UDC 530.14

Matrix Integrals and Gluings of Regular $2n$-gons

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This review is concerned applications of matrix models in combinatorics. We will discuss counting of orientable and nonorientable gluings of regular $2n$-gons using gaussian matrix integrals.

Key words and phrases: matrix integrals, generalized Catalan numbers, generating function of gluings, virtual Euler characteristic.

1. Introduction

First time a gaussian matrix model was proposed by E.Wigner to describe the excited states of nuclei. Nowadays matrix models have many applications. In high energy physics the method of $1/N$-expansion was presented by G.t' Hooft [1]. He demonstrated that in the limit of a large number of colors N quantum $U(N)$ Yang-Mills theory is described by the diagrams, which can be drawn on a plane or a sphere. Such diagrams are called planar. Contribution (for example, the partition function) of each chart is included with the factor N in the degree of the Euler characteristic of the surface on which you can draw graphs. Therefore, the contributions of other diagrams are suppressed by $1/N$.

As is well known, in the main (or planar) approximation of N in the gaussian hermitian matrix model the correlation functions are the Catalan numbers. At the same time, the number for ways of gluings $2n$-gon to get the sphere, too, will give the n^{th} Catalan number. This coincidence is not accidental. Moreover, J.Harer and D.Zagier [2–4] was found the generating function for the number for orientable gluings of all sorts. It has a surprisingly simple form. The number of gluings $2n$-gon, giving a surface of genus g, can be regarded as generalized Catalan numbers $\epsilon_g(n)$. These numbers were obtained by the recurrence relations. Correlations in this model are even or odd polynomials of degree N. For odd n, we obtain a polynomial of even degree and vice versa.

By the way, the main achievement of [2] is even more interesting result, namely the computation of virtual Euler characteristic of the moduli space for the two-dimensional surface of genus g with n marked points. The moduli space is the orbifold for which a cell decomposition has the form R^k/G_k, where G_k are finite groups. This implies that its is generally not an integer this is true for the manifold. Euler characteristics are expressed in terms of Bernoulli numbers, and are particularly remarkable form of $\zeta(1-2g)$ for a surface of genus g with one marked point.

In this report also we will discuss gluings regular $2n$-gon for nonorientable surfaces. They are described by gaussian integrals on symmetric matrices. We will present examples of calculations for correlators in this model. They already have the form of polynomials in N with non-zero coefficients for all the terms of the lowest degrees.

2. Orientable Case

At first the generating function for orientable gluings of regular $2n$-gons was obtained by Harer and Zagier [2]. These results were generalized by many authors [5–7] and so on.

Total number of orientable gluings for regular $2n$-gon is $(2n-1)!!$.

Received 11^{th} June, 2012.

It is possible to count the number of orientable gluings using gaussian hermitian matrix model. The measure of this model with hermitian matrix H is

$$\mathrm{d}\mu(H) = \frac{1}{2^{\frac{N}{2}}(\pi)^{\frac{N^2}{2}}} e^{-\frac{1}{2}\mathrm{tr}H^2} \prod_{i=1}^{N} \mathrm{d}x_{ii} \prod_{i<j} \mathrm{d}x_{ij}\mathrm{d}y_{ij},$$

where $h_{ij} = x_{ij} + iy_{ij}$. Here correlation functions are given $\langle h_{ij}h_{kl}\rangle = \delta_{il}\delta_{jk}$.

Let us define a sequence of polynomials

$$T_n(N) = \sum_{g=0} \epsilon_g(n)N^{n+1-2g} = \sum_{g=0} \epsilon_g(n)N^{n-1+\chi(g)},$$

where $\chi(g)$ is Euler characteristic.

Then their first few values are

$$T_1(N) = N^2, \quad T_2(N) = 2N^3 + N,$$
$$T_3(N) = 5N^4 + 10N^2, \quad T_4(N) = 14N^5 + 70N^3 + 21N \ldots.$$

First numbers are Catalan numbers 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 .... They obey the following recursion formula $C_{n+1} = C_0C_n + C_1C_{n-1} + ... + C_nC_0$ and

$$C_n = \frac{2n!}{n!(n+1)!}.$$

Consider generating function $T(N,s) = 1 + 2Ns + 2s \sum\limits_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!}s^n$

J. Harer and D. Zagier (1986) [2] using matrix integrals

$$T_n(N) = \int tr(H^{2n})\,\mathrm{d}\mu(H)T_n(N) = \langle \mathrm{tr}H^{2n}\rangle = \int \mathrm{tr}(H^{2n})\mathrm{d}\mu(H)$$

obtained that

$$T(N,s) = \left(\frac{1+s}{1-s}\right)^N. \tag{1}$$

For generalized Catalan numbers ϵ_g recursion formula takes the form

$$(n+1)\epsilon_g(n) = (4n-2)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2).$$

B. Lass [8] obtained pure combinatorial proof of the formula (1). A. Morozov and Sh. Shakirov [7] introduced another generating function for polynomials and calculated that

$$\sum_{N,i=0}^{\infty} T_{2i}(N)\frac{s^{2i}\lambda^N}{(2i-1)!!} = \frac{\lambda}{1-\lambda}\frac{1}{1-\lambda-(1+\lambda)s^2}.$$

They also calculated generating functions for two-and three-point correlators. Surprisingly, these generating functions are elementary.

Moreover authors [2] found virtual Euler characteristics of moduli space $\mathcal{M}_{g,n}$ for curve with genus g and with n punctured points

$$\chi(\mathcal{M}_{g,n}) = (-1)^n\frac{(2g-3+n)!(2g-1)}{(2g)!}B_{2g},$$

where B_{2g} are Bernulli numbers. Here, the virtual Euler characteristic is not an integer, because the moduli space is the orbifold.

Especially simple this formula looks for one punctured point

$$\chi(\mathcal{M}_{g,1}) = \zeta(1-2g) = -B_{2g}/2g.$$

Penner [9, 10] developed a matrix model that calculate $\chi(\mathcal{M}_{g,n})$. By the way famous Kontsevich model [11] computes insertion indices on the moduli spaces.

3. General Case

Total number for nonorientable gluings [12] of regular $2n$-gon is $2^n(2n-1)!!$. It can be demonstrated that the total number of gluings is described symmetric gaussian matrix model. Matrix models with symmetric matrix were studied many authors [13–15].

For correlators of elements in the case of symmetric matrix S we can obtain

$$\langle s_{ij} s_{kl} \rangle = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}.$$

Here the correlators or polynomials take the form [16, 17]

$$\int tr(S^2)\,\mathrm{d}\mu(S) = N^2 + N, \quad \int tr(S^4)\,\mathrm{d}\mu(S) = 2N^3 + 5N^2 + 5N,$$
$$\int tr(S^6)\,\mathrm{d}\mu(S) = 5N^4 + 22N^3 + 52N^2 + 41N,$$
$$\int tr(S^8)\,\mathrm{d}\mu(S) = 14N^5 + 93N^4 + 374N^3 + 690N^2 + 509N.$$

Here as before first numbers are Catalan numbers. Surfaces with the same Euler characteristic are taken into account in a suitable coefficient of the polynomial. To illustrate this let us take gluings of an 4-gon with surfaces of zero Euler characteristics. As a result we have one torus and four Klein bottle.

Similarly, the oriented case let us define the coefficients of these polynomials as

$$U_n(N) = \sum_{g=0} \mu_g(n) N^{n-1+\chi(g)}.$$

Then these numbers $\mu_g(n)$ have the following recurrent formula [16, 17] for every $n \geqslant 4$

$$(n+1)\mu_g(n) = (8n-2)\mu_{g-1}(n-1) - (4n-1)\mu_g(n-1) + n(2n-3)(10n-9)\mu_g(n-2) -$$
$$-8(2n-3)\mu_{g-2}(n-2) + 8(2n-3)\mu_{g-1}(n-2) - 10(2n-3)(2n-4)(2n-5)\mu_{g-1}(n-3) +$$
$$+5(2n-3)(2n-4)(2n-5)\mu_g(n-3) - 2(2n-3)(2n-4)(2n-5)(2n-6)(2n-7)\mu_g(n-4).$$

4. Conclusion and Outlook

We have considered the current state of the Harer-Zagier problem for general case with nonorientable gluings of a regular $2n$-gon. Virtual Euler characteristics for nonorientable two-dimensional surfaces via symmetric Penner model were counted in [15]. Symmetric Kontsevich model is not investigated.

Author would like to thank V.O. Manturov for discussions, L.O. Chekhov for providing the article [15] and A.V. Borisov for reading manuscript and remarks.

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УДК 530.14

Матричные интегралы и склейки правильных $2n$-угольников

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Рассматриваются приложения матричных моделей в комбинаторике. Обсуждается подсчёт ориентируемых и неориентируемых склеек правильных $2n$-угольников с помощью гауссовых интегралов по ортогональным матрицам.

Ключевые слова: матричные интегралы, обобщённые числа Каталана, производящая функция склеек, виртуальная эйлерова характеристика.