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Nonstandard Finite Difference Schemes for Reaction-Diffusion Equations

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A nonstandard, implicit finite difference scheme for reaction-diffusion equation was constructed. This scheme is an extension of the method of Mickens. Conditions of positivity and boundedness are established.

Key words and phrases: reaction-diffusion equation, nonstandard finite difference scheme, positivity, boundedness.

1. Introduction

As is known, [1, 2], a large class of important phenomena in the natural and engineering sciences can be represented by a system of nonlinear reaction-diffusion PDEs in one-space dimension:

$$\frac{\partial u_i}{\partial t} = D_i \frac{\partial^2 u_i}{\partial x^2} + f_i(u_1, u_2, \ldots, u_n) u_i, \quad i = 1, 2, \ldots, n, \tag{1}$$

where D_i are nonnegative diffusion coefficients, and the f_i are rational functions of $(u_1, u_2, \ldots, u_n)$. We restrict the discussion to systems modeled by (1) for which the $u_i(x, t)$ represent density variables. Consequently,

$$u_i(x, 0) \geqslant 0 \quad \Rightarrow \quad u_i(x, t) \geqslant 0,$$

which is a positivity condition on the dependent variables.

We will consider the system of equations (1) along with the initial condition

$$u_i(x, 0) = \varphi_i(x) \geqslant 0, \quad x \in (0, l), \tag{2}$$

and either Dirichlet boundary conditions

$$u_i(0, t) = \omega_i(t), \quad u_i(l, t) = \vartheta_i(t), \tag{3}$$

or Neumann boundary conditions

$$\frac{\partial u_i}{\partial x}(0, t) = -\omega_i(t), \quad \frac{\partial u_i}{\partial x}(l, t) = \vartheta_i(t), \tag{4}$$

where $\omega_i(t) \geqslant 0$, $\vartheta_i(t) \geqslant 0$ are given functions.

We will use the following notations:

$$t \to t_k = k\tau, \quad k = 0, 1, \ldots, \quad x \to x_m = mh, \quad m = 0, 1, \ldots, N,$$

where τ and h are, respectively, the time and space step-sizes; and the discrete approximations to the dependent variables are

$$u_i\left(x_m, t_k\right) \to [y_i]_m^k, \quad i = 1, 2, \ldots, n.$$

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2. The Nonstandard Finite Difference Schemes

In this article we consider a nonstandard finite difference schemes of the form

$$\frac{[y_i]_m^{k+1}-[y_i]_m^k}{\tau} = D_i\left((1-\sigma)[y_i]_{\bar{x}x,m}^k + \sigma[y_i]_{\bar{x}x,m}^{k+1}\right) + \\ + dp_i(y)[y_i]_m^k - \left((d-1)p_i(y)+q_i(y)\right)[y_i]_m^{k+1}, \quad (5)$$

where $\sigma \in [0,1]$ and $d \geqslant 1$ are parameters and $p_i(y)$ and $q_i(y)$ are, respectively, the collection of functions obtained from the positive and negative coefficient terms in f_i. Note that $p_i(y)$ and $q_i(y)$ are evaluated at t_k and x_m i.e., in detail

$$p_i(y) = p_i\left([y_1]_m^k, [y_2]_m^k, \dots, [y_n]_m^k\right),$$
$$q_i(y) = q_i\left([y_1]_m^k, [y_2]_m^k, \dots, [y_n]_m^k\right).$$

The functions $p_i(y)$ and $q_i(y)$ also have the important property

$$p_i(y) \geqslant 0, \quad q_i(y) \geqslant 0 \quad \text{if} \quad [y_i]_m^k \geqslant 0, \quad i = 1, 2, \dots, n.$$

Note that the schemes (5) are implicit in general and we rewrite them as follows:

$$-a_m[y_i]_{m-1}^{k+1} + c_m[y_i]_m^{k+1} - b_m[y_i]_{m+1}^{k+1} = F\left([y_i]_m^k\right), \quad (6)$$

where

$$a_m = b_m = \sigma\gamma D_i, \quad \gamma = \frac{\tau}{h^2},$$
$$c_m = 1 + 2\sigma\gamma D_i + \tau\left((d-1)p_i(y) + q_i(y)\right),$$
$$F\left([y_i]_m^k\right) = (1-\sigma)\gamma D_i\left([y_i]_{m-1}^k + [y_i]_{m+1}^k\right) + \left(1 + d\tau p_i(y) - 2(1-\sigma)\gamma D_i\right)[y_i]_m^k. \quad (7)$$

The discrete initial conditions are

$$[y_i]_m^0 = \varphi_i\left(x_m\right), \quad m = 0, 1, \dots, N. \quad (8)$$

The discrete boundary conditions corresponding to (3), (4) are

$$[y_i]_0^k = \omega_i\left(t_k\right), \quad [y_i]_N^k = \vartheta_i\left(t_k\right), \quad k = 0, 1, \dots, \quad (9)$$

and

$$\frac{[y_i]_1^k - [y_i]_{-1}^k}{2h} = -\omega_i\left(t_k\right), \quad \frac{[y_i]_{N+1}^k - [y_i]_{N-1}^k}{2h} = \vartheta_i\left(t_k\right), \quad (10)$$

respectively. The equations (6) together with (8), (9) or (8), (10) must be form a closed system.

We first consider the discrete Dirichlet boundary condition (9). Since $[y_i]_0^{k+1}$ and $[y_i]_N^{k+1}$ are known via the condition (9), we can transfer these quantities from equation (6) for $m = 1$ and $m = N-1$ to the right hand side respectively. As a result we obtain

$$\begin{cases} c_1[y_i]_1^{k+1} - b_1[y_i]_2^{k+1} = F\left([y_i]_1^k\right), \\ -a_m[y_i]_{m-1}^{k+1} + c_m[y_i]_m^{k+1} - b_m[y_i]_{m+1}^{k+1} = F\left([y_i]_m^k\right), \quad m = 2, \dots, N-2, \\ -a_{N-1}[y_i]_{N-2}^{k+1} + c_{N-1}[y_i]_{N-1}^{k+1} = F\left([y_i]_{N-1}^k\right), \end{cases} \quad (11)$$

where

$$F\left([y_i]_1^k\right) = a_1\omega_i\left(t_{k+1}\right) + (1-\sigma)\gamma D_i\left(\omega_i\left(t_k\right) + [y_i]_2^k\right) +$$

$$+\left(1+d\tau p_i(y)-2(1-\sigma)\gamma D_i\right)[y_i]_1^k, \quad (12)$$

$$F\left([y_i]_{N-1}^k\right)=b_{N-1}\vartheta\left(t_{k+1}\right)+(1-\sigma)\gamma D_i\left(\vartheta\left(t_k\right)+[y_i]_{N-2}^k\right)+$$
$$+\left(1+d\tau p_i(y)-2(1-\sigma)\gamma D_i\right)[y_i]_{N-1}^k.$$

Now we proceed to the discrete Neumann boundary condition (10). As in [3, 4] the discrete boundary conditions (10) are incorporated by introducing the auxiliary points $x_{-1}=x_0-h$ and $x_{N+1}=x_N+h$. By using the difference equation (6) at $m=0$ and first relation in (10), we can eliminate the unknown $[y_i]_{-1}^k$. And, of course, the unknown $[y_i]_{N+1}^k$ can be eliminated in a similar way. As a result, we obtain

$$\begin{cases} c_0[y_i]_0^{k+1}-2b_0[y_i]_1^{k+1}=F\left([y_i]_0^k\right), \\ -a_m[y_i]_{m-1}^{k+1}+c_m[y_i]_m^{k+1}-b_m[y_i]_{m+1}^{k+1}=F\left([y_i]_m^k\right), \quad m=1,\dots,N-1, \\ -2a_{N-1}[y_i]_{N-1}^{k+1}+c_N[y_i]_N^{k+1}=F\left([y_i]_N^k\right), \end{cases} \quad (13)$$

where

$$F\left([y_i]_0^k\right)=2h\gamma D_i\left(\sigma\omega_i(t_{k+1})+(1-\sigma)\omega_i\left(t_k\right)\right)+2(1-\sigma)\gamma D_i[y_i]_1^k+$$
$$+\left(1+d\tau p_i(y)-2(1-\sigma)\gamma D_i\right)[y_i]_0^k, \quad (14)$$

$$F\left([y_i]_N^k\right)=2h\gamma D_i\left(\sigma\vartheta_i\left(t_{k+1}\right)+(1-\sigma)\vartheta_i\left(t_k\right)\right)+2(1-\sigma)\gamma D_i[y_i]_{N-1}^k+$$
$$+\left(1+d\tau p_i(y)-2(1-\sigma)\gamma D_i\right)[y_i]_N^k.$$

We are ready to show that

Theorem 1. *The positivity condition holds under condition*

$$(1-\sigma)\gamma \leqslant \frac{1+d\tau p_i}{2D_i}, \quad i=1,\dots,n. \quad (15)$$

Proof. The systems of equations (11) and (13) can be rewritten as a matrix from

$$A\,[y_i]^{k+1}=F\left([y_i]^k\right).$$

It is easy to see that the tri-diagonal matrix A is a strictly diagonally dominant and monotone type. Moreover, from (7),(12) and (14), it follows that

$$F\left([y_i]\right)^k \geqslant 0,$$

under the assumptions $[y_i]_m^k \geqslant 0$ and (15). The Theorem 1 has proved. □

Now we consider a boundedness property of the solution to (1), (2) and (3)–(4).

Theorem 2. *Assume that*

(*i*) $0 \leqslant [y_i]_m^k \leqslant M, \quad m=0,1,\dots,N,$

(*ii*) *the condition (1.21) holds,*

(*iii*) $\left(1+d\tau p_i(y)-2(1-\sigma)\gamma D_i\right)\left(M-[y_i]_m^k\right) \geqslant M\tau\left(p_i(y)-q_i(y)\right),$
$m=1,2,\dots,N-1.$

Then

$$0 \leqslant [y_i]_m^{k+1} \leqslant M \quad \textit{for all} \quad m=0,1,\dots,N.$$

Proof. Theorem 1 is valid under assumptions (i) and (ii), i.e., $[y_i]_m^{k+1} \geqslant 0$ for all $m = 0, 1, \ldots, N$. Moreover, we have [5]

$$\max_m [y_i]_m^{k+1} \leqslant \frac{F\left([y_i]_m^k\right)}{r_m},$$

where

$$r_m = c_m - a_m - b_m.$$

It is easy to seen that the condition (iii) is sufficient for

$$\frac{F\left([y_i]_m^k\right)}{r_m} \leqslant M.$$

It means that $[y_i]_m^{k+1} \leqslant M$ for all $m = 0, 1, \ldots, N$, which completes the proof of Theorem 2. □

Remark. The positivity condition (15) is valid for any time and space step sizes τ, h for implicit scheme (5) with $\sigma = 1$.

3. Applications

We consider some particular cases of problem (1), (2) and (3).

The Nagumo equation. Now, we consider the reaction-diffusion equation

$$u_t = Du_{xx} + \left(\alpha - \beta u^2\right) u, \quad 0 \leqslant x \leqslant L, \quad t > 0$$

with $D > 0$, $\alpha > 0$, $\beta > 0$ and

$$0 \leqslant u(x,t) \leqslant \sqrt{\frac{\alpha}{\beta}}\ .$$

The difference scheme is

$$\frac{y_m^{k+1} - y_m^k}{\tau} = D\left((1-\sigma)y_{\bar{x}x,m}^k + \sigma y_{\bar{x}x,m}^{k+1}\right) + d\alpha y_m^k - \left((d-1)\alpha + \beta\left(y_m^k\right)^2\right) y_m^{k+1}.$$

The positivity and boundedness of the solutions is satisfied if the condition

$$(1-\sigma)\gamma \leqslant \frac{1 + (d-2)\tau\alpha}{2D}$$

holds.

Finally, we consider a set of PDEs:

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + \left[f\left(u_1\right) - u_2\right] u_1, \\ \frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} + \left[u_1 - g\left(u_2\right)\right] u_2, \end{aligned} \tag{16}$$

where

$$f\left(u_1\right) = a\left(a_1 + b_1 u_1 - u_1^2\right), \quad g\left(u_2\right) = a_2 + b_2 u_2,$$

and a, a_1, a_2, b_1, b_2 are nonnegative parameters. This system is a model for the description of the patchy distributions of microscopic aquatic organisms and was constructed by Mimura and Mirray.

The finite difference scheme for this set of equations is

$$\frac{y_m^{k+1}-y_m^k}{\tau} = D_1\left[(1-\sigma)y^k_{\bar{x}x,m}+\sigma y^{k+1}_{\bar{x}x,m}\right] + da\left(a_1+b_1y_m^k\right)y_m^k - \\ -\left[(d-1)a\left(a_1+b_1y_m^k\right)+a\left(y_m^k\right)^2+w_m^k\right]y_m^{k+1},$$

$$\frac{w_m^{k+1}-w_m^k}{\tau} = D_2\left[(1-\sigma)w^k_{\bar{x}x,m}+\sigma w^{k+1}_{\bar{x}x,m}\right] + dy_m^kw_m^k - \\ -\left[(d-1)y_m^k+a_2+b_2w_m^k\right]w_m^{k+1}.$$

The positivity condition becomes

$$\gamma(1-\sigma)\leqslant\frac{1}{2D},\quad D=\min(D_1,D_2).$$

One can check that the sufficient boundedness requirement (*iii*) with $M=1$ is fulfilled if

$$a>0,\quad a_2\geqslant 1,\quad b_2\geqslant 0,\quad a_1+b_1\leqslant 1,\quad d=\frac{a_1+b_1}{b_1}\geqslant 1. \tag{17}$$

From (17), we can see that the parameter $d\geqslant 1$ plays as a control parameter. So, by choosing the value of d in suitable way, one can satisfy boundedness requirement for solution of (17).

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Нестандартные конечно-разностные схемы для уравнений реакции-диффузии

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Построена нестандартная, неявная конечно-разностная схема для уравнения реакции–диффузии. Эта схема является обобщением метода Микенса. Установлены условия положительности и ограниченности решения.

Ключевые слова: уравнения реакции-диффузии, нестандартная конечно-разностная схема, положительность и органиченность.