Физика

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On Two-Field Solitons in 2 and 3 Dimensions

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We study two- and three-dimensional stationary solitons with non-trivial topology in gauge-invariant nonlinear sigma models (NSMs) describing interaction of scalar unit S^N fields with gauge vector $SU(N-1)$ fields, $N = 2, 3$.

Key words and phrases: nonlinear sigma models, solitons.

1. Introduction

Nonlinear sigma models (NSMs) are of great importance in the modern mathematical physics, which is due to their universality: they appear in various branches of fundamental science. Classical NSMs describe evolution in time of N-component unit isovector field $s_a(\mathbf{x}, t)$ in $(D+1)$-dimensional space-time $(a = 1, \dots, N+1)$; field manifolds of these models are unit spheres S^N. The most interesting cases correspond to $D = 2, 3$ and $N = 2, 3$.

Below we discuss the $A3M$ model with $(N = 2, D = 2)$, introduced in [1], and the $A4YM$ model with $(N = 3, D = 3)$, introduced in [2]. The $A4YM$ model is the straightforward extension of the $A3M$ model. On the other hand, one can see deep resemblance of the $A4YM$ model with the bosonic sector of the reduced electroweak Salam-Weinberg theory, widely known as SU2-Higgs model, in which radial degree of freedom of the Higgs field is frozen (see the A4YM Lagrangian in Sec. 2). In fact, our gauged NSMs include: i) unit length scalar $(N+1)$-component field, with values on $S^N : {s_1}^2 + ...{s_{N+1}}^2 = 1$ $(N = 2, 3)$, interacting with ii) vector field with $U(1)$ or $SU(2)$ symmetry (Maxwell or Yang-Mills).

2. A3M model in 2 dimensions

Consider minimal interaction of the S^2 scalar field (A3-field) with the Maxwell field $A_\mu(x)$. The resulting "A3M model" is described by the gauge-invariant Lagrangian:

$$\begin{gathered}\mathcal{L} = \bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3 - V(s_a) - \frac{1}{4} F_{\mu\nu}^2, \\ \bar{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - ieA_\mu, \\ s_+ = s_1 + is_2, \quad s_- = s_1 - is_2, \\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta(1 - s_3^2),\end{gathered} \tag{1}$$

where β, e are coupling constants and $\mu, \nu = 0, 1, 2$. The localized distributions of unit isovector $s_a(\mathbf{x})$ in this model are divided into classes with different topological inidices ("charges") Q_t; solitons with nonzero topological charges are referred to as

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"topological solitons" [1]. We look for the topological solitons of the A3M model using the "hedgehog-like" ansatz for the A3-field

$$s_1 = \cos m\chi \sin\vartheta(R), \quad s_2 = \sin m\chi \sin\vartheta(R), \quad s_3 = \cos\vartheta(R),$$
$$\sin\chi = \frac{y}{R}, \quad \cos\chi = \frac{x}{R}, \quad R^2 = x^2 + y^2,$$

where m is an integer number. We use also the standard "vortex" ansatz for the vector field A_μ, describing localized distributions of a stationary magnetic field:

$$A_0 = 0, \quad A_1 = A_x = -ma(R)\frac{y}{R^2}, \quad A_2 = A_y = ma(R)\frac{x}{R^2}.$$

For them $Q_t = m$.

After rescaling ($a = \alpha e^{-1}, \quad R = re^{-1}$,) we calculate $\delta H/\delta\vartheta$ and $\delta H/\delta\alpha$, arriving at coupled equations for $\vartheta(r)$ and $\alpha(r)$

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\vartheta}{\mathrm{d}r} - \sin\vartheta\cos\vartheta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0, \tag{2}$$

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\alpha}{\mathrm{d}r} + 2\sin^2\vartheta(1-\alpha) = 0, \tag{3}$$

to be solved under the following boundary conditions:

$$\vartheta(0) = \pi, \quad \vartheta(\infty) = 0, \tag{4}$$

$$\alpha(0) = 0, \quad \frac{\mathrm{d}\alpha}{\mathrm{d}r}(\infty) = 0. \tag{5}$$

Using series expansion of $\vartheta(r)$ and $\alpha(r)$ at $r \to 0$, we find from Eqs. (2) and (3) for $m = 1$

$$\vartheta(r) = \pi - C_1 r + o(r),$$
$$\alpha(r) = r^2\left(E_1^2 - \frac{1}{4}C_1^2 r^2\right) + o(r^4),$$

and for $m = 2$

$$\vartheta(r) = \pi - C_2 r^2 + o(r^2),$$
$$\alpha(r) = r^2\left(E_2^2 - \frac{1}{12}C_2^2 r^4\right) + o(r^6).$$

The asymptotic form of the soliton solution for $r \to \infty$ is:

$$\vartheta(r) \approx \frac{T}{\sqrt{r}}\exp(-\sqrt{p}r), \quad T = \text{const},$$
$$\alpha(r) \approx \alpha_\infty - (1-\alpha_\infty)\frac{T^2}{2rp}\exp(-2\sqrt{p}r).$$

We studied the problem (2)–(5) by various numerical methods, among them shooting technique, stabilization method. The method based on power and asymptotic series and on the analytic continuation technique (re-expansions and Pade approximants) was used as well [3].

Solutions exist and are stable for the values of dimensionless anisotropy parameter

$$0 < p < p_{cr} \approx 0.41.$$

The plots of radial functions $\alpha(r)$ and $\vartheta(r)$ and corresponding distributions of energy density and magnetic field have been presented in [1]. Later we have found the dependence of $Q_t = 1$ soliton energy on parameter p [4]. It is interesting to note that the dependence $\alpha(\infty)$ on p proved to be surprisingly symmetric (see Fig. 1). Presently the only way to explain such a symmetry is to refer to high $(U(1) \otimes Z(2))$ symmetry of the $A3M$ model (1). Then we studied $Q_t = 2$ solitons. We have found that for all $0 < p < p_{cr} \approx 0.41$ their energies turned out to satisfy inequality $E_{sol}(Q_t = 2, p) < 2 * E_{sol}(Q_t = 1, p)$. This means that two $Q_t = 1$ solitons attract to each other, forming the $Q_t = 2$ bound states as a result of initial configuration evolution.

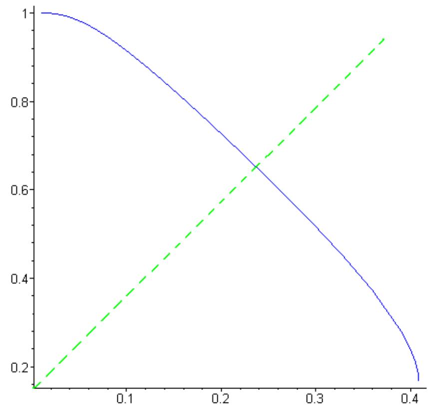

Figure 1. $\alpha(\infty)$ **vs** p **for the** $A3M$ **model.**

3. A4YM model for D = 3

Further we shall consider another gauged sigma model, which describe minimal interaction of the easy-axis 4-component unit isovector field $q^\alpha(x^\mu)$ ("the A4-field") interacting with the vector $SU(2)$ Yang-Mills field $A^a_\mu(x^\nu)$.

The Lagrangian density of this ("the A4YM") model is:

$$\mathcal{L} = \mathcal{D}_\mu q^a \mathcal{D}^\mu q^a + \partial_\mu q^0 \partial^\mu q^0 - V(q^0) - \frac{1}{4}(F^a_{\mu\nu})^2,$$
$$\mathcal{D}_\mu q^a = \partial_\mu q^a + g\varepsilon^{abc} A^b_\mu q^c,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu,$$
$$V(q^0) = \beta\left[1 - (q^0)^2\right],$$

where $\alpha, \mu, \nu = 0, 1, 2, 3;\ a, b, c = 1, 2, 3; \beta, g$ are coupling constants.

First we looked for stationary topological solitons of the A4YM model using the following ansatz for the A4- and the $SU(2)$ Yang-Mills fields:

$$q^0 = \cos\vartheta(R), \quad q^a = \sin\vartheta(R)\frac{x^a}{R}, \quad R^2 = x^2 + y^2 + z^2,$$
$$A^a_0 = 0, \quad A^a_i = c(R)\varepsilon^{iak} x^k.$$

Then the Hamiltonian density distributions of localized field bunches are spherically symmetric:

$$\mathcal{H}_{st}(R) = \left(\frac{\mathrm{d}\vartheta}{\mathrm{d}R}\right)^2 + \frac{2\sin^2\vartheta}{R^2} + 4gc\sin^2\vartheta + 2g^2c^2R^2\sin^2\vartheta + 6c^2 + \\ + \left(\frac{\mathrm{d}c}{\mathrm{d}R}\right)^2 R^2 + \frac{1}{2}g^2c^4R^4 + 4Rc\frac{\mathrm{d}c}{\mathrm{d}R} + 2gR^2c^3 + \beta\sin^2\vartheta.$$

Introduce dimensionless variables $r = gR$, $b(r) = g^{-1}cr^2$.

Calculating $\delta\mathcal{H}/\delta\vartheta$ and $\delta\mathcal{H}/\delta r$, we get coupled equations $\left(P = \dfrac{\beta}{g^2}\right)$

$$\begin{gathered} \frac{\mathrm{d}^2\vartheta}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\vartheta}{\mathrm{d}r} - \sin\vartheta\cos\vartheta\left[\frac{2(b+1)^2}{r^2} + P\right] = 0, \\ \frac{\mathrm{d}^2 b}{\mathrm{d}r^2} - \frac{2b}{r^2} - 2\sin^2\vartheta(1+b) - \frac{b^2}{r^2}(b+3) = 0. \end{gathered} \tag{6}$$

When searching for localized solutions we set the following boundary conditions:

$$\vartheta(0) = \pi, \quad \vartheta(\infty) = 0, \quad b(0) = 0, \quad b(\infty) = G. \tag{7}$$

Solutions to above problem (6)–(7) would define localized distributions $q^\alpha(x^k)$ ($\alpha = 0, 1, 2, 3$, and $k = 1, 2, 3$) of the A4-field with unit topological charge, $Q_t = 1$. Here Q_t is the "mapping degree" of continuous maps $R^3_{comp} \to S^3$. However such solutions have not been found. Because of that we look for more general ansatz.

More general ansatz keeps the "hedgehog" form for q^α and a generalized expression for A_i:

$$A^a_i(x) = \epsilon_{aij}n_j C(R)R + (\delta_{ai} - n_a n_i)\frac{B(R)}{R} + n_a n_i \frac{D(R)}{R}. \tag{8}$$

However such ansatz should respect Lorentz gauge. Equating $\dfrac{\partial A^a_i}{\partial x_i} = 0$, we find

$$B(R) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}R}(R^2 D(R)). \tag{9}$$

Finally we obtain the ansatz defined by equations (8) and (9).

We calculate the hamiltonian density $\mathcal{H}_{st}(R)$ for such ansatz using the computer algebra system Maple [5]. Equating variational derivatives $\dfrac{\delta\mathcal{H}_{st}(R)}{\delta C}, \dfrac{\delta\mathcal{H}_{st}(R)}{\delta D}, \dfrac{\delta\mathcal{H}_{st}(R)}{\delta\vartheta}$ to 0, we obtain coupled equations for radial functions $C(R), D(R), \vartheta(R)$. Their solutions (if exist) define localized soliton solutions to A4YM model. The study of coupled equations for $C(R), D(R), \vartheta(R)$ is in progress.

4. Conclusions

In this paper we discussed the existence and properties of localized solutions of the $A3M$ model ($D = 2$) and the A4YM model ($D = 3$). Topological solitons of these models can be considered as soliton analogues of the so-called defect solutions: $2D$ strings-vortices in the Abelian Higgs model [6, 7] and $3D$ 't Hooft-Polyakov "hedgehogs"-monopoles [8–10] correspondingly.

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Двухполевые солитоны в двух- и трехмерном пространстве

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В работе исследуются двух- и трехмерные стационарные солитоны с нетривиальной топологией в калибровочно-инвариантных нелинейных сигма-моделях, описывающих взаимодействие скалярных полей со значениями на сферах S^N с калибровочными векторными $SU(N-1)$ полями ($N = 2, 3$).

Ключевые слова: нелинейные сигма-модели, солитоны.