# Математическое моделирование

UDC 519.633.6, 519.688

## Numerical Algorithm for Simulation of Thermal Processes in Four Layer Cylindrical Object

**A. S. Ayriyan*, J. Pribiš†**

* *Laboratory of Information Technologies*
*Joint Institute for Nuclear Research*
*6, Joliot-Curie, Dubna, Moscow region, Russia, 141980*

† *Technical University in Košice*
*9, Letná, Košice, Slovak Republic, 04001*

In the paper the algorithm is proposed for the numerical simulation of the thermal conductivity for the design and optimization of cryogenic cells pulsed (in the millisecond range) feeding the working gases into the electron-stringed source of multiply charged ions. Heating process comes when the electric current passed through one of the layer. A model of the cryogenic cell with four layers (materials) is investigated. The heat transfer into the object is described by the system of heat equations with temperature dependent discontinuous thermal coefficients. The discontinuous thermal coefficients are given by experimental data and approximated by the least-squares method using the polynomial analytical functions. Conjugation condition between materials is considered to be ideal. The results are reported for a common configuration of the cell. The parallel algorithm for modeling thermal processes into four layer model was developed and speedup of the algorithm in depending on number of CPUs is shown.

**Key words and phrases:** heat transfer, explicit finite difference scheme, non-uniform grid, parallel algorithm, discontinuous thermal coefficients.

## 1. Introduction

The object shown in Fig. 1 is considered to be a cryogenic cell pulsed (millisecond range) feeding the working gases into the electron-string source of highly charged ions [1]. This model consists four layers (materials). Two-layer model have been investigated in [2, 3]. The heat conductivity modeling in the composite object, when the object is heated by the electric current passing through one of the layer (graphite) is complex, especially if one of the layers is from 50 up to 1200 times less then the other ones. The main goal of this work is implementation a model of thermal processes for a given object for systematic studies.

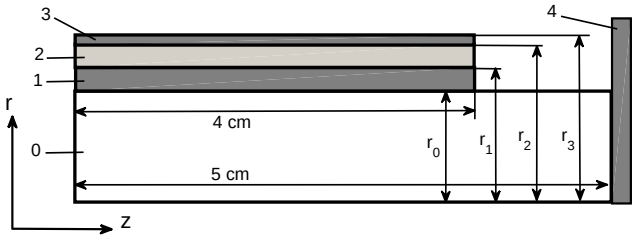


**Figure 1. Schematic view of the object slice. The slice of the object: 0 — cooler, 1 — electrical insulator, 2 — heat source (conductive layer), 3 — external insulator, 4 — liquid helium temperature terminal with** $T = 4.2$ K

Received 28th January, 2014.
Work partially supported by RFBR grant no. 14-01-31227 and JINR grant no. 14-602-01.

## 2. Equations and Boundary Conditions

The heat transfer into the object can be described by the following system of partial differential equations with temperature dependent discontinuous thermal coefficients [4]:

$$\rho_m C_{V\,m}(T)\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda_m(T)\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\lambda_m(T)\frac{\partial T}{\partial z}\right) + X_m(T), \qquad (1)$$

where $r \in [0, r_{\max}]$, $z \in [0, z_{\max}$ and $t \geqslant 0$.

Index $m$ is introduced for each material, $m = 0$ — cooler (copper), $m = 1$ — electrical insulator, $m = 2$ — heat source (graphite), $m = 3$ — external insulator. The radius $r_{\max} = r_3^* = 0.1301$ cm and space variable $z_{\max} = 5$ cm. The intermediate radii $r_0^* = 0.12$ cm, $r_1^* = 0.125$ cm, $r_2^* = 0.13$ cm and intermediate space variable $z_1 = 4$ cm. Source power $X(T)$ is nonzero only in the layer 2, for $r \in [r_1^*, r_2^*]$ and $z \in [0, z_1]$.

Coefficients $C_{V\,m}$ and $\lambda_m$ are specific heat capacity and thermal conductivity, respectively. For the chosen materials the corresponding data is obtained from the [5], it is measured in the temperature range from $T = 4.2$ K to $T = 60$ K. Their temperatures dependences were approximated by the least-squares method using the polynomial analytical functions [6]. Source dependence was approximated by function $X(T) = I^2/S\chi(T)$, where $\chi(T) = 1.8/\sqrt{T}$ and constant $I^2/S = 10^6$ [2].

The initial condition is given in the form

$$T(r, z, t = 0) = T_0, \qquad (2)$$

where $T_0 \equiv 4.2$ K (liquid helium temperature) and the boundary ones are taken as

$$\frac{\partial T(r = 0, 0 \leqslant z \leqslant z_{\max}, t)}{\partial r} = \frac{\partial T(r = r_0^*, z_1 \leqslant z \leqslant z_{\max}, t)}{\partial r} = \\ = \frac{\partial T(r = r_{\max}^*, 0 \leqslant z \leqslant z_1, t)}{\partial r} = 0, \qquad (3)$$

$$\frac{\partial T(0 \leqslant r \leqslant r_{\max}, z = 0, t)}{\partial z} = \frac{\partial T(r_0^* \leqslant r \leqslant r_{\max}, z = z_1, t)}{\partial z} = 0,$$

$$T(r, z = z_{\max}, t) = T_0. \qquad (4)$$

The functions $C_V$ and $\lambda$ have discontinuities of first kind at the following points: $r_0^*$, $r_1^*$ and $r_2^*$ for $z \in [0, z_1]$. Conjugation condition between materials is given:

$$\lambda_m(T)\left.\frac{\partial T}{\partial r}\right|_{r=r_m^*-0} = \lambda_{m+1}(T)\left.\frac{\partial T}{\partial r}\right|_{r=r_m^*+0}, \qquad (5)$$

where $m = 0, 1, 2$ and $r_m^*$ is a point of the border between the materials $m$ and $m+1$ (discontinuity points).

## 3. Numerical Algorithm

Numerical calculations are carried out on a non-uniform grid of variables $r$, $z$ and $t$ with constant spatial steps $h_m$ and time step $\tau$, respectively: $\omega_\tau = \{t_i = i \cdot \tau,\, i = 0, \ldots, n_\tau\}$, $\omega_z = \{z_i = i \cdot h_z,\, i = 0, \ldots, n_z\}$, $\omega_r = \{r_i, i = 0, \ldots, \sum_{m=0}^{3} n_m\}$, $\tau = t_{\max}/n_\tau$, $h_z = z_{\max}/n_z$, $h_0 = r_0/n_0$, $h_m = (r_m - r_{m-1})/n_m$ for $m \geqslant 1$, where $r \in [0, r_{\max}]$, $z \in [0, z_{\max}]$, $t \in [0, t_{\max}]$ and $n_1$ and $n_2$ are numbers of partitions of variable $r$ , $n_z$

and $n_\tau$ are numbers of partitions of variables $z$ nad $t$. The nonuniform step $h_m$ has been introduced to speed up the calculation time. Since the layers 3 much thinner than layers 1–2 and the layers 1–2 much thinner than layer 0, we can choose the steps $h_0 \gg h_1 = h_2 \gg h_3$.

For the numerical solution the initial-boundary-value problem (1)–(4) was approximated by the following explicit finite difference scheme [7]:

$$\rho {C_V}^k_{i,j} \frac{T^{k+1}_{i,j} - T^k_{i,j}}{\tau} = \frac{1}{r_i} \Lambda_i \left[ r_i \lambda^k_{i,j} T^k_{i,j} \right] + \Lambda_j \left[ \lambda^k_{i,j} T^k_{i,j} \right] + X^k_{i,j}, \tag{6}$$

where spatial finite difference operators are:

$$\Lambda_i \left[ r_i \lambda^k_{i,j} T^k_{i,j} \right] = \frac{1}{\hbar_i} \left[ r_{i+\frac{1}{2}} \lambda^k_{i+\frac{1}{2},j} \frac{T^k_{i+1,j} - T^k_{i,j}}{h_{i+1}} - r_{i-\frac{1}{2}} \lambda^k_{i-\frac{1}{2},j} \frac{T^k_{i,j} - T^k_{i-1,j}}{h_i} \right],$$

$$\Lambda_j \left[ \lambda^k_{i,j} T^k_{i,j} \right] = \frac{1}{h_z^2} \left[ \lambda^k_{i,j+\frac{1}{2}} \left( T^k_{i,j+1} - T^k_{i,j} \right) - \lambda^k_{i,j-\frac{1}{2}} \left( T^k_{i,j} - T^k_{i,j-1} \right) \right], \tag{7}$$

$$T^k_{i,j} = T(r_i, z_j, t_k), \quad {C_V}^k_{i_j} = C_V(T^k_{i,j}), \quad \lambda^k_{i,j} = \lambda(T^k_{i,j}), \quad X^k_{i,j} = X(r_i, z_j, t_k),$$

$$h_i = h_1, \ i = 0, \ldots, n_1 - 1, \quad h_i = h_1, \ i = n_1, \ldots, n_2, \quad \hbar_i = \frac{h_i + h_{i+1}}{2},$$

$$r_{i\pm\frac{1}{2}} = r_i \pm \frac{h_i}{2}, \quad \lambda^k_{i\pm\frac{1}{2},j} = \lambda \left( \frac{T^k_{i,j} + T^k_{i\pm1,j}}{2} \right), \quad \lambda^k_{i,j\pm\frac{1}{2}} = \lambda \left( \frac{T^k_{i,j} + T^k_{i,j\pm1}}{2} \right).$$

The algorithm of numerical solution is described, for instance, in [7]. Boundary conditions (2) have been approximated by the following formulas:

$$T^{k+1}_{0,j} = \frac{4T^{k+1}_{1,j} - T^{k+1}_{2,j}}{3} \ \text{ for } r = 0, \ z \in [0, z_{\max}],$$

$$T^{k+1}_{n,j} = \frac{4T^{k+1}_{n-1,j} - T^{k+1}_{n-2,j}}{3} \ \text{ for } r = r_0, \ z \in [z_1, z_{\max}] \ \text{ and } r = r_{\max}, \ z \in [0, z_1].$$

Similar formulas can be used for approximation of boundary conditions (3), for $z = 0$ and $z = z_1$. At the right exterior boundary ($z = z_{\max}$) the temperature values are known and equal to the temperature of liquid helium, i.e. $T^{k+1}_{i,n} = T_0$ The same temperature is at the initial time inside the whole object (see (3)).

The conjugation conditions are approximated using one-sided derivatives:

$$\lambda_m(T^k_{i^*,j}) \frac{T^{k+1}_{i^*,j} - T^{k+1}_{i^*-1,j}}{h_i} = \lambda_{m+1}(T^k_{i^*,j}) \frac{T^{k+1}_{i^*+1,j} - T^{k+1}_{i^*,j}}{h_{i+1}},$$

where $i^*$ is corresponding index of discontinuity point, $m = 0, 1, 2$ and the values of temperatures at these points are:

$$T^{k+1}_{i^*,j} = \frac{\lambda_m(T^k_{i^*,j}) h_{i+1} T^{k+1}_{i^*-1,j} - \lambda_{m+1}(T^k_{i^*,j}) h_i T^{k+1}_{i^*+1,j}}{\lambda_m(T^k_{i^*,j}) h_{i+1} + \lambda_{m+1}(T^k_{i^*,j}) h_i}.$$

## 4. Results and Conclusions

The explicit schema (6) is stable only in case if $\tau < \frac{|h|^2}{2} \left| \frac{\rho C_V(T)}{\lambda(T)} \right|_{\min}$, where $|h| = \min\limits_{m=0..3} \{h_m, h_z\}$. The result shown in Fig. 2 was obtained at a time 0.5 ms for $h_0 = 10^{-4}$ cm, $h_{1,2,3} = 10^{-5}$ cm, $h_z = 10^{-2}$ cm and $\tau = 10^{-11}$ s, when the schema (6) is stable. For numerical simulation, the parallel algorithm for modeling thermal processes into four layer cylindrical object was developed. This algorithm is based on approach described in [2,3]. The speedup of the parallel algorithm is shown in Fig. 3. It shows approximately 90% of speedup up to 25 CPUs.

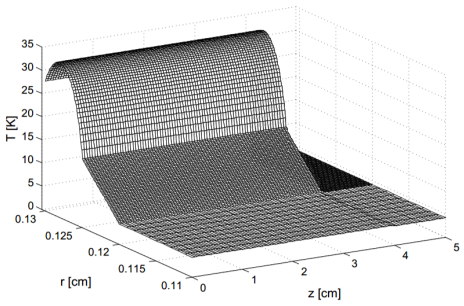


**Figure 2. Temperature in the object at the moment $t = 0.5$ ms**

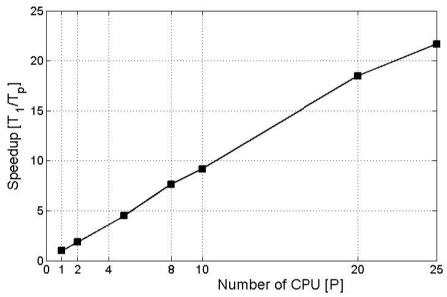


**Figure 3. Speedup VS number of CPUs**

It is necessary to make systematic studies in order to find an optimal construction and materials of the object. To make so, a great number of calculations is needed. One of the ways for such research can be construction of an implicit scheme, which is stable for a larger time step $\tau$.

## References

1. Электронно-струнные источники многозарядных ионов с линейной и трубчатой геометрией струны / Д. Е. Донец, Е. Д. Донец, Е.Е. Донец и др. // Прикладная физика. — 2010. — № 3. — С. 34–42. [Electron-String Ion Sources of Highly Charged Ions with Linear and Tubular Ge- ometry of String / D. E. Donets, E. D. Donets, E. E. Donets et al. // Applied Physics. — 2010. — No 3. — Pp. 34–42. — (in russian). ]
2. Numerical Simulation of Heat Conductivity in Composite Object with Cylindrical Symmetry / A. Ayriyan, E. Ayryan, E. Donets, J. Pribiš // Lecture Notes in Computer Science. — 2012. — Vol. 7125. — Pp. 264–269.
3. *Айриян А. С., Прибиш Я.* Моделирование процесса теплопроводности в составном образце с цилиндрической симметрией // Математическое моделирование. — 2012. — Т. 24, № 12. — С. 113–118. [Ayriyan A., Ján Pribiš. Mathematical Simulation of Heat Conductivity in Compos- ite Object with Cylindrical Symmetry // Mathematic Modeling. — 2012. — Vol. 24, No 12. — Pp. 113–118. ]
4. *Samarskii A. A., Vabishchevich P. N.* Computational Heat Transfer. — Chichester, England: John Wiley & Sons Ltd., 1995. — Vol. 1, Mathematical Modelling.
5. National Institute Of Standards And Technology. — `http://www.nist.com/`.
6. *Ayriyan A.* Model with One Spatial Variable for Design of a Technical Device // Bulletin of PFUR. Series “Mathematics. Information Sciences. Physics”. — 2013. — No 2. — Pp. 89–93.
7. *Samarskii A. A.* The Theory of Difference Schemes. — New York: Marcel Dekker Inc., 2001.

УДК 519.633.6, 519.688

# Алгоритм численного моделирования тепловых процессов в четырехслойном цилиндрическом объекте

**А. С. Айриян*, Я. Прибиш†**

* *Лаборатория информационных технологий*
*Объединённый институт ядерных исследований*
*ул. Жолио-Кюри, д. 6, Дубна, Московская область, Россия, 141980*
† *Технический университет Кошице*
*ул. Летна, д. 9, Кошице, Словакия, 04001*

В работе предложен алгоритм для моделирования процесса теплопроводности для проектирования и оптимизации криогенной ячейки, импульсно подающей рабочие газы (в миллисекундном диапазоне) в электронно-струнный источник высокозарядных ионов. Рассмотрена модель криогенной ячейки с четырьмя слоями (материалами). Тепловые процессы в исследуемом объекте возникают при пропускании электрического тока через один из проводящих слоев. Тепловые процессы описываются уравнением теплопроводности с зависящими от температуры разрывными теплофизическими коэффициентами. Коэффициенты материалов при криогенных температурах даны таблично и аппроксимированы аналитическими функциями. Условия сопряжения сред считаются идеальными. В результатах представлен расчет температурного поля для определенной конфигурации ячейки. Для ускорения расчетов разработан параллельный алгоритм, приведено ускорение алгоритма в зависимости от числа процессоров.

**Ключевые слова:** теплопередача, явная разностная схема, неравномерная сетка, параллельный алгоритм, разрывные термические коэффициенты.