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Solving the Hysteresis Loop Calculation Problem for Josephson Junction Stacks

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A detailed investigation of the IVC breakpoint and the breakpoint region width gives important information concerning the peculiarities of stacks with a finite number of intrinsic Josephson junctions. The current-voltage characteristics for a stack of n Josephson junctions is defined from solving the system of n nonlinear differential equations. The current voltage characteristic has the shape of a hysteresis loop. On the back branch of the Hysteresis loop, near the breakpoint I_b, voltage $V(I)$ decreases to zero rapidly. The goal of this work is to accelerate the computation of IVC based on numerical solution of the system. A numerical-analytical method was proposed in. This method showed perfect results in IVC calculations for a stack of 9 and 19 intrinsic Josephson junctions and the computation time reduced by five times approximately. The question of choosing a change-over point from "analytical" to numerical calculation was open. In testing computations the change-over point was taken equal to $2I_b$. In the case of periodic boundary conditions an equation, determining the approximate location of I_b, was obtained. This moment we succeeded to develop an algorithm determining the approximate value I_b in more complicated technically case of non-periodic boundary conditions with $\gamma = 1$. All calculations were performed using the REDUCE 3.8 system.

Key words and phrases: stack of Josephson junctions, computation of current-voltage characteristics, hysteresis loop, Cauchy problem for a system of nonlinear differential equations, fourth-order Runge-Kutta method, long-time asymptotic formulas, a numerical-analytical method, computation of formulas using the REDUCE 3.8 system.

1. Introduction

Solving the system

$$\varphi_l = \sum_{l'=1}^{n} A_{l,l'}(I - \sin(\varphi_{l'}) - \beta\dot{\varphi}_{l'}), \quad l = 1, \dots, n, \tag{1}$$

for different $I : I = I_0 + k\Delta I \leqslant I_{\max}$; $I = I_{\max} - k\Delta I$, the current-voltage characteristics of stacks as hysteresis loops are found [1]. For initial value of the current ($I = I_0$) the system (1) is solved with zero initial data on an interval $[0, T_{\max}]$. For each next $I : I = I_{k+1}$, found already $\varphi_l(I_k, T_{\max}), \dot{\varphi}_l(I_k, T_{\max})$ are used as initial data. On the back branch of the Hysteresis loop, near the breakpoint I_b, voltage $V(I)$ decreases to zero rapidly. The goal of this search is to accelerate the computation of IVC.

In the case of periodic boundary conditions the A matrix is

$$\begin{pmatrix} 1+2\alpha & -\alpha & 0 & \dots & 0 & -\alpha \\ -\alpha & 1+2\alpha & -\alpha & 0 & \dots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -\alpha & 1+2\alpha & -\alpha \\ -\alpha & 0 & \dots & 0 & -\alpha & 1+2\alpha \end{pmatrix}, \tag{2}$$

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square matrix of order n. And in the case of nonperiodic boundary conditions the A matrix is

$$\begin{pmatrix} 1+\alpha(1+\gamma) & -\alpha & 0 & \dots & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & \dots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & 0 & \dots & 0 & -\alpha & 1+\alpha(1+\gamma) \end{pmatrix}, \tag{3}$$

where $\gamma = s/s_0 = s/s_n$ and s, s_0, s_n are sickness of middle, first and last superconducting layers respectively [1]. The parameter α gives the coupling between junctions, β is the dissipation parameter. The dynamics of phase differences $\varphi_l(t)$ had been simulated by solving the equation system (2) using the fourth order Runge-Kutta method [2]. After simulation of the phase differences dynamics the voltages on each junction were calculated as

$$\partial\varphi_l/\partial t = \sum_{l'=1}^{n} A_{l,l'} V_{l'}. \tag{4}$$

The average of the voltage $\bar{V}_l$ is given by

$$\bar{V}_l = \frac{1}{T_{\max} - T_{\min}} \int_{T_{\min}}^{T_{\max}} V_l \partial t. \tag{5}$$

Finally the total voltage V of the stack is obtained by summing these averages:

$$V = \sum_{l=1}^{n} \bar{V}_l. \tag{6}$$

The calculation can be simplified using specific properties of the matrices (2), (3). These matrices are symmetric. They have complete systems of orthonormal eigenvectors E_l with real eigenvalues λ_l . The fundamental matrices D (whose columns are E_l) reduce the A-matrices to the diagonal form: $D'AD = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

After changing the variables

$$\varphi_l = \sum_{l'=1}^{n} d_{l,l'} \psi_{l'}, \quad V_l = \sum_{l'=1}^{n} d_{l,l'} W_{l'}$$

we get a system:

$$\ddot{\psi}_l = -\lambda_l \beta \dot{\psi}_l + \lambda_l \cdot I \cdot S_l - \lambda_l \sum_{l'=1}^{n} d_{l',l} \sin(\varphi_{l'}),$$

where $l = 1, \dots, n$ and S_l is the sum of E_l elements: $S_l = d_{1,l} + d_{2,l} + \dots + d_{n,l}$.

Relations (4), (5), (6) result in

$$\frac{\partial\psi_l}{\partial t} = \lambda_l W_l, \quad \bar{W}_l = \frac{\psi_l(T_{\max}) - \psi_l(T_{\min})}{\lambda_l (T_{\max} - T_{\min})}, \quad V = \sum_{l=1}^{n} S_l \cdot \bar{W}_l \tag{7}$$

respectively.

2. Periodic Boundary Conditions

In the case of periodic boundary conditions, the eigenvalue problem of A has the solution:

$$\lambda_l = 1 + 2\alpha(1 - \cos(\varphi_l)), \quad \varphi_l = \frac{2\pi(l-1)}{n}, \quad l = 1, \dots, ns, ;$$

where ns is integer part of $(n+1)/2$, $ns = [(n+1)/2]$,

$$E_1 = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad E_{2(l-1)} = \sqrt{\frac{2}{n}} \begin{bmatrix} \sin(\varphi_l) \\ \sin(2\varphi_l) \\ \vdots \\ \sin(n\varphi_l) \end{bmatrix}, \quad E_{2l-1} = \sqrt{\frac{2}{n}} \begin{bmatrix} \cos(\varphi_l) \\ \cos(2\varphi_l) \\ \vdots \\ \cos(n\varphi_l) \end{bmatrix},$$

$$l = 2, \dots, ns.$$

When n is even, $n = 2 \cdot ns$, matrix A has additional eigenvalue $\lambda_{ns+1} = 1 + 4\alpha$, the corresponding $\varphi_{ns+1} = \pi$ and corresponding eigenvector is

$$E_n = \frac{1}{\sqrt{n}}[-1, 1, -1, 1, \dots, -1, 1]^*.$$

So every of λ_l, $2 \leqslant l \leqslant ns$, has a pair of eigenvectors. Let me remind that S_l is a sum of E_l elements. This time we have $S_1 = \sqrt{n}$, $S_l = 0, l = 2, \dots, n$.

As a result, the hysteresis calculation problem is reduced [3] to solving the unique equation

$$\eta(t) = \xi_1 + \frac{(\xi_2 - \omega)}{\beta}(1 - \mathrm{e}^{-\beta t}) + \omega t - \frac{1}{\beta}\int\limits_0^t (1 - \mathrm{e}^{-\beta(t-s)}) \sin(\eta(s)) \mathrm{d}s. \tag{8}$$

Solving this equation we find $\psi_1(t) = \sqrt{n}\eta(t)$. The rest components $\psi_j(t)$, $j = 2, \dots, n$ are equal zeros. The equation (8) is solved by the simple iterations method. Starting from $\eta_0 = 0$. we obtained at the third iteration step

$$\eta_3(t) = \left(\omega - \frac{\cos(\vartheta)}{2\omega(\omega^2+\beta^2)} + \frac{\sin(\vartheta)}{2\beta(\omega^2+\beta^2)}\right) t + A - \frac{\cos(A+\vartheta)}{\beta\omega} - \\ - \frac{\cos(2A+\vartheta)}{4\beta\omega^3} + \frac{\sin(\omega t + A + \vartheta + \mathrm{atg})}{\omega\sqrt{\beta^2+\omega^2}} + O(\omega^{-4} + \mathrm{e}^{-\beta t}). \tag{9}$$

Here $\omega = I/\beta$, $A = \xi_1 + (\xi_2 - \omega)/\beta$, $\mathrm{atg} = \mathrm{arctg}(\beta/\omega)$, $\vartheta = -\cos(A)/(\omega\beta)$.

Remark that $V(I, n) = \sqrt{n}\bar{W}_1(I)$ (see (7)) and

$$\bar{W}_1(I) = \sqrt{n}(\eta(I, T_{\max}) - \eta(I, T_{\min}))/(T_{\max} - T_{\min}).$$

The approximate break point location ω can be found [4] from (9) as a solution of the equation $F(\omega) = 0$, where

$$F(\omega) = \omega + \frac{\sin(\vartheta - \mathrm{atg})}{2\beta\omega\sqrt{\omega^2+\beta^2}} + \\ + \frac{2\sin(\omega(T_{\max} - T_{\min})/2)\cos(\omega(T_{\max} + T_{\min})/2 + A + \vartheta + \mathrm{atg})}{\omega\sqrt{\beta^2+\omega^2}(T_{\max} - T_{\min})}.$$

Here $A = -\omega/\beta$, according to $\xi_1 = \xi_2 = 0$. The polynomial $P(x) = 4\beta^2\omega^4(\omega^2+\beta^2)-1$ has the unique positive root $xt = 1.35232$. We find that $F(xt) = 1.447\ldots$ and $F(1) = -1.434\ldots$. After this no difficulty is to calculate the approximate break point location using the interval bisection method, $\tilde{I}_b = 0.210248\ldots$. Roughly spiking the jump to numerical calculations must be done at $2\tilde{I}_b$. In our calculations we put $T_{\min} = 50$, $T_{\max} = 1000$, $\Delta I = 0.05$. The step in the Runge-Kutta method was $h = 0.1$. All calculations were performed by using the REDUCE 3.8 system [5] .

3. Nonperiodic Boundary Conditions

In the case of nonperiodic with $\gamma = 1$ boundary conditions, the A matrix has following eigenvalues and eigenvectors: $\lambda_j = 1+2\alpha(1-\cos(j\vartheta)), \vartheta = \pi/(n+1), \quad j = 1,\ldots,n$;

$$E_j = cn[\sin(j\vartheta), \sin(2j\vartheta), \ldots, \sin(nj\vartheta)]^T,\ j = 1,\ldots,n, \quad cn = \sqrt{2/(n+1)}.$$

For even j, $\quad j = 2,4,\ldots,2k \leqslant n$, $S_j = 0$. And for odd j

$$S_j = cn\,\mathrm{ctg}(j\vartheta/2), \quad j = 1,3,\ldots,2k-1 \leqslant n.$$

We proved [3] that in the case of non-periodic boundary conditions with $\gamma = 1$ the problem of Hysteresis loop calculation reduces to solving the following system of ns integral equations:

$$\psi_{2l-1} = \omega_{2l-1}t + \xi_1(2l-1) + \frac{\xi_2(2l-1)-\omega_{2l-1}}{\beta\lambda_{2l-1}}\left(1-\exp(-\beta\lambda_{2l-1}t)\right)$$

$$-\frac{1}{\beta}\int\limits_0^t \left(1-\exp\left(-\beta\lambda_{2l-1}(t-s)\right)\right)\sum_{m=1}^{n} d_{m,2l-1}\sin\left(\sum_{k=1}^{ns} d_{m,2k-1}\psi_{2k-1}\right)\mathrm{d}s, \qquad (10)$$

where $l = 1,2,\ldots,ns$, ns is integer part of $(n+1)/2$ and $\omega_{2l-1} = S_{2l-1}I/\beta$.

For each I and given initial data $\xi_1(2l-1)$, $\xi_2(2l-1)$ the system (10) was solved using simple iterations starting at zero. The results obtained after three iterations are regarded as "asymptotics" of the solution for large t. These "asymptotics" were used in [3].

In result of a number of useless attempts to find approximate breakpoint location by analogy with [4], we concluded that this can be done as follows. It is sufficient to calculate $V(I)$ for different $I, I = 0.5 - 0.05\cdot j$, solving the system (10) with zero initial data "analytically", until I_0, satisfying $V(I_0)\cdot V(I_0+0.05) < 0$ was obtained. Found I_0 is taken for the approximate breakpoint location. In the case of the non-periodic boundary conditions we succeeded to realize this algorithm only after refusing a number of algebraic manipulations "eating" the time and leading to computing errors accumulation as well. The calculating time decreased more than 9 times (against 5 declared in [3]). Below we present result of calculating $\tilde{I}_b = 0.3$ for the stack 0f 19 Josephson junctions.

```
vb :={{0.5,33.7167375273},{0.45,40.235725455},{0.4,63.5053115073},
{0.35,62.9218862448}, {0.3, - 0.0454510614144}}
```

In Fig. 1 the pictures of the back way of the hysteresis loop are shown. The solid and dotted lines refer to numerical and "analytical" calculations respectively. In Fig. 2 the solid line is the same as in Fig. 1, while the circles on this line refer to calculation performed by the following mixed numerical-analytical method. The right way of the hysteresis loop and the back way on the interval $1.45 > I > 0.45 = 1.5\cdot\tilde{I}_b$ are computed using the "asymptotic" formulas. The rest points of the hysteresis loop are computed numerically. The calculations were performed for $\alpha = 0.2, \beta = 0.2$ using the REDUCE 3.8 system [5] with $T_{\min} = 50, T_{\max} = 1000$ and the step $h = 0.1$ was chosen in the numerical calculations.

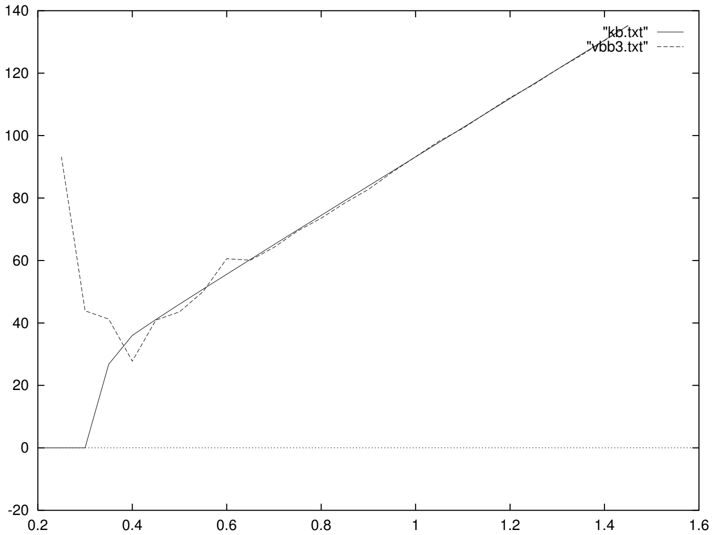

Figure 1. The solid line refers to the back branch of the hysteresis loop for $n = 19$, calculated numerically using the fourth order Runge-Kutta method. The dotted line refers to the back branch of the hysteresis loop, calculated "analytically" using the "asymptotic" formulas

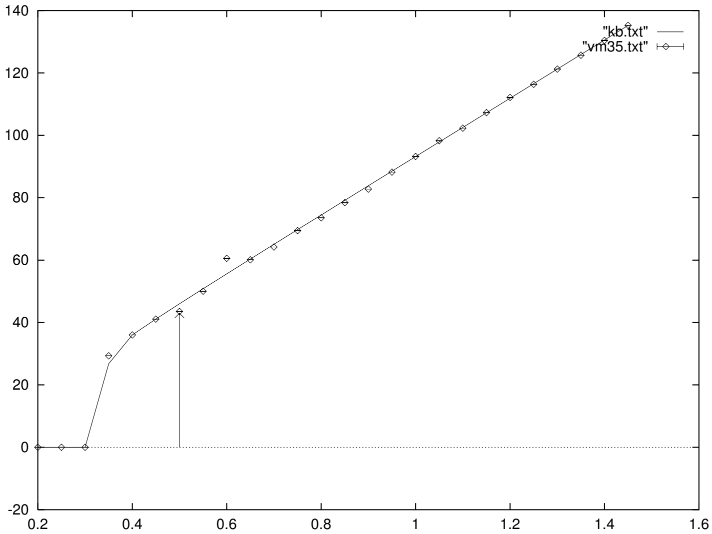

Figure 2. The solid line refers to the back branch of the hysteresis loop calculated numerically for $n = 19$. The circles on this line refer to calculation performed by the mixed analytical- numerical method: the whole right branch of the hysteresis loop, together with the back branch at $1.45 > I > 0.45 = 1.5 \cdot \tilde{I}_b$, have been computed using the "asymptotic" formulas. The points at $0.45 \geqslant I \geqslant 0.2$ were computed numerically. The point $(0.5, 43.561\ldots)$, marked in Fig. 2 by the arrow, is the last point of the hysteresis loop calculated "analytically"

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Решение проблемы вычисления петли гистерезиса для систем джозефсоновских переходов

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Детальное исследование критических точек ВАХ и оценка области их влияния представляют большой интерес для изучения свойств систем с конечным числом внутренних джозефсоновских переходов. Вольт-амперная характеристика для системы n внутренних джозефсоновских переходов определялась по решению системы n нелинейных дифференциальных уравнений. Вольт-амперная характеристика (сокращённо ВАХ) имеет вид петли гистерезиса. На обратной ветви петли гистерезиса, при подходе к точке излома I_b, напряжение $V(I)$ резко спадает к нулю. Цель этой работы — ускорить процесс вычисления ВАХ, основанный на численном решении системы. Был предложен смешанный численно-аналитический алгоритм. Этот метод показал прекрасные результаты при вычислении ВАХ для систем 9 и 19 внутренних джозефсоновских переходов. При этом время счета по смешанному методу сократилось приблизительно в пять раз. Оставался открытым вопрос выбора точки перехода от «аналитического» счета к численному. При тестовых расчётах точка перехода принималась равной $2I_b$. В случае периодических граничных условий было получено уравнение, определяющее приближенное значение I_b. В настоящий момент удалось разработать алгоритм, определяющий приближенное значение I_b в более сложном технически случае непериодических граничных условий. Все вычисления производились с использованием системы REDUCE 3.8.

Ключевые слова: система джозефсоновских переходов, вычисление вольт-амперных характеристик, петля гистерезиса, задача Коши для систем нелинейных дифференциальных уравнений, метод Рунге–Кутта четвёртого порядка точности, асимптотические формулы решения задачи Коши при больших t, численно-аналитический метод вычисления ВАХ, вывод асимптотических формул, используя систему REDUCE 3.8.