UDC 530.12:531.51

A Static Generalization of the Schwarzschild Solution, that Gives not Asymptotically Dipole Term

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In this paper we study the static axisymmetric solutions of the vacuum Einstein equations. Among static axisymmetric vacuum solutions of the most interest are the asymptotically flat solutions reducing to the Schwarzschild solution.
The purpose of this paper is to obtain a static solution which turned out to be appropriate for describing the gravitational field around an axisymmetric mass distribution.
In this paper the method of singular sources os considered and some new applications are presented. By mean of the method of singular sources it is possible to construct gravitational multipoles which generalize the Schwarzschild solution.
The linearity of the gravistatic equations makes it possible to solve the problem of superposition of two or several known solutions. The obtained static vacuum axisymmetric generalization of the Schwarzschild solution near two points of horizon has coordinate singularities.
In the obtained solution the dipole term is absent, and we have found the corresponding condition.
If one considers axially symmetric solutions of gravistatics, then construction of gravitational multipoles becomes ambiguous. It means that different solutions can give asymptotically the same Newtonian limit.

Key words and phrases: gravistatic, Schwarzschild solution, Einstein equation, Weyl metric, asymptotically flat.

1. Introduction

The vacuum static Einstein equations for the case of spherical symmetry were considered in 1916 by Schwarzschild [1] who obtained a solution which turned out to be appropriate for describing the gravitational field around a spherically symmetric mass distribution. Further research in the field of exact solutions of the general relativity equations was appreciably influenced by the work of Weyl [1]. Weyl's static axially symmetric vacuum equations formed a basis for obtaining new exact solutions by Chazy and Curzon, Erez and Rozen, Gutsunaev and Manko [1]. By means of the method of singular sources it is possible to construct gravitational multipoles which generalize the Schwarzchild solution.

2. Basic Equations

For axially symmetric vacuum static gravitational fields, the line element reduces to the Weyl metric [1]

$$\mathrm{d}s^2 = \frac{1}{f}[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2] - f\mathrm{d}t^2, \tag{1}$$

where ρ, z, φ and t are Weyl's canonical coordinates and time. Here $f = f(\rho, z)$ and $\gamma = \gamma(\rho, z)$.

Received 25th November, 2013.

The Einstein equations for an axially symmetric gravitational field outside the sources read

$$f\Delta f = (\vec{\nabla} f)^2. \tag{2}$$

The operators Δ and $\vec{\nabla}$ are defined by the formulae

$$\Delta \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}, \quad \vec{\nabla} = \vec{\rho}_0 \frac{\partial}{\partial\rho} + \vec{z}_0 \frac{\partial}{\partial z},$$

($\vec{\rho}_0$ and $\vec{z}_0$ being unit vectors),

$$4\frac{\partial\gamma}{\partial\rho} = \rho f^{-2}\left[\left(\frac{\partial f}{\partial\rho}\right)^2 - \left(\frac{\partial f}{\partial z}\right)^2\right],$$

$$2\frac{\partial\gamma}{\partial z} = \rho f^{-2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z}. \tag{3}$$

With the substitution $f = e^{2\Psi}$, (2) becomes linear:

$$\Delta\Psi = \frac{\partial^2\Psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{\partial^2\Psi}{\partial z^2} = 0. \tag{4}$$

3. Method of Singular Sources

The right-hand side of (4) contains zero though actually there should be a certain singular function characterizing the distribution of sources.

Let $\sigma(\rho, z)$ denote the mass density of such sources, and let us rewrite (4) in the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) + \frac{\partial^2\Psi}{\partial z^2} = -\sigma(\rho, z). \tag{5}$$

This equation has the solution

$$\Psi(\rho, z) = \frac{1}{4\pi}\int\limits_{V'} \frac{\sigma(\rho', z')\,\mathrm{d}V'}{|\vec{r} - \vec{r}'|}. \tag{6}$$

The solution (6) in Weyl's coordinates has the form

$$\Psi(\rho, z) = \frac{1}{4\pi}\int\limits_{\rho'=0}^{\infty}\int\limits_{z'=0}^{\infty}\int\limits_{\varphi'=0}^{2\pi} \frac{\sigma(\rho', z')\rho'\,\mathrm{d}\rho'\,\mathrm{d}\varphi'\,\mathrm{d}z'}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi') + (z - z')^2}}. \tag{7}$$

Since the left-hand side of (7) does not depend on φ, we can set $\varphi = 0$ in the integral.

If we choose

$$\sigma(\rho', z') = \frac{\delta(\rho' - \rho_0)}{\rho'}\sigma(\rho, z'),$$

$$\Psi = \frac{1}{4\pi}\int\limits_{z'=-\infty}^{\infty}\int\limits_{\varphi'=0}^{2\pi} \frac{\sigma(\rho_0, z')\,\mathrm{d}\varphi'\,\mathrm{d}z'}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\varphi' + (z - z')^2}}. \tag{8}$$

With the full elliptic integral of the first kind $K(k)$ [2] we have

$$\Psi(\rho,z)=\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\mathrm{d}z'\frac{\sigma(\rho_0,z')}{\sqrt{(\rho+\rho_0)^2+(z-z')^2}}\cdot K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+(z-z')^2}}\right). \tag{9}$$

Let us examine some opportunities arising from the case $\sigma(\rho_0,z')=\sigma_0(z')\vartheta(z')$, where $\vartheta(z')$ is the step function

$$\vartheta(z')=\begin{cases}0, & m<z'<-m,\\ 1, & -m<z'<m.\end{cases} \tag{10}$$

The solution (9) under these assumptions transforms to

$$\Psi(\rho,z)=\frac{1}{\pi}\int\limits_{-m}^{m}\frac{\sigma_0(z')\,\mathrm{d}z'}{\sqrt{(\rho+\rho_0)^2+(z-z')^2}}\cdot K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+(z-z')^2}}\right). \tag{11}$$

4. Applications

Example 1. $\sigma(\rho_0,z')=-2m\delta(z')$, where δ is Dirac's δ function. Integration of (9) then leads to

$$\Psi=-\frac{2}{\pi}\frac{m}{\sqrt{(\rho+\rho_0)^2+z^2}}\cdot K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+z^2}}\right). \tag{12}$$

In case $\rho_0=0$ (12) leads to the Chazy–Curzon solution

$$\Psi(\rho,z)=-\frac{m}{\sqrt{\rho^2+z^2}}. \tag{13}$$

Example 2. Let $\sigma_0(z')=-1$. With this choice we arrive at the following solution [3]:

$$\Psi=-\frac{1}{\pi}\int\limits_{-m}^{m}\frac{\mathrm{d}z'}{\sqrt{(\rho+\rho_0)^2+(z-z')^2}}\cdot K\left(\sqrt{\frac{4\rho\rho_0}{(\rho+\rho_0)^2+(z-z')^2}}\right). \tag{14}$$

Integration of (14) leads to

$$\begin{aligned}\Psi(\rho,z)=\frac{1}{2}\ln\frac{z-m+\sqrt{(\rho+\rho_0)^2+(z-m)^2}}{z+m+\sqrt{(\rho+\rho_0)^2+(z+m)^2}}-\\ -\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{(2n+1)!!}{2(n+1)!!}\right]^2\left[\frac{4\rho\rho_0}{(\rho+\rho_0)^2}\right]^{n+1}\times\sum_{r=0}^{n}\frac{(-1)^{r+1}}{2r+1}\frac{n!}{(n-r)!r!}\times\\ \times\left[\left(\frac{z-m}{\sqrt{(\rho+\rho_0)^2+(z-m)^2}}\right)^{2r+1}-\left(\frac{z+m}{\sqrt{(\rho+\rho_0)^2+(z+m)^2}}\right)^{2r+1}\right].\end{aligned} \tag{15}$$

In case $\rho_0 = 0$, with the coordinate transformation

$$\frac{r}{m} - 1 = \frac{1}{2m}\left[\sqrt{\rho^2 + (z+m)^2} + \sqrt{\rho^2 + (z-m)^2}\right],$$
$$\cos\vartheta = \frac{1}{2m}\left[\sqrt{\rho^2 + (z+m)^2} - \sqrt{\rho^2 + (z-m)^2}\right],$$

we obtain the Schwarzschild solution

$$f = 1 - \frac{2m}{r}. \tag{16}$$

In the coordinates

$$\begin{cases} \rho = m\sqrt{(x^2-1)(1-y^2)}, \\ z = mxy, \end{cases}$$

the metric function f near points $x = 1$, $y = \pm 1$ ($\vartheta = 0, \pi$; $r = 2m$ — horizon) has coordinate singularities.

To examine the asymptotic behaviour of the potential f let us write it down in the coordinates (r, ϑ):

$$\begin{cases} x = \dfrac{r}{k_0} - 1, \\ y = \cos\vartheta, \end{cases}$$

Expanding f in inverse powers of r, we obtain the asymptotic of this solution:

$$f \approx 1 - \frac{2k_0}{r} - \frac{2k_0^2\rho_0\sin\vartheta}{mr^2}. \tag{17}$$

Example 3. Let us consider the solution

$$f = \left(1 + \frac{a_0}{r}\right)^{\delta} e^{2\psi_1 - 2\psi_2}, \tag{18}$$

$$\psi_{1,2} = -\frac{1}{\pi}\int\limits_{-m_{1,2}}^{m_{1,2}} \frac{dz'}{\sqrt{\left(\rho + \rho_0^{(1,2)}\right)^2 + (z-z')^2}} \cdot K\left(\sqrt{\frac{4\rho\rho_0^{(1,2)}}{\left(\rho + \rho_0^{(1,2)}\right)^2 + (z-z')^2}}\right).$$

In the coordinates

$$\begin{cases} \rho = k_0\sqrt{(x^2-1)(1-y^2)}, \\ z = k_0 xy, \end{cases}$$

the metric function f near points $x = 1$ (horizon), $y = \pm 1$ has coordinates singularities.

With the coordinate transformation $x = \frac{r}{k_0} - 1$, $y = \cos\vartheta$ the asymptotic behaviour of the metric coefficient f takes the form:

$$f \approx 1 - \frac{2M}{r} + 0\left(\frac{1}{r^3}\right). \tag{19}$$

Here $\delta = 1$, $k_0 = M$, $a_0 = -2M$, $\dfrac{\rho_0^{(1)}}{m_1} = \dfrac{\rho_0^{(2)}}{m_2}$.

5. Conclusion

What can be said about the physical interpretation of the solution (18), (19)? Apparently it can be considered as describing the generalization of Schwarzschild solution with monopole and quadrupole terms.

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УДК 530.12:531.51

Об одном обобщении решения Шварцшильда, не имеющего на асимптотике дипольного члена

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В статье мы изучаем статические аксиально-симметричные решения вакуумных уравнений Эйнштейна. Среди статических аксиально-симметричных вакуумных решений наибольший интерес вызывают асимптотически плоские решения, которые переходят в решение Шварцшильда.

Цель этой статьи — это получение статического решения, которое описывает гравитационное поле вокруг аксиального распределения масс.

В статье рассматривается метод сингулярных источников и некоторые его новые приложения. Методом сингулярных источников возможно построение гравитационных мультиполей, обобщающих решение Шварцшильда.

Линейность уравнений гравистатики позволяет строить суперпозицию из двух и более известных решений статических вакуумных аксиально-симметричных уравнений Эйнштейна. Полученное статическое вакуумное аксиально-симметричное обобщение решения Шварцшильда имеет две сингулярные точки на горизонте событий. В полученном решении отсутствует дипольный член, и мы нашли соответствующее условие в явном виде. Если рассматривать аксиально-симметричные вакуумные решения гравистатики, то построение гравитационных мультиполей является неоднозначной задачей. Это означает, что различные решения могут иметь на асимптотике одинаковый ньютоновский предел.

Ключевые слова: гравистатика, решение Шварцшильда, уравнения Эйнштейна, метрика Вейля, асимптотически плоский.