear Volterra integral equations of the first and third kinds with smooth kernels, the existence of a multiparameter family of solutions was proved in [7]. The regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the first kind were investigated in [4,5]. In this paper, while analizing the following integral equation

$$\int\limits_a^b K(t,x,y)u(t,y)\mathrm{d}y + \int\limits_{t_0}^T Q(t,x,s)u(s,x)\mathrm{d}s + \int\limits_{t_0}^t \int\limits_a^x C(t,x,s,y)u(s,y)\mathrm{d}y\mathrm{d}s =$$

$$= f(t,x),\ (t,x) \in G = \left\{(t,x) \in R^2 : t_0 \leqslant t \leqslant T,\ a \leqslant x \leqslant b\right\}, \quad (1)$$

where

$$K(t,x,y) = \begin{cases} A(t,x,y),\ t_0 \leqslant t \leqslant T,\ a \leqslant y \leqslant x \leqslant b; \\ B(t,x,y),\ t_0 \leqslant t \leqslant T,\ a \leqslant x \leqslant y \leqslant b, \end{cases} \quad (2)$$

$$Q(t,x,s) = \begin{cases} M(t,x,s),\ t_0 \leqslant s \leqslant t \leqslant T,\ a \leqslant x \leqslant b, \\ N(t,x,s),\ t_0 \leqslant t \leqslant s \leqslant T,\ a \leqslant x \leqslant b, \end{cases} \quad (3)$$

where $A(t,x,y), B(t,x,y), M(t,x,s), N(t,x,s), C(t,x,s,y)$ are given continuous functions, respectively on the domains

$$G_1 = \{(t,x,y) : t_0 \leqslant t \leqslant T,\ a \leqslant y \leqslant x \leqslant b\};$$

$$G_2 = \{(t,x,y) : t_0 \leqslant t \leqslant T,\ a \leqslant x \leqslant y \leqslant b\};$$

$$G_3 = \{(t,x,s) : t_0 \leqslant s \leqslant t \leqslant T,\ a \leqslant x \leqslant b\};$$

$$G_4 = \{(t,x,s) : t_0 \leqslant t \leqslant s \leqslant T,\ a \leqslant x \leqslant b\};$$

$$G_5 = \{(t,x,s,y) : t_0 \leqslant t \leqslant s \leqslant T,\ a \leqslant x \leqslant b\}.$$

$u(t,x)$ and $f(t,x)$ are the unknown and given functions respectively, $(t,x) \in G$.

2. Uniqueness of Solutions of Integral Equations

Using $A(t,x,y), B(t,x,y), M(t,x,s)$ and $N(t,x,s)$ we define the following functions

$$\begin{cases} P(t,x,y) = A(t,x,y) + B(t,y,x), (t,x,y) \in G_1; \\ H(t,x,s) = M(t,x,s) + N(s,x,t), (t,x,s) \in G_3. \end{cases} \tag{4}$$

Assume that the following conditions are satisfied:
(i).

$$\begin{aligned}
P(t,b,a) \in C[t_0,T],\ P(t,b,a) \geqslant 0 \quad &\text{for all} \quad t \in [t_0,T],\\
P'_y(s,y,a) \in C(G), \quad P'_y(s,y,a) \leqslant 0 \quad &\text{for all} \quad (s,y) \in G,\\
P'_z(s,b,z) \in C(G), \quad P'_z(s,b,z) \geqslant 0 \quad &\text{for all} \quad (s,z) \in G,\\
P''_{zy}(s,y,z) \in C(G_1), \quad P''_{zy}(s,y,z) \leqslant 0 \quad &\text{for all} \quad (s,y,z) \in G_1,\\
H(T,y,t_0) \in C[a,b], \quad H(T,y,t_0) \geqslant 0 \quad &\text{for all} \quad y \in [a,b],\\
H'_s(s,y,t_0) \in C(G), \quad H'_s(s,y,t_0) \leqslant 0 \quad &\text{for all} \quad (s,y) \in G,\\
H'_\tau(T,y,\tau) \in C(G), \quad H'_\tau(T,y,\tau) \geqslant 0 \quad &\text{for all} \quad (y,\tau) \in G,\\
H''_{\tau s}(s,y,\tau) \in C(G_3), \quad H''_{\tau s}(s,y,\tau) \leqslant 0 \quad &\text{for all} \quad (s,y,\tau) \in G_3;
\end{aligned}$$

(ii).

$$C(T,b,t_0,a) \geqslant 0, \quad C'_s(s,b,t_0,a) \in C[t_0,T], \quad C'_s(s,b,t_0,a) \leqslant 0 \quad \text{for all} \quad s \in [t_0,T],$$

$$\begin{aligned}
C'_\tau(T,b,\tau,a) \in C[t_0,T], \quad C'_\tau(T,b,\tau,a) \geqslant 0 \quad &\text{for all} \quad \tau \in [t_0,T],\\
C'_y(T,y,t_0,a) \in C[a,b], \quad C'_y(T,y,t_0,a) \leqslant 0 \quad &\text{for all} \quad y \in [a,b],\\
C'_z(T,b,t_0,z) \in C[a,b], \quad C'_z(T,b,t_0,z) \geqslant 0 \quad &\text{for all} \quad z \in [a,b],\\
C''_{sy}(s,y,t_0,a) \in C(G), \quad C''_{sy}(s,y,t_0,a) \geqslant 0 \quad &\text{for all} \quad (s,y) \in G,\\
C''_{\tau y}(T,y,\tau,t_0) \in C(G), \quad C''_{\tau y}(T,y,\tau,t_0) \leqslant 0 \quad &\text{for all} \quad (y,\tau) \in G,\\
C''_{zs}(s,b,t_0,z) \in C(G), \quad C''_{zs}(s,b,t_0,z) \leqslant 0 \quad &\text{for all} \quad (s,z) \in G,\\
C''_{\tau z}(T,b,\tau,z) \in C(G), \quad C''_{\tau z}(T,b,\tau,z) \geqslant 0 \quad &\text{for all} \quad (\tau,z) \in G,\\
C'''_{\tau sy}(s,y,\tau,a) \in C(G_3), \quad C'''_{\tau sy}(s,y,\tau,a) \geqslant 0 \quad &\text{for all} \quad (s,y,\tau) \in G_3,\\
C'''_{\tau zs}(s,b,\tau,z) \in C(G_3), \quad C'''_{\tau zs}(s,b,\tau,z) \leqslant 0 \quad &\text{for all} \quad (s,z,\tau) \in G_3,\\
C'''_{zsy}(s,y,t_0,z) \in C(G_1), \quad C'''_{zsy}(s,y,t_0,z) \geqslant 0 \quad &\text{for all} \quad (s,y,z) \in G_1,
\end{aligned}$$

$$C'''_{\tau zy}(T,y,\tau,z)\in C(G_1),\quad C'''_{\tau zy}(T,y,\tau,z)\leqslant 0\quad\text{for all}\quad(\tau,y,z)\in G_1,$$
$$C^{IV}_{\tau zsy}(s,y,\tau,z)\in C(G_4),\quad C^{IV}_{\tau zsy}(s,y,\tau,z)\geqslant 0\quad\text{for all}\quad(s,y,\tau,z)\in G_4,$$
$$C''_{\tau s}(s,b,\tau,a)\in C(G_5),\quad C''_{\tau s}(s,b,\tau,a)\leqslant 0\quad\text{for all}$$
$$(s,\tau)\in G_5=\{(s,\tau): t_0\leqslant\tau\leqslant s\leqslant T\},$$
$$C''_{zy}(T,y,t_0,z)\in C(G_6),\quad C''_{zy}(T,y,t_0,z)\leqslant 0\quad\text{for all}$$
$$(y,z)\in G_6=\{(y,z): a\leqslant z\leqslant y\leqslant b\};$$

(iii). For almost all $(s,y,\tau,z)\in G_5$ the quadratic form

$$L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\frac{1}{(s-t_0)(y-a)}\left\{-P'_y(s,y,a)\alpha_1^2-\right.$$
$$-H'_s(s,y,t_0)\alpha_2^2-2C(s,y,t_0,a)\alpha_1\alpha_2-(s-t_0)\left[H''_{\tau s}(s,y,\tau)\alpha_3^2+2C'_\tau(s,y,\tau,a)\alpha_3\alpha_1\right]-$$
$$\left.-(y-a)\left[2C'_z(s,y,t_0,z)\alpha_2\alpha_4+P''_{zy}(s,y,z)\alpha_4^2\right]-2(s-t_0)(y-a)C''_{\tau z}(s,y,\tau,z)\alpha_3\alpha_4\right\}$$

is nonnegative, i.e. $L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\geqslant 0$ for all $\alpha_1,\alpha_2,\alpha_3\alpha_4\in R$;

(iv). If for almost all $(s,y,\tau,z)\in G_5$ $L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=0$, then it follow that $\alpha_1=0$, or $\alpha_2=0$, or $\alpha_3=0$, or $\alpha_4=0$.

Theorem. *Let conditions (i)–(iv) be satisfied. Then the solution of the equation (1) is unique in* $L_2(G)$.

Proof. Taking into account (2) and (3) from (1) we have

$$\int_a^x A(t,x,y)u(t,y)\mathrm{d}y+\int_x^b B(t,x,y)u(t,y)\mathrm{d}y+\int_{t_0}^t M(t,x,s)u(s,x)\mathrm{d}x+$$
$$+\int_t^T N(t,x,s)u(s,x)\mathrm{d}s+\int_{t_0}^t\int_a^x C(t,x,s,y)u(s,y)\mathrm{d}y\mathrm{d}s=f(t,x),\ (t,x)\in G.\quad(5)$$

Taking the multiplication of both sides of the equation (5) with $u(t,s)$, integrating the results on G, we obtain

$$\int_a^b\int_{t_0}^T\int_a^y A(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}s\mathrm{d}y+\int_a^b\int_{t_0}^T\int_y^b B(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}s\mathrm{d}y+$$
$$+\int_a^b\int_{t_0}^T\int_{t_0}^s M(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y+\int_a^b\int_{t_0}^T\int_s^T N(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y+$$
$$+\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y=\int_a^b\int_{t_0}^T f(s,y)u(s,y)\mathrm{d}s\mathrm{d}y.\quad(6)$$

Using the Dirichlet formula and taking into account (4) from (6), we have

$$\int_{t_0}^T\int_a^b\int_a^y P(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}y\mathrm{d}s+\int_a^b\int_{t_0}^T\int_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y+$$

$$+\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s\int\limits_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y=\int\limits_a^b\int\limits_{t_0}^T f(s,y)u(s,y)\mathrm{d}s\mathrm{d}y. \quad (7)$$

Integrating by parts and using the Dirichlet formula we obtain

$$\int\limits_{t_0}^T\int\limits_a^b\int\limits_a^y P(s,y,z)u(s,z)u(s,y)\mathrm{d}z\mathrm{d}y\mathrm{d}s=$$

$$=-\int\limits_{t_0}^T\int\limits_a^b\int\limits_a^y P(s,y,z)\frac{\partial}{\partial z}\left(\int\limits_z^y u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}z u(s,y)\mathrm{d}y\mathrm{d}s=$$

$$=\frac{1}{2}\int\limits_{t_0}^T\int\limits_a^b P(s,y,a)\left[\frac{\partial}{\partial y}\left(\int\limits_a^y u(s,\nu)\mathrm{d}\nu\right)^2\right]\mathrm{d}y\mathrm{d}s+$$

$$+\frac{1}{2}\int\limits_{t_0}^t\int\limits_a^b\int\limits_z^b P'_z(s,y,z)\left[\frac{\partial}{\partial y}\left(\int\limits_z^y u(s,\nu)\mathrm{d}\nu\right)^2\right]\mathrm{d}y\mathrm{d}z\mathrm{d}s=$$

$$=\frac{1}{2}\int\limits_{t_0}^T P(s,b,a)\left(\int\limits_a^b u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}s-\frac{1}{2}\int\limits_{t_0}^T\int\limits_a^b P'_y(s,y,a)\left(\int\limits_a^y u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}s+$$

$$+\frac{1}{2}\int\limits_{t_0}^T\int\limits_a^b P'_y(s,b,y)\left(\int\limits_y^b u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}y\mathrm{d}s-$$

$$-\frac{1}{2}\int\limits_{t_0}^T\int\limits_a^b\int\limits_a^y P''_{zy}(s,y,z)\left(\int\limits_z^y u(s,\nu)\mathrm{d}\nu\right)^2\mathrm{d}z\mathrm{d}y\mathrm{d}s, \quad (8)$$

where $P'_y(s,b,y)=[P'_z(s,b,z)]|_{z=y}$. Similarly integrating by parts and using the Dirichlet formula we have

$$\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s H(s,y,\tau)u(\tau,y)u(s,y)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y=-\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s H(s,y,\tau)\frac{\partial}{\partial\tau}\left(\int\limits_\tau^s u(\xi,y)\mathrm{d}\xi\right)\times$$

$$\times\mathrm{d}\tau u(s,y)\mathrm{d}s\mathrm{d}y=\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T H(s,y,t_0)\left[\frac{\partial}{\partial s}\left(\int\limits_{t_0}^s u(\xi,y)\mathrm{d}\xi\right)^2\right]\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T\int\limits_\tau^T H'_\tau(s,y,\tau)\left[\frac{\partial}{\partial s}\left(\int\limits_\tau^s u(\xi,y)\mathrm{d}\xi\right)^2\right]\mathrm{d}s\mathrm{d}\tau\mathrm{d}y=$$

$$=\frac{1}{2}\int\limits_a^b H(T,y,t_0)\left(\int\limits_{t_0}^T u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}y-\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T H'_s(s,y,t_0)\left(\int\limits_{t_0}^s u(\xi,y)\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T H'_s(T,y,s)\left(\int_s^T u(\xi,y)\mathrm{d}\xi\right)^2 \mathrm{d}s\mathrm{d}y-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s H''_{\tau s}(s,y,\tau)\left(\int_\tau^s u(\xi,y)\mathrm{d}\xi\right)^2 \mathrm{d}\tau\mathrm{d}s\mathrm{d}y, \quad (9)$$

where $H'_s(T,y,s)=(H'_\tau(T,y,\tau))|_{\tau=s}$.

Further we use the following formula

$$C\nu''_{\tau z}=(C\nu)''_{\tau z}-(C'_\tau\nu)'_z-(C'_z\nu)'_\tau+C''_{\tau z}\nu.$$

Then integrating by parts and using the Dirichlet formula, we have

$$\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}y\mathrm{d}s=$$

$$=\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)\frac{\partial^2}{\partial\tau\partial z}\left(\int_\tau^s\int_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)\mathrm{d}z\mathrm{d}\tau u(s,y)\mathrm{d}s\mathrm{d}y=$$

$$=\int_a^b\int_{t_0}^T C(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)u(s,y)\mathrm{d}s\mathrm{d}y+$$

$$+\int_a^b\int_{t_0}^T\int_\tau^T C'_\tau(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)u(s,y)\mathrm{d}s\mathrm{d}\tau\mathrm{d}y+$$

$$+\int_a^b\int_{t_0}^T\int_z^b C'_z(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)u(s,y)\mathrm{d}y\mathrm{d}s\mathrm{d}z+$$

$$+\int_a^b\int_{t_0}^T\int_\tau^T\int_z^b C''_{\tau z}(s,y,\tau,z)\left(\int_\tau^s\int_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)u(s,y)\mathrm{d}y\mathrm{d}s\mathrm{d}\tau\mathrm{d}z. \quad (10)$$

Further we apply the following formula

$$C\nu\nu''_{sy}=\frac{1}{2}(C\nu^2)''_{sy}-\frac{1}{2}(C'_s\nu^2)'_y-\frac{1}{2}(C'_y\nu^2)'_s+\frac{1}{2}(C''_{sy}\nu^2-C\nu'_y\nu'_s$$

and using the Dirichlet formula we have

$$\int_a^b\int_{t_0}^T\int_{t_0}^s\int_a^y C(s,y,\tau,z)u(\tau,z)u(s,y)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y=$$

$$=\frac{1}{2}C(T,b,t_0,a)\left(\int_a^b\int_{t_0}^T u(\xi,\nu)\mathrm{d}\xi\mathrm{d}\nu\right)^2-\frac{1}{2}\int_{t_0}^T C'_s(s,b,t_0,a)\left(\int_a^b\int_{t_0}^s u(\nu,\xi)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}s-$$

$$-\frac{1}{2}\int_a^b C'_y(T,y,t_0,a)\left(\int_a^y\int_{t_0}^T u(\nu,\xi)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}y+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T C''_{sy}(s,y,t_0,a)\left(\int_{t_0}^s\int_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y-$$

$$-\int_a^b\int_{t_0}^T C(s,y,t_0,a)\left(\int_{t_0}^s u(\xi,y)\mathrm{d}\xi\right)\left(\int_a^y u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}y\mathrm{d}s+$$

$$+\frac{1}{2}\int_{t_0}^T C'_s(T,b,s,a)\left(\int_s^T\int_a^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}s-$$

$$-\frac{1}{2}\int_{t_0}^T\int_{t_0}^s C''_{\tau s}(s,b,\tau,a)\left(\int_\tau^s\int_a^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T C''_{sy}(T,y,s,t_0)\left(\int_s^T\int_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_{t_0}^s C'''_{\tau sy}(s,y,\tau,a)\left(\int_\tau^s\int_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s\mathrm{d}y-$$

$$-\int_a^b\int_{t_0}^T\int_{t_0}^s C'_\tau(s,y,\tau,a)\left(\int_\tau^s u(\xi,y)\mathrm{d}\xi\right)\left(\int_a^y u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}\tau\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int_a^b C'_y(T,b,t_0,y)\left(\int_{t_0}^T\int_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}y-$$

$$-\frac{1}{2}\int_a^b\int_{t_0}^T C''_{ys}(s,b,t_0,y)\left(\int_{t_0}^s\int_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}s\mathrm{d}y-$$

$$-\frac{1}{2}\int_a^b\int_a^y C''_{zy}(T,y,t_0,z)\left(\int_{t_0}^T\int_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}z\mathrm{d}y+$$

$$+\frac{1}{2}\int_a^b\int_{t_0}^T\int_a^y C'''_{zsy}(s,y,t_0,z)\left(\int_{t_0}^s\int_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}y\mathrm{d}s\mathrm{d}z-$$

$$-\int_a^b\int_{t_0}^T\int_a^y C'_z(s,y,t_0,z)\left(\int_{t_0}^s u(\xi,y)\mathrm{d}\xi\right)\left(\int_z^y u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}z\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T C''_{\tau z}(T,b,\tau,z)\left(\int\limits_\tau^T\int\limits_z^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}z-$$

$$-\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T\int\limits_a^y C'''_{szy}(T,y,s,z)\left(\int\limits_s^T\int\limits_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}z\mathrm{d}s\mathrm{d}y-$$

$$-\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s C'''_{\tau ys}(s,b,\tau,y)\left(\int\limits_\tau^s\int\limits_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s\mathrm{d}y+$$

$$+\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s\int\limits_a^y C^{(IV)}_{\tau zsy}(s,y,\tau,z)\left(\int\limits_\tau^s\int\limits_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y-$$

$$-\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s\int\limits_a^y C''_{\tau z}(s,y,\tau,z)\left(\int\limits_\tau^s u(\xi,y)\mathrm{d}\xi\right)\left(\int\limits_z^y u(s,\nu)\mathrm{d}\nu\right)\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y,\quad (11)$$

where $C'_s(T,b,s,a)=(C'_\tau(T,b,\tau,a))|_{\tau=s}, C'_y(T,b,t_0,y)=(C'_z(T,b,t_0,z))|_{z=y}$.

Taking into account (8), (9) and (10) from (7) we obtain

$$\frac{1}{2}C(T,b,t_0,a)\left(\int\limits_a^b\int\limits_{t_0}^T u(\xi,\nu)\mathrm{d}\xi\mathrm{d}\nu\right)^2+\frac{1}{2}\int\limits_{t_0}^T\left\{P(s,b,a)\left(\int\limits_a^b u(s,\nu)\mathrm{d}\nu\right)^2-\right.$$

$$\left.-C'_s(s,b,t_0,a)\left(\int\limits_a^b\int\limits_{t_0}^s u(\nu,\xi)\mathrm{d}\nu\mathrm{d}\xi\right)^2+C'_s(T,b,s,a)\left(\int\limits_s^T\int\limits_a^b u(\xi,\nu)\mathrm{d}\xi\mathrm{d}\nu\right)^2\right\}\mathrm{d}s+$$

$$+\frac{1}{2}\int\limits_a^b\left\{H(T,y,t_0)\left(\int\limits_{t_0}^T u(\xi,y)\mathrm{d}\xi\right)^2-C'_y(T,y,t_0,a)\left(\int\limits_a^y\int\limits_{t_0}^T u(\nu,\xi)\mathrm{d}\nu\mathrm{d}\xi\right)^2+\right.$$

$$\left.+C'_y(T,b,t_0,y)\left(\int\limits_{t_0}^T\int\limits_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\right\}\mathrm{d}y+$$

$$+\frac{1}{2}\int\limits_a^b\int\limits_{t_0}^T\int\limits_{t_0}^s\int\limits_a^y\left\{L(s,y,\tau,z,\int\limits_a^y u(s,\nu)\mathrm{d}\nu,\int\limits_{t_0}^s u(\xi,y)\mathrm{d}\xi,\int\limits_\tau^s u(\xi,y)\mathrm{d}\xi,\int\limits_z^y u(s,\nu)\mathrm{d}\nu)+\right.$$

$$+\frac{1}{(s-t_0)(y-a)}\left[P'_y(s,b,y)\left(\int\limits_y^b u(s,\nu)\mathrm{d}\nu\right)^2+H'_s(T,y,s)\left(\int\limits_s^T u(\xi,y)\mathrm{d}\xi\right)^2\right]+$$

$$+\frac{1}{(s-t_0)(y-a)}\left[C''_{sy}(s,y,t_0,a)\left(\int\limits_{t_0}^s\int\limits_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2-\right.$$

$$- C''_{sy}(T,y,s,a)\left(\int\limits_s^T\int\limits_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2 - C''_{ys}(s,b,t_0,y)\left(\int\limits_{t_0}^s\int\limits_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2 +$$

$$+C''_{sy}(T,b,s,y)\left(\int\limits_s^T\int\limits_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\Bigg] + \frac{1}{y-a}\Bigg[C'''_{\tau sy}(s,y,\tau,a)\left(\int\limits_\tau^s\int\limits_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2 -$$

$$-C'''_{\tau ys}(s,b,\tau,y)\left(\int\limits_\tau^s\int\limits_y^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\Bigg] + \frac{1}{s-t_0}\Bigg[C'''_{zsy}(s,y,t_0,z)\left(\int\limits_{t_0}^s\int\limits_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2 -$$

$$- C'''_{szy}(T,y,s,z)\left(\int\limits_s^T\int\limits_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\Bigg] +$$

$$+ C^{(IV)}_{\tau syz}(s,y,\tau,z)\left(\int\limits_\tau^s\int\limits_z^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\Bigg\}\mathrm{d}z\mathrm{d}\tau\mathrm{d}s\mathrm{d}y-$$

$$-\frac{1}{2}\int\limits_{t_0}^T\int\limits_{t_0}^s C''_{\tau s}(s,b,\tau,a)\left(\int\limits_\tau^s\int\limits_a^b u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}\tau\mathrm{d}s - \frac{1}{2}\int\limits_a^b\int\limits_a^y C''_{zy}(T,y,t_0,z)\times$$

$$\times\left(\int\limits_{t_0}^T\int\limits_a^y u(\xi,\nu)\mathrm{d}\nu\mathrm{d}\xi\right)^2\mathrm{d}z\mathrm{d}y = \int\limits_a^b\int\limits_{t_0}^T f(s,y)u(s,y)\mathrm{d}s\mathrm{d}y. \quad (12)$$

Let $f(t,x)=0$ for all $(t,x)\in G$. Then by virtue of conditions (i)-(iv), from (11) we have $\int_{t_0}^y u(s,\nu)\mathrm{d}\nu = 0$ for all $(s,y)\in G$, or $\int_a^s u(\xi,y)\mathrm{d}\xi = 0$ for all $(s,y)\in G$. Here $u(t,x)=0$ for all $(t,x)\in G$. The Theorem is proved. □

Example. We consider the equation (1) for $a=t_0=0, b=T=1$,

$$A(t,x,y)=\alpha(1+t)(1-x)(1+y),(t,x,y)\in G_1,$$

$$B(t,x,y)=\alpha(1+t)(1-y)x,(t,x,y)\in G_2,$$

$$M(t,x,s)=\alpha(1+x)(1-t)(1+s),(t,x,y)\in G_3,$$

$$N(t,x,s)=\alpha(1+x)(1-s)t,(t,x,y)\in G_4.$$

$$C(t,x,y,s)=\beta,(t,x,s,y)\in G_5.$$

where $\alpha,\beta\in R,\alpha>0,\alpha>|\beta|$.
In this case

$$P(t,x,y)=\alpha(1+t)(1-x)(1+2y),(t,x,y)\in G_1,$$

$$H(t,x,s)=\alpha(1+x)(1-t)(1+2s),(t,x,s)\in G_3,$$

$$L(s,y,\tau,z,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\frac{1}{sy}\left\{[\alpha(1+t)\alpha_1^2+\alpha(1+x)\alpha_2^2-2\beta\alpha_1\alpha_2]+\right.$$

$$\left.+2\alpha s(1+x)\alpha_3^2+2\alpha y(1+t)\alpha_4^2\right\}.$$

Then the conditions (i)-(iv) be satisfied.

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Единственность решений для одного класса интегральных уравнений первого рода с двумя независимыми переменными

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Данная статья посвящена исследованию единственности решений линейных интегральных уравнений первого рода с двумя независимыми переменными в которых оператор, порожденный ядрами, не является компактным оператором.

Актуальность проблемы обусловлена потребностями в разработке новых подходов для регуляризации и единственности решения линейных интегральных уравнений первого рода с двумя независимыми переменными. В качестве приближённых решений таких задач, устойчивых к малым изменениям исходных данных, используются решения, получаемые методом регуляризации, которые принадлежат к классу некорректно поставленных задач. Один из классов таких некорректных задач составляют интегральные уравнения первого рода с двумя независимыми переменными.

Целью работы является доказательства теорем единственности для решения линейных интегральных уравнений первого рода с двумя независимыми переменными и доказательство теорем единственности.

В работе доказана теорема единственности решения интегральных уравнений первого рода с двумя независимыми переменными. Для получения сформулированных автором задач использованы методы функционального анализа и метод неотрицательных квадратичных форм. Полученные результаты являются новыми. Достоверность результата установлена доказательствами и иллюстрируется примерами.

Работа носит теоретический характер. Полученные теоретические результаты могут быть применены в различных областях науки и техники.

Ключевые слова: линейный, интегральные уравнения, первого рода, двух переменных, решение, единственность.