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On Nonstationary Solutions to Yang–Mills Equations

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We study Yang–Mills fields with $SU(2)$ symmetry generated by classical field sources. It is shown that in this case the Yang–Mills equations can be regarded as a reasonable nonlinear generalization of the equations of Maxwell's electrodynamics. We seek new classes of solutions to the examined Yang–Mills equations and find their nontrivial solutions in the case of nonstationary spherically symmetric sources and a wide class of their non-Abelian wave solutions.

Key words and phrases: Yang–Mills equations, $SU(2)$ symmetry, classical field sources, nonstationary spherically symmetric solutions, non-Abelian wave solutions.

1. Introduction

Let us study the Yang–Mills equations with $SU(2)$ symmetry. They can be represented as [1, 2]

$$D_\mu F^{k,\mu\nu} \equiv \partial_\mu F^{k,\mu\nu} + g\varepsilon_{klm} F^{l,\mu\nu} A^m_\mu = (4\pi/c) J^{k,\nu}, \tag{1}$$

$$F^{k,\mu\nu} = \partial^\mu A^{k,\nu} - \partial^\nu A^{k,\mu} - g\varepsilon_{klm} A^{l,\mu} A^{m,\nu}, \tag{2}$$

where $\mu,\ \nu = 0,\ 1,\ 2,\ 3,\ \ k,\ l,\ m = 1,\ 2,\ 3,\ \ D_\mu$ is the Yang–Mills covariant derivative, $A^{l,\mu}$, $F^{k,\mu\nu}$ are potentials and strengths of a Yang–Mills field, respectively, ε_{klm} is the antisymmetric tensor, $\varepsilon_{123} = 1$, g is the constant of electroweak interactions, $J^{k,\nu}$ are three four-dimensional vectors of current densities, and $\partial_\mu \equiv \partial/\partial x^\mu$, where x^μ are orthogonal space-time coordinates of the Minkowsky geometry.

Consider Eqs. (1)–(2) in the case of the following field sources:

$$J^{1,\nu} = J^\nu, \quad J^{2,\nu} = J^{3,\nu} = 0, \tag{3}$$

where J^ν is a classical four-dimensional vector of current densities.

Then the Yang–Mills equations (1)–(2) have trivial solutions in which $A^{2,\nu} = A^{3,\nu} = 0$, $F^{2,\mu\nu} = F^{3,\mu\nu} = 0$ and the potentials $A^{1,\nu}$ and strengths $F^{1,\mu\nu}$ satisfy the Maxwell equations with the sources J^ν. Besides, the expressions for the Lagrangian and energy-momentum tensor of the Yang–Mills field are similar to those of the Maxwell field. That is why the considered Yang–Mills equations with the classical field sources (3) can be regarded as a reasonable nonlinear generalization of the Maxwell equations. This nonlinear theory was studied in our works [3–5], where several classes of exact solutions to Eqs. (1)–(3) were found. In our monograph [6] these solutions are applied to a number of anomalous phenomena that remain still unexplained within the framework of the linear Maxwell theory.

It should be noted that the Yang–Mills equations (1)–(2) with the field sources (3) are not independent. Namely, from (3) and the well-known identities for the Yang–Mills covariant derivative D_μ [1, 2] we have that when $\delta_1 = 1$, $\delta_2 = \delta_3 = 0$,

$$\delta_k D_\nu [D_\mu F^{k,\mu\nu} - (4\pi/c) J^{k,\nu}] \equiv 0. \tag{4}$$

That is why in Refs. [3–6] one more equation to the Yang–Mills equations (1)–(2) with the field sources of the form (3) was proposed to uniquely determine the field

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strengths $F^{k,\mu\nu}$. This equation has the form

$$I^{k,\nu}I_{k,\nu} = J^{k,\nu}J_{k,\nu}, \quad I^{k,\nu} = J^{k,\nu} - (gc/4\pi)\varepsilon_{klm}F^{l,\mu\nu}A^m_\mu, \tag{5}$$

where, as follows from Eq. (1), the components $I^{k,\nu}$ satisfy the charge conservation equations $\partial_\nu I^{k,\nu} = 0$ and can be regarded as four-dimensional densities of full currents which include not only source current densities $J^{k,\nu}$ but also current densities of field virtual particles.

The additional equation (5) implies the conservation of the intrinsic energy in a small part of a field source when charged particles are created inside the source [3–6].

Using Eq. (1), we can represent Eq. (5) in the form

$$\partial_\alpha F^{k,\alpha\nu}\partial^\beta F_{k,\beta\nu} = (4\pi/c)^2 J^{k,\nu}J_{k,\nu}. \tag{6}$$

Further we will seek exact solutions to the Yang–Mills equations (1)–(2) with the field sources (3) that also satisfy the additional equation (6).

2. Nonstationary solutions to the Yang–Mills equations with spherically symmetric sources

Consider the Yang–Mills equations (1)–(2) with the following spherically symmetric sources:

$$\begin{aligned}(4\pi/c)J^{1,0} &= j^0(\tau,r), \quad (4\pi/c)J^{1,n} = x^n j(\tau,r), \quad n=1,2,3,\\ J^{2,\nu} &= J^{3,\nu} = 0, \quad \tau = x^0, \quad r = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2},\end{aligned} \tag{7}$$

where $\tau = ct$, t is time, and r is distance from the source center.

Let us seek the field potentials $A^{k,\nu}$ in the form

$$A^{k,0} = \beta^k(\tau,r), \quad A^{k,n} = x^n\alpha^k(\tau,r). \tag{8}$$

Then from Eq. (2) we find

$$F^{k,0n} = x^n u^k(\tau,r), \quad F^{k,in} = 0, \quad k,i,n = 1,2,3, \tag{9}$$

where

$$u^k = \partial\alpha^k/\partial\tau + (1/r)\partial\beta^k/\partial r + g\varepsilon_{klm}\alpha^l\beta^m. \tag{10}$$

Substituting expressions (7)–(9) into the Yang–Mills equations (1), we derive

$$r\partial u^k/\partial r + 3u^k - gr^2\varepsilon_{klm}u^l\alpha^m = -j^0\delta_k, \quad \delta_1 = 1, \quad \delta_2 = \delta_3 = 0, \tag{11}$$

$$\partial u^k/\partial\tau + g\varepsilon_{klm}u^l\beta^m = j\delta_k. \tag{12}$$

As is well-known, the Yang–Mills equations (1)–(2) have the following consequence [1,2]:

$$D_\nu J^{k,\nu} = 0. \tag{13}$$

From (3) and (13) we find

$$\partial j^0/\partial\tau + r\partial j/\partial r + 3j = 0, \tag{14}$$

$$j^0\beta^k - r^2 j\alpha^k = 0, \tag{15}$$

where $k = 2, 3$. However, since $J^{2,\nu} = J^{3,\nu} = 0$, we can impose one gauge condition on the field potentials and choose the gauge so as to have Eq. (15) satisfied for $k = 1$. That is why we will further consider Eq. (15) fulfilled for $k = 1, 2, 3$.

Let us now multiply Eq. (11) by j and Eq. (12) by j^0 and then add the products. Then using (15), we derive

$$j^0 \partial u^k/\partial \tau + j(r\partial u^k/\partial r + 3u^k) = 0. \tag{16}$$

Multiplying Eq. (11) by u^k and summing over k, we find

$$\sum_{k=1}^{3} u^k (r\partial u^k/\partial r + 3u^k) = -j^0 u^1. \tag{17}$$

Eq. (11) also gives two equations for α^k.

Let us turn to Eq. (16). To solve it we introduce the function

$$q(\tau, r) = \int\limits_0^r r^2 j^0(\tau, r)\,\mathrm{d}r. \tag{18}$$

From (18) we find, using equality (14),

$$\partial q/\partial \tau = \int\limits_0^r r^2 \frac{\partial j^0}{\partial \tau}\,\mathrm{d}r = -\int\limits_0^r r^2 \left(r\frac{\partial j}{\partial r} + 3j \right) \mathrm{d}r = -\int\limits_0^r \frac{\partial (r^3 j)}{\partial r}\,\mathrm{d}r = -r^3 j, \tag{19}$$

$$\partial q/\partial r = r^2 j^0. \tag{20}$$

From (19) and (20) we have

$$j^0 \partial q/\partial \tau + rj\partial q/\partial r = 0. \tag{21}$$

Using now the equaliy (21), we obtain the following solution of Eq. (16):

$$u^k = P^k(q)/r^3, \tag{22}$$

where $P^k(q)$ are arbitrary differentiable functions of the argument q.

Indeed, from (21) and (22) we derive

$$j^0 \partial u^k/\partial \tau + j(r\partial u^k/\partial r + 3u^k) = \frac{1}{r^3}\frac{\mathrm{d}P^k}{\mathrm{d}q} \big(j^0 \partial q/\partial \tau + rj\partial q/\partial r \big) = 0. \tag{23}$$

Hence, formula (22) gives solutions to Eq. (16). This formula describes general solutions to Eq. (16) since it contains three arbitrary differentiable functions $P^k(q)$ and Eq. (16) presents three partial differential equations of the first order.

Substituting formula (22) into Eq. (17), we obtain

$$\sum_{k=1}^{3} P^k \frac{\mathrm{d}P^k}{\mathrm{d}q}\frac{\partial q}{\partial r} = -r^2 j^0 P^1. \tag{24}$$

Using formula (20), from (24) we find

$$\sum_{k=1}^{3} P^k \mathrm{d}P^k/\mathrm{d}q = -P^1. \tag{25}$$

Let us turn now to Eq. (6). From it and formulas (7) and (9) we derive

$$\sum_{k=1}^{3}[(r\partial u^k/\partial r + 3u^k)^2 - r^2(\partial u^k/\partial\tau)^2] = (j^0)^2 - r^2 j^2. \tag{26}$$

Substituting formula (22) into Eq. (26), we obtain

$$\sum_{k=1}^{3}\left(\mathrm{d}P^k/\mathrm{d}q\right)^2\left[(\partial q/\partial r)^2 - (\partial q/\partial\tau)^2\right] = r^4[(j^0)^2 - r^2 j^2]. \tag{27}$$

Using formulas (19) and (20), from Eq. (27) we find

$$\sum_{k=1}^{3}(\mathrm{d}P^k/\mathrm{d}q)^2 = 1. \tag{28}$$

Let us seek solutions to Eqs. (25) and (28) in the following form, taking into account that $J^{2,\nu} = J^{3,\nu} = 0$ and hence the equivalence of the second and third gauge axes:

$$P^1 = -P\cos\xi, \quad P^2 = P^3 = -2^{-1/2}P\sin\xi, \quad P = P(q), \quad \xi = \xi(q). \tag{29}$$

Then from (25) and (28) we derive

$$\mathrm{d}P/\mathrm{d}q = \cos\xi, \quad (\mathrm{d}P/\mathrm{d}q)^2 + P^2(\mathrm{d}\xi/\mathrm{d}q)^2 = 1. \tag{30}$$

From these equations we obtain

$$\mathrm{d}P = \cos\xi\mathrm{d}q, \quad P\mathrm{d}\xi = \pm\sin\xi\mathrm{d}q. \tag{31}$$

Eqs. (31) give

$$\mathrm{d}P/P = \pm\cot\xi\mathrm{d}\xi. \tag{32}$$

Integrating this equation and choosing the sign '+' to have no singularity at $\xi = 0$, we find

$$P = K_0\sin\xi, \quad K_0 = \text{const}. \tag{33}$$

Substituting this formula into Eqs. (31), we obtain

$$\mathrm{d}\xi/\mathrm{d}q = 1/K_0, \quad \xi = q/K_0 + K_1, \quad K_1 = \text{const}. \tag{34}$$

As follows from formulas (18), (22), and (29), $q(0) = 0$ and $P(0) = 0$. That is why we choose $K_1 = 0$ in order to satisfy formula (33) at$r = 0$. Then from (33) and (34) we find

$$P = K_0\sin(q/K_0), \quad \xi = q/K_0. \tag{35}$$

Substituting these expressions for P and ξ into formulas (29) and then (22) and (9), we come to the following formulas for the field strengths $F^{k,\mu\nu}$:

$$F^{1,n0} = K\sin\left(\frac{q(\tau,r)}{K}\right)\frac{x^n}{r^3}, \quad K = \frac{K_0}{2} = \text{const},$$

$$F^{2,n0} = F^{3,n0} = \frac{\sqrt{2}}{2} K \left[1 - \cos\left(\frac{q(\tau, r)}{K} \right) \right] \frac{x^n}{r^3}, \tag{36}$$

$$F^{k,in} = 0, \quad k, i, n = 1, 2, 3.$$

As follows from (7) and (18), the function $q(\tau, r)$ presents the charge of the part of the field source situated in the spherical region of radius r at time $t = \tau/c$.

From (36) we find

$$F^{1,n0} = q_{\text{eff}}(\tau, r) \frac{x^n}{r^3}, \quad q_{\text{eff}}(\tau, r) = K \sin\left(\frac{q(\tau, r)}{K} \right). \tag{37}$$

Here $q_{\text{eff}}(\tau, r)$ can be regarded as an effective charge at the time $t = \tau/c$ in the spherical region of the radius r which includes not only the source charge $q(\tau, r)$ but also charged quanta of the Yang–Mills field.

The constant K should be considered as some sufficiently large charge. Then when $|q/K| \ll 1$ we have $q_{\text{eff}} = q$ and formula (37) describes the classical electric field. Therefore, this formula can be regarded as a nonlinear generalization of the classical formula in the cases of spherical sources with sufficiently large charges.

Formula (37) was applied in Refs. [3,6] to explain the phenomenon of ball lightning, where a relation between its maximum diameter and the constant K was found. Using the known estimate of the maximum diameter of the ball lightning which is about 100 cm [7], from this relation we obtain that the constant $K \sim 10^7$ coul.

It should be noted that in Ref. [8] a nonlinear model of the Earth ionosphere is proposed in which strong electric fields are taken into account and described by formula (37). Besides, as shown in [8], just the obtained esimate of the constant $K \sim 10^7$ coul provides good agreement of density distributions in the ionosphere computed by means of the proposed model with experimental data derived from artificial satellites.

Let us now study another class of exact solutions to the examined Yang–Mills equations.

3. Non-Abelian expanding waves

Consider the Yang–Mills equations (1)–(2) in the region outside field sources where

$$J^{k,\nu} = 0. \tag{38}$$

Let us seek their wave solutions in the form

$$A^{k,0} = u^k(y_0, y_1, y_2, y_3), \quad A^{k,n} = \frac{x^n}{r} A^{k,0}, \quad y_0 = x^0 - r, \quad y_n = x^n, \tag{39}$$
$$k, n = 1, 2, 3, \quad r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2},$$

where u^k are some functions of the wave phase $y_0 = x^0 - r$ and of the spatial coordinates $y_n = x^n$.

Substituting these expressions for the potentials $A^{k,\nu}$ into formula (2) for the strengths $F^{k,\mu\nu}$, we readily find

$$F^{k,0n} = \frac{\partial u^k}{\partial y_n}, \quad F^{k,in} = \frac{1}{r} \left(y_i \frac{\partial u^k}{\partial y_n} - y_n \frac{\partial u^k}{\partial y_i} \right), \quad k, i, n = 1, 2, 3. \tag{40}$$

As will be shown below, these field strengths satisfy Eq. (6) in the considered case (38).

Let us now substitute formulas (38)–(40) for $J^{k,\nu}$, $A^{k,\nu}$, and $F^{k,\mu\nu}$ into the Yang–Mills equations (1).

Then when the index $\nu = 0$ we obtain

$$\sum_{i=1}^{3} \left(\frac{\partial^2 u^k}{\partial y_i^2} - \frac{y_i}{r} \frac{\partial^2 u^k}{\partial y_0 \partial y_i} + g \frac{y_i}{r} \varepsilon_{klm} u^l \frac{\partial u^m}{\partial y_i} \right) = 0 \tag{41}$$

and when the index $\nu = n = 1, 2, 3$ we derive after reductions

$$\frac{y_n}{r} \sum_{i=1}^{3} \left(y_i \frac{\partial^2 u^k}{\partial y_0 \partial y_i} - r \frac{\partial^2 u^k}{\partial y_i^2} + \frac{y_i}{r} \frac{\partial u^k}{\partial y_i} - g \varepsilon_{klm} y_i u^l \frac{\partial u^m}{\partial y_i} \right) + \\ + \frac{\partial}{\partial y_n} \left(\sum_{i=1}^{3} y_i \frac{\partial u^k}{\partial y_i} \right) = 0. \tag{42}$$

It should be noted that Eqs. (41) and (42) can be represented in the form

$$\partial_\mu F^{k,\mu 0} = g \sum_{i=1}^{3} \frac{y_i}{r} \varepsilon_{klm} u^l \frac{\partial u^m}{\partial y_i}\,, \quad \partial_\mu F^{\mu n} = g \frac{y_n}{r} \sum_{i=1}^{3} \frac{y_i}{r} \varepsilon_{klm} u^l \frac{\partial u^m}{\partial y_i}\,. \tag{43}$$

From (43) we readily find that the field strengths $F^{k,\mu\nu}$ of the form (40) satisfy Eq. (6) in the considered case (38).

Let us denote

$$p^k = \sum_{i=1}^{3} y_i \frac{\partial u^k}{\partial y_i}\,, \quad q^k = \sum_{i=1}^{3} \frac{\partial^2 u^k}{\partial y_i^2}\,. \tag{44}$$

Then from (41) and (42) we find

$$q^k = \frac{1}{r} \left(\frac{\partial p^k}{\partial y_0} - g \varepsilon_{klm} u^l p^m \right), \quad r = \sqrt{y_1^2 + y_2^2 + y_3^2}\,, \tag{45}$$

$$y_n \left(\frac{1}{r} \frac{\partial p^k}{\partial y_0} - q^k + \frac{p^k}{r^2} - \frac{g}{r} \varepsilon_{klm} u^l p^m \right) + \frac{\partial p^k}{\partial y_n} = 0, \quad n = 1, 2, 3. \tag{46}$$

As follows from (40) and (44), in the case $p^k = 0$ the considered expanding waves are transverse. At the same time when $p^k \neq 0$, these waves also have longitudinal components.

Let us substitute expression (45) for q^k into Eqs. (46). Then we easily obtain

$$\frac{y_n p^k}{r^2} + \frac{\partial p^k}{\partial y_n} = 0, \quad n = 1, 2, 3. \tag{47}$$

As can be readily verified, these equations have the following solution:

$$p^k = \frac{s^k(y_0)}{r}, \tag{48}$$

where s^k are arbitrary differentiable functions of the argument y_0.

From (44), (45), and (48) we get

$$\sum_{i=1}^{3} y_i \frac{\partial u^k}{\partial y_i} = \frac{s^k(y_0)}{r}, \tag{49}$$

$$\sum_{i=1}^{3}\frac{\partial^2 u^k}{\partial y_i^2}=\frac{1}{r^2}\left[\frac{ds^k(y_0)}{dy_0}-g\varepsilon_{klm}u^l s^m(y_0)\right]. \tag{50}$$

As can be readily verified, Eq. (49) has the following solution:

$$u^k(y_0,y_1,y_2,y_3)=-\frac{s^k(y_0)}{r}+f^k(y_0,\xi_1,\xi_2,\xi_3),\quad \xi_i=\frac{y_i}{r},\quad i=1,2,3, \tag{51}$$

where $r=\sqrt{y_1^2+y_2^2+y_3^2}$ and f^k are arbitrary differentiable functions.

Actually, from (51) we derive

$$\frac{\partial u^k}{\partial y_i}=\frac{s^k(y_0)y_i}{r^3}+\frac{1}{r}\frac{\partial f^k}{\partial \xi_i}-\frac{y_i}{r^3}\sum_{n=1}^{3}y_n\frac{\partial f^k}{\partial \xi_n},\quad \sum_{i=1}^{3}y_i\frac{\partial u^k}{\partial y_i}\equiv\frac{s^k(y_0)}{r}. \tag{52}$$

Consider Eq. (50) using formula (51). For the functions $f^k(y_0,\xi_1,\xi_2,\xi_3)$ we have

$$\frac{\partial f^k}{\partial y_i}=\frac{1}{r}\sum_{j=1}^{3}\frac{\partial f^k}{\partial \xi_j}\left(\delta_{ij}-\xi_i\xi_j\right),\quad i=1,2,3,\quad \xi_i=\frac{y_i}{r},$$
$$\delta_{ii}=1,\quad \delta_{ij}=0 \ \text{ when } j\neq i,$$

$$\frac{\partial^2 f^k}{\partial y_i^2}=\frac{1}{r^2}\sum_{j,n=1}^{3}\frac{\partial^2 f^k}{\partial \xi_j\partial \xi_n}\left(\delta_{ij}-\xi_i\xi_j\right)\left(\delta_{in}-\xi_i\xi_n\right)-$$
$$-\frac{1}{r^2}\sum_{j=1}^{3}\frac{\partial f^k}{\partial \xi_j}\left[\xi_j(1-3\xi_i^2)+2\xi_i\delta_{ij}\right]. \tag{53}$$

Let us substitute expression (51) for the functions u^k into Eq. (50) and take into account that the function $1/r$ is harmonic. Then using (53) and the evident equality $\xi_1^2+\xi_2^2+\xi_3^2=1$, we obtain

$$\sum_{i=1}^{3}\left[(1-\xi_i^2)\frac{\partial^2 f^k}{\partial \xi_i^2}-2\xi_i\frac{\partial f^k}{\partial \xi_i}\right]-\sum_{\substack{i,j=1\\ j\neq i}}^{3}\xi_i\xi_j\frac{\partial^2 f^k}{\partial \xi_i\partial \xi_j}=\frac{ds^k(y_0)}{dy_0}-g\varepsilon_{klm}f^l s^m(y_0). \tag{54}$$

The arguments $\xi_i=y_i/r$ of the functions f^k are not independent, since $\xi_1^2+\xi_2^2+\xi_3^2=1$. That is why instead of ξ_1,ξ_2, ξ_3 we can choose two independent arguments. Let us choose the following two arguments θ and σ:

$$f^k(y_0,\xi_1,\xi_2,\xi_3)=h^k(y_0,\theta,\sigma),\quad \theta=\frac{1}{2}\ln\left(\frac{1+\xi_1}{1-\xi_1}\right),\quad \sigma=\arctan\left(\frac{\xi_2}{\xi_3}\right). \tag{55}$$

Then we have

$$\frac{\partial f^k}{\partial \xi_1}=\beta\frac{\partial h^k}{\partial \theta},\quad \frac{\partial f^k}{\partial \xi_2}=\gamma\xi_3\frac{\partial h^k}{\partial \sigma},\quad \frac{\partial f^k}{\partial \xi_3}=-\gamma\xi_2\frac{\partial h^k}{\partial \sigma},\quad \beta=\frac{1}{1-\xi_1^2},\quad \gamma=\frac{1}{\xi_2^2+\xi_3^2},$$

$$\frac{\partial^2 f^k}{\partial \xi_1^2}=\beta^2\left(\frac{\partial^2 h^k}{\partial \theta^2}+2\xi_1\frac{\partial h^k}{\partial \theta}\right),\quad \frac{\partial^2 f^k}{\partial \xi_2^2}=\gamma^2\xi_3\left(\xi_3\frac{\partial^2 h^k}{\partial \sigma^2}-2\xi_2\frac{\partial h^k}{\partial \sigma}\right),$$

$$\frac{\partial^2 f^k}{\partial \xi_3^2} = \gamma^2 \xi_2 \left(\xi_2 \frac{\partial^2 h^k}{\partial \sigma^2} + 2\xi_3 \frac{\partial h^k}{\partial \sigma} \right), \quad \frac{\partial^2 f^k}{\partial \xi_1 \partial \xi_2} = \beta\gamma\xi_3 \frac{\partial^2 h^k}{\partial \theta \partial \sigma},$$

$$\frac{\partial^2 f^k}{\partial \xi_1 \partial \xi_3} == -\beta\gamma\xi_2 \frac{\partial^2 h^k}{\partial \theta \partial \sigma}, \quad \frac{\partial^2 f^k}{\partial \xi_2 \partial \xi_3} = -\gamma^2 \left(\xi_2 \xi_3 \frac{\partial^2 h^k}{\partial \sigma^2} + (\xi_3^2 - \xi_2^2) \frac{\partial h^k}{\partial \sigma} \right) \tag{56}$$

and as can be readily verified, the left-hand side of (54) acquires the form

$$\sum_{i=1}^{3} \left[(1-\xi_i^2) \frac{\partial^2 f^k}{\partial \xi_i^2} - 2\xi_i \frac{\partial f^k}{\partial \xi_i} \right] - \sum_{\substack{i,j=1 \\ j \neq i}}^{3} \xi_i \xi_j \frac{\partial^2 f^k}{\partial \xi_i \partial \xi_j} =$$
$$= \frac{1}{1-\xi_1^2} \frac{\partial^2 h^k}{\partial \theta^2} + \frac{1}{\xi_2^2 + \xi_3^2} \frac{\partial^2 h^k}{\partial \sigma^2} . \tag{57}$$

Since the variables $\xi_i = y_i/r$ satisfy the equality $\xi_2^2 + \xi_3^2 = 1 - \xi_1^2$, from (54), (55), and (57) we come to the following equation:

$$\frac{\partial^2 h^k}{\partial \theta^2} + \frac{\partial^2 h^k}{\partial \sigma^2} = (1-\xi_1^2) \left[\frac{\mathrm{d}s^k(y_0)}{\mathrm{d}y_0} - g\varepsilon_{klm} h^l s^m(y_0) \right], \quad \xi_1 = \tanh\theta. \tag{58}$$

Let us put

$$h^k = v^k(y_0, \theta, \sigma) + \varkappa(y_0) s^k(y_0) \ln(\cosh\theta) + d^k(y_0), \tag{59}$$

where $v^k(y_0,\theta,\sigma)$, $\varkappa(y_0)$, and $d^k(y_0)$ are some functions.

Then substituting (59) into (58) and taking into account that ε_{klm} are antisymmetric, we get

$$\frac{\partial^2 v^k}{\partial \theta^2} + \frac{\partial^2 v^k}{\partial \sigma^2} =$$
$$= (1-\tanh^2\theta) \left[\frac{\mathrm{d}s^k(y_0)}{\mathrm{d}y_0} - \varkappa(y_0) s^k(y_0) - g\varepsilon_{klm} \left(v^l + d^l(y_0) \right) s^m(y_0) \right]. \tag{60}$$

Let us require that the four functions $\varkappa(y_0)$ and $d^k(y_0)$ should satisfy the following system of three algebraic equations which are linear with respect to them:

$$\frac{\mathrm{d}s^k(y_0)}{\mathrm{d}y_0} - \varkappa(y_0) s^k(y_0) - g\varepsilon_{klm} d^l(y_0) s^m(y_0) = 0. \tag{61}$$

Then from (60) we get

$$\frac{\partial^2 v^k}{\partial \theta^2} + \frac{\partial^2 v^k}{\partial \sigma^2} = -g(1-\tanh^2\theta) \varepsilon_{klm} v^l s^m(y_0), \quad v^k = v^k(y_0,\theta,\sigma). \tag{62}$$

After multiplying (61) by $s^k(y_0)$ and summing it over the index k, we derive the following simple formula for the function $\varkappa(y_0)$:

$$\varkappa(y_0) = \frac{1}{s(y_0)} \frac{\mathrm{d}s(y_0)}{\mathrm{d}y_0}, \quad \big(s(y_0)\big)^2 = \sum_{k=1}^{3} \big(s^k(y_0)\big)^2. \tag{63}$$

Let us turn to Eq. (62). We will seek its solution in the following form:

$$v^k(y_0,\theta,\sigma) = \Re \sum_{n=0}^{N} V_n^k(y_0,\theta)\exp\bigl(-n(\theta+i\sigma)\bigr), \qquad (64)$$

where $V_n^k(y_0,\theta)$ are some complex functions and n, N are nonnegative integers. Then substituting (64) into Eq. (62), we come to the equations

$$\frac{\partial^2 V_n^k}{\partial\theta^2} - 2n\frac{\partial V_n^k}{\partial\theta} = -g(1-\tanh^2\theta)\varepsilon_{klm}V_n^l s^m(y_0). \qquad (65)$$

Let us choose the variable $\varphi = \tanh\theta$ instead of θ and put

$$V_n^k = V_n^k(y_0,\varphi), \quad \varphi = \tanh\theta \equiv \xi_1. \qquad (66)$$

Then Eq. (65) acquires the following form, since $\mathrm{d}\varphi/\mathrm{d}\theta = 1-\tanh^2\theta = 1-\varphi^2$:

$$(1-\varphi^2)\frac{\partial^2 V_n^k}{\partial\varphi^2} - 2(n+\varphi)\frac{\partial V_n^k}{\partial\varphi} = -g\varepsilon_{klm}V_n^l s^m(y_0), \quad |\varphi| \leqslant 1. \qquad (67)$$

Putting

$$\eta = (1-\varphi)/2, \quad 0 \leqslant \eta \leqslant 1, \quad -1 \leqslant \varphi \leqslant 1, \qquad (68)$$

from (67) and (68) we get

$$\eta(\eta-1)\frac{\partial^2 V_n^k}{\partial\eta^2} - (n+1-2\eta)\frac{\partial V_n^k}{\partial\eta} - g\varepsilon_{klm}V_n^l s^m(y_0) = 0, \quad V_n^k = V_n^k(y_0,\eta). \qquad (69)$$

Let us seek solutions $V_n^k(y_0,\eta)$ to Eq. (69) in the form

$$V_n^k = \sum_{j=0}^{\infty} \lambda_{j,n}^k(y_0)\eta^j, \qquad (70)$$

where $\lambda_{j,n}^k(y_0)$ are some complex functions. Then substituting (70) into (69), we obtain the recurrence relation for $\lambda_{1,n}^k, \lambda_{2,n}^k, \lambda_{3,n}^k, \ldots$.

$$\lambda_{j+1,n}^k = \frac{j(j+1)\lambda_{j,n}^k - g\varepsilon_{klm}\lambda_{j,n}^l s^m(y_0)}{(j+1)(j+1+n)}, \quad j = 0,1,2,\ldots, \qquad (71)$$

where the complex functions $\lambda_{0,n}^k = \lambda_{0,n}^k(y_0)$ may be assigned arbitrarily.

From (71) we can easily derive that the sequence $\left|\lambda_{j,n}^k\right|$ is bounded for any y_0.

Actually, let us denote

$$L(y_0) = \max_{1\leqslant k,l\leqslant 3} |g\varepsilon_{klm}s^m(y_0)| \qquad (72)$$

and consider (71) when $j > L(y_0)-1$ for an arbitrary y_0. Then we find

$$\max_{1\leqslant k\leqslant 3}\left|\lambda_{j+1}^k\right| \leqslant \frac{j(j+1)+L(y_0)}{(j+1)(j+1+n)} \max_{1\leqslant k\leqslant 3}\left|\lambda_j^k\right| \leqslant \max_{1\leqslant k\leqslant 3}\left|\lambda_j^k\right|. \qquad (73)$$

This formula precisely proves that that the sequence $\left|\lambda_j^k\right|$ is bounded for any y_0.

From (73) we also get that the values $\max_{1\leqslant k\leqslant 3}\left|\lambda_j^k\right|$, $0\leqslant j<\infty$, are bounded by their maximum when $0\leqslant j\leqslant L(y_0)$.

Consider the case in which the source of non-Abelian waves under examination is situated along the axis x^1. Then for the waves outside the source we have $-\infty<\theta<\infty$ and from (66) and (68) we find

$$0<\eta<1. \tag{74}$$

Since, as shown above, the sequence $\left|\lambda_j^k\right|$ is bounded for any y_0, the considered power series (70) is absolutely convergent when $0\leqslant\eta<1$. Therefore, in the case $|\theta|<\infty$ under eximination the functions $V_n^k(y_0,\eta)$ can be determined by formula (70). After that we can find the functions $v^k(y_0,\theta,\sigma)$ and $h^k(y_0,\theta,\sigma)$ using formulas (64) and (59). Then applying formulas (55) and (51) we determine the functions $u^k(y_0,y_1,y_2,y_3)\equiv u^k(x^0-r,x^1,x^2,x^3)$ describing non-Abelian expanding waves radiated from the considered source situated along the axis x^1.

As indicated above, the considered wave solutions to the Yang–Mills equations can have longitudinal components when the functions p^k of the form (48) are non-zero. This property of the found non-Abelian waves can be used to detect cosmic sources of Yang–Mills fields.

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О нестационарных решениях уравнений Янга–Миллса

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Исследуются поля Янга–Миллса с $SU(2)$ симметрией, создаваемые классическими полевыми источниками. Показывается, что в данном случае уравнения Янга–Миллса могут быть рассмотрены как естественное нелинейное обобщение уравнений максвелловской электродинамики. Ищутся новые классы решений исследуемых уравнений Янга–Миллса и находятся их нетривиальные решения в случае нестационарных сферически-симметричных источников и широкий класс их неабелевых волновых решений.

Ключевые слова: уравнения Янга–Миллса, $SU(2)$ симметрия, классические источники поля, нестационарные сферически-симметричные решения, неабелевые волновые решения.