Hamiltonian Formulation of Ideal Ferrohydrodynamics

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The Hamiltonian description of ferrohydrodynamics equations for a ideal nonconducting compressible magnetic fluid with frozen magnetization is presented.

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1.   Introduction

An interesting class of intelligent materials are magnetic fluids, or ferrofluids, which without magnetic field are homogeneous colloidal suspensions of ferromagnetic nanoparticles coated with surface-active dispersive medium (typical diameters of particles range from 5 to 10 nm) in a carrier liquids [1]. Ferrohydrodynamics describes evolution of a magnetic fluid, carrying a magnetic field. The continuous models of ferrohydrodynamics have been studied in recent years from different points of view. Dynamic processes in a magnetic fluid can be described in continual approximation for two limiting cases. One of them corresponds to the equilibrium magnetization of the magnetic fluid [2], i.e., the case where the relaxation time characterizing the magnetization relaxation to the equilibrium value is infinitely small. The other limiting case corresponds to the situation where the magnetic fluid possesses a frozen magnetization [3], i.e., the case where the relaxation time is infinitely large. A complete system of equations describing an ideal non-conducting magnetic fluid with the frozen magnetization was obtained in [3]. As shown in [4,5], the linear approximation of ferrohydrodynamics equations for the fluid with frozen magnetization describes adequately experimental data for the anisotropy of ultrasonic velocity in magnetized magnetic fluids based on the various liquids. The complete system of equations describing an ideal nonconducting magnetic fluid of density ρ with the frozen-in magnetization $\vec{M} = \rho\vec{m}$ has the form [3]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0\,,$$

$$\rho \frac{\partial v_i}{\partial t} = -\frac{\partial P}{\partial x_i} + (H_i^{eg} - H_i)\frac{\partial (M_j)}{\partial x_j} + M_k \frac{\partial H_i^{eg}}{\partial x_k}\,,$$

$$\frac{dm_i}{dt} = m_j \frac{\partial v_i}{\partial x_j}\,, \quad \frac{\partial s}{\partial t} + v_i \frac{\partial s}{\partial x_i} = 0\,, \tag{1}$$

$$H_i^{eq} = \left(\frac{\partial \epsilon}{\partial m_i}\right)_{\rho,s}\,, \quad p = \left(\rho^2 \frac{\partial \epsilon}{\partial \rho}\right)_{s,m}\,, \quad H_i = -\frac{\partial \Psi}{\partial x_i}\,, \quad \nabla^2 \Psi = 4\pi \frac{\partial (\rho m_j)}{\partial x_j}\,.$$

The system of equations is closed by setting a specific form of the internal energy density per unit mass $\epsilon = \epsilon(\rho, s, m_i)$ which depends on fluid density $\rho = \rho(\vec{x}, t)$, the specific entropy $s = s(\vec{x}, t)$, and the components of the magnetization per unit mass $m_i = m_i(\vec{x}, t)$. The specific feature of system (1) is the equation for the magnetization that express the condition of the magnetization is frozen in a magnetic fluid. The latter

two equations of system (1) are the Maxwell magnetostatic equations, where Ψ is a scalar potential of the magnetic field.

The purpose of this work is to obtain the Hamiltonian equations of ferrohydrodynamics with frozen magnetization.

2.   Hamiltonian Description of the Ideal Nonconducting Magnetic Fluid with Frozen Magnetization

The functional of the total energy of the magnetized nonconducting fluid is represented in the form

$$W\left(\rho, s, \vec{m}, \vec{v}\right) = \int \mathrm{d}\vec{x} \left[\rho \frac{v^2}{2} + \rho\epsilon\left(\rho, s, \vec{m}\right) - \rho(\vec{m}, \vec{H}) - \frac{H^2}{8\pi}\right]. \tag{2}$$

Then, the action functional with the Lagrangian with constraints is determined by the formula

$$S = \int L \mathrm{d}t = \int \mathrm{d}t \mathrm{d}\vec{r} \left\{ \rho \frac{v^2}{2} - \rho\epsilon\left(\rho, s, \vec{m}\right) - \rho\left(\vec{m}\vec{H}\right) + \frac{H^2}{8\pi} + \alpha \left[\frac{\partial s}{\partial t} + v_k \frac{\partial s}{\partial x_k}\right] + \right.$$
$$\left. + \varphi \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho v_k\right)\right] + \lambda_n \left[\frac{\partial m_n}{\partial t} + v_k \frac{\partial m_n}{\partial x_k} - m_k \frac{\partial v_n}{\partial x_k}\right] \right\},$$

where the functions α , φ and λ_n, $(n = 1, 2, 3)$ are the Lagrangian multipliers, and by the twice repeating index, summation from 1 to 3 is performed.

The extended Lagrangian set of equations [6, 7] follow upon setting the functional derivative of the action functional to zero:

$$\frac{\delta S}{\delta a_i} = 0, \tag{3}$$

where $a_i = \{\varphi_i, v_i, p_i\}$, $\varphi_i = (\rho, s, m_n)$, $p_i = (\varphi, \alpha, \lambda_n)$.

Clebsch representation for the hydrodynamic momentum density is determined by the formula

$$\frac{\delta S}{\delta v_k} = 0 \Longrightarrow \pi_k = \rho v_k = \rho \frac{\partial \varphi}{\partial x_k} - \alpha \frac{\partial s}{\partial x_k} - \lambda_n \frac{\partial m_n}{\partial x_k} - \frac{\partial \left(\lambda_n m_n\right)}{\partial x_k}. \tag{4}$$

If we then introduce a generalized momentum conjugate to generalized coordinates $\varphi_i = (\rho, s, m_n)$ and construct the extended Hamiltonian of the system as the Legendre transformation, then we obtain that (i) the Lagrange multipliers in the extended Hamiltonian formalism play the role of generalized momentum p_i and (ii) the extended Hamiltonian does not contain nonphysical variables, or Lagrangian multipliers, and coincides with the functional of the total energy (2). As a result, taking into account (4), the system (3) is equivalent to the extended Hamiltonian system

$$\frac{\partial \varphi_i}{\partial t} = \frac{\delta H}{\delta p_i}, \quad \frac{\partial p_i}{\partial t} = -\frac{\delta H}{\delta \varphi_i}. \tag{5}$$

3.   Poisson Brackets Method in Ferrohydrodynamics

Let us show the way in which the Hamiltonian equations for physical variables is obtained in the context of the method of Poisson brackets. The results presented below are obtained using formula (2) for the Hamiltonian of ferrohydrodynamics, formula (4)

for the hydrodynamic momentum, and a known property of the Poisson bracket (see, e.g., [6, 8]):

$$\{H(F_1 F_2, \ldots, F_n), F_k\} = \sum_i^h \int \mathrm{d}\vec{x}' \frac{\delta H}{\delta F_i(\vec{x}')}\{F_i(\vec{x}'), F_k\}. \tag{6}$$

Calculating the reciprocal Poisson brackets for physical fields $\varphi_i = (\rho, s, m_n)$ taking into account (4) and requiring the resultant density of hydrodynamic forces to be independent of velocity, we obtain (see also [6, 8, 9]):

$$\{\rho, \rho'\} = 0, \quad \{\rho, s'\} = 0, \quad \{m_n, m_k'\} = 0, \quad \{\rho, \pi_k'\} = \rho(x') \frac{\partial}{\partial x_k'} \delta(x' - x),$$

$$\{s, \pi_k'\} = -\frac{\partial s(x')}{\partial x_k'} \delta(x' - x), \quad \{\pi_i, \pi_k'\} = \partial_i'(\pi_k' \delta) - \partial_k(\pi_i \delta),$$

$$\{m_n, \pi_k'\} = -\delta(x' - x) \frac{\partial m_n}{\partial x_k} - \delta_{nk} \frac{\partial}{\partial x_\alpha'}[m_\alpha(x') \delta(x' - x)].$$

Therefore, the Hamiltonian equations of motion are formulated taking into account (2) and the set of commutation relations presented above. They have the following final form:

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} = -\frac{\partial}{\partial x_k}(\rho v_k),$$

$$\frac{\partial m_n}{\partial t} = \{H, m_n\} = v_k \frac{\partial m_n}{\partial x_k}, \quad \frac{\partial s}{\partial t} = \{H, s\} = -v_k \frac{\partial s}{\partial x_k}, \tag{7}$$

$$\frac{\partial \pi_i}{\partial t} = \{H, \pi_i\} = -\frac{\partial}{\partial x_i}\left(\rho^2 \frac{\partial \epsilon}{\partial \rho}\right) - \partial_k(\pi_i v_k) +$$

$$+ \rho m_\alpha \frac{\partial}{\partial x_a}\left(\frac{\partial \epsilon}{\partial m_i}\right) + \left(\frac{\partial \epsilon}{\partial m_i} - H_i\right) \frac{\partial \rho m_j}{\partial x_j}.$$

It is easy to notice that these Hamiltonian equations coincide with the equations of system (1).

4.   Conclusion

Therefore, in this work, the Hamiltonian set of equations of ferrohydrodynamics (5) is for the first time constructed with the use of Hamiltonian (2) of the system and of the method of Poisson brackets.

We can apparently affirm that the experimental verification of the suggested Hamiltonian theory will allow one to create the complete theory of ferrohydrodynamics.

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Гамильтонова формулировка идеальной феррогидродинамики

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Дано гамильтоново описание уравнений феррогидродинамики для идеальной непроводящей сжимаемой магнитной жидкости с вмороженной намагниченностью.

Ключевые слова: формализм Гамильтона, феррогидродинамика, магнитная жидкость.