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Symbolic Solving of Differential Equations with Partial Derivatives

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An algorithm for the symbolic solving of systems of linear partial differential equations by means of multivariate Laplace–Carson transform (LC) is produced. Considered is a system of K linear equations with M as the greatest order of partial derivatives and right hand parts of a special type, that permits a symbolic Laplace–Carson transform. Initial conditions are input. As a result of Laplace–Carson transform of the system according to the initial conditions, we obtain an algebraic system of equations. There exist efficient methods to solve large size systems of such types. It gives a possibility to implement the method for solving the large PDE systems. A method to obtain compatibility conditions is discussed. The application of LC allows one to execute it in a symbolic way.

Ключевые слова: systems of partial differential equations, Laplace-Carson transform, symbolic solving.

1. Introduction

The Laplace transform has been useful in various problems of differential equations theory, including problems of partial equations (for example [**?**, 1–5]). On the other hand, there are many ways to use computer algebra systems for numerical or symbolic solving of PDE systems, for example the well known usage of MAPLE for characteristics method that permits to simplify equations in many cases (for instance [6, 7]).

We produce an algorithm for symbolic solving of systems of linear partial differential equations by means of multivariate Laplace–Carson transform. Considered are the systems of arbitrary number K of unknown functions and equations of arbitrary order M of derivatives in the cases, described in section 2, under conditions a)-b). The method allows one not to reduce (or to reduce to canonical form) the problem at initial stage, it reduces it to solving a linear algebraic system with polynomial coefficients where efficient methods were developed (for example [8–10]). So large systems of linear PDE may be solved in real time.

The application of Laplace–Carson transform permits to obtain compatibility conditions in a symbolic way for many types of PDE equations and systems of PDE equations.

2. Method

Consider the space S of functions $f(x)$, $x = (x_1, \dots, x_n) \in \mathbf{R}^n_+$, $\mathbf{R}^n_+ = \{x : x_i \geqslant 0,\ i = 1, \dots, n\}$, for which $\mathcal{M} > 0, a = (a_1, \dots, a_n) \in \mathbf{R}^n$, $a_i > 0$, $i = 1, \dots, n$, exist such that for all $X \in \mathbf{R}^n_+$ the following is true: $|f(x)| \leqslant \mathcal{M}e^{ax}$, $\quad ax = \sum\limits_{i=1}^{n} a_i x_i$.

On the space S the Laplace–Carson transform (LC) is defined as follows:

$$LC : f(x) \mapsto F(p) = p^1 \int\limits_0^\infty e^{-px} f(x) \mathrm{d}x,$$

$$p = (p_1, \dots, p_n), \quad p^1 = p_1 \dots p_n, \quad px = \sum_{i=1}^{n} p_i x_i, \quad \mathrm{d}x = \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

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Denote $\widetilde{m} = (m_1, \ldots, m_n)$. Consider a system

$$\sum_{k=1}^{K} \sum_{m=0}^{M} \sum_{\widetilde{m}} a^j_{\widetilde{m}k} \frac{\partial^m}{\partial^{m_1} x_1 \ldots \partial^{m_n} x_n} u_k(x) = f_j(x), \tag{1}$$

where $j = 1, \ldots, K$, $u_k(x)$, $k = 1, \ldots, K$, — are unknown functions of $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$, $f_j \in S$, $a^j_{\widetilde{m}k}$ are real numbers, m is the order of a derivative, and k — the number of an unknown function. Here and further summing by $\widetilde{m} = (m_1, \ldots, m_n)$ is executed for $m_1 + \ldots + m_n = m$.

We solve a problem with initial conditions for each variable. Introduce notations for them. Denote by Γ^ν a set of vectors $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $\gamma_\nu = 1$, $\gamma_i = 0$, if $i < \nu$, and γ_i equals 0 or 1 in all possible combinations for $i > \nu$. The amount of elements in Γ^ν equals $2^{\nu-1}$.

Denote $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i = 0, \ldots, m_i$, a set of indexes such that the derivative of $u^k(x)$ of the order β_i with respect to the variables with numbers i equals $u^k_{\beta,\gamma}(x^{(\gamma)})$ at the point $x = x^\gamma$ with zeros at the positions μ for which the coordinates γ_μ of γ equal 1. For example, if zeros stand only at the places with the numbers $1, 2, 3$, then $\gamma = (1, 1, 1, 0, \ldots, 0)$.

Let $LC : u^k \mapsto U^k, u^k_{\beta,\gamma}(x^{(\gamma)}) \mapsto U^k_{\beta,\gamma}(p^{(\gamma)}), f_j \mapsto F_j$, the notation $p^{(\gamma)}$ is correspondent to the notation $x^{(\gamma)}$. Denote by $\|\gamma\|$ the "length" of γ — the number of units in γ, $p^{\widetilde{m}} = p_1^{m_1} \ldots p_n^{m_n}$.

Then

$$LC : \frac{\partial^m}{\partial^{m_1} x_1 \ldots \partial^{m_n} x_n} u_k(x) \ \mapsto$$
$$\mapsto p^{\widetilde{m}} U^k(p) \ + \sum_{\nu=1}^{n} \sum_{\beta_\nu=0}^{m_\nu} \sum_{\gamma \in \Gamma^\nu} (-1)^{\|\gamma\|} p_1^{m_1-\beta_1-\gamma_1} \ldots p_n^{m_n-\beta_n-\gamma_n} U^k_{\beta,\gamma}(p^{(\gamma)}).$$

Denote

$$\Phi^j_{mk} = \sum_{\widetilde{m}} a^j_{\widetilde{m}k} \sum_{\nu=1}^{n} \sum_{\beta_\nu=0}^{m_\nu} \sum_{\gamma \in \Gamma^\nu} (-1)^{\|\gamma\|} p_1^{m_1-\beta_1-\gamma_1} \ldots p_n^{m_n-\beta_n-\gamma_n} U^k_{\beta,\gamma}(p^{(\gamma)}).$$

As a result of Laplace–Carson transform of the system (1) according to initial conditions we obtain an algebraic system relative to U^k

$$\sum_{k=1}^{K} \sum_{m=0}^{M} \sum_{\widetilde{m}} a^j_{\widetilde{m}k} p^{\widetilde{m}} U^k(p) = F_j - \sum_{k=1}^{K} \sum_{m=0}^{M} \Phi^j_{mk}, \quad j = 1, \ldots, K. \tag{2}$$

The algorithm component is the definition of compatible initial conditions. The system (1) should be solved under such conditions.

Denote by D the determinant of the system (2), D_i — the maximal order minors of the extended matrix of (2). A case when there is a set $\mathcal{Q}$ of zeros of D with infinite limit point at $\operatorname{Re} p_k > 0$, $k = 1, \ldots, n$ is of most interest. Solving the system (2), we obtain U^k as fractions with D in the denominators. The inverse Laplace–Carson transform is possible if α_k, $k = 1, \ldots, n$ exist such that these functions are holomorphic in the domain $\operatorname{Re} p_k > \alpha_k$,. So we make a demand: $D_i = 0$ at $\mathcal{Q}$. This demand produces requirements to the LC images of initial conditions functions, and after LC^{-1} transform to initial conditions. They turns to be dependent. We obtain the so-called compatibility conditions.

The algorithm of solving the system (1) consists of four main steps:

1. Laplace–Carson transform of the system (1).

2. Solving of the algebraic system (2).
3. Establishing of compatibility conditions.
4. Inverse Laplace–Carson transform of the solutions of (2) — it is the solution of the system (1).

At the present stage of research we guarantee symbolic computations if we may carry out the following:

a) ***For LC***:

Represent input functions as sums of exponents with polynomial coefficients.

Call a rational fraction *“a proper fraction”* if the degree of each variable (over $\mathbf{C}$) in nominator is less than its degree in denominator.

b) ***For LC***$^{-1}$:

– represent the solutions of algebraic system as sums of proper fractions with exponential coefficients;
– reduce the denominator of these proper fractions to a product of functions linear with respect to each variable.

Note that the class **b)** does not exhaust all cases that admit pure symbolic computations. We produce two very simple examples: for a case from **b)** and out of it. It is convenient in these cases to change notations for unknown functions, their Laplace transform, variables, initial conditions.

3. Examples

1. Take a system of two equations with two unknown functions on $\mathbf{R}^2_+$.

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = x, \quad \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} = y, \quad f = f(x,y); \quad g = g(x,y).$$

Initial conditions:

$$f(0,y) = a(y); \quad f(x,0) = b(x); \quad g(0,y) = c(y); \quad g(x,0) = d(x),$$

$$LC : f(x,y) \mapsto u(p,q), \quad g(x,y) \mapsto v(p,q),$$

$$a(y) \mapsto \alpha(q), \quad b(x) \mapsto \beta(p), c(y) \mapsto \delta(q), \quad d(x) \mapsto \gamma(p),$$

$$pu - p\alpha(q) + qv - q\gamma(p) = 1/p, \quad qu - q\beta(p) + pv - p\delta(q) = 1/q.$$

Then

$$u = -\frac{-\alpha p^2 + \beta q^2 + (\delta - \gamma)pq}{p^2 - q^2}, \quad v = -\frac{-p^2 + q^2 + (\alpha - \beta)p^2q^2 - (\delta p^2 - \gamma q^2)pq}{pq(p^2 - q^2)}.$$

The denominator D: $D(p,q) = pq(p^2 - q^2)$.
The set of zeros of D with infinite limit points at the right half-plane is $q = p$.
Substituting $q = p$ into the nominator of u and v, we obtain the compatibility condition: $\alpha - \beta + \gamma - \delta = 0$.
For example we may take $\beta = 0$; $\gamma = \dfrac{2}{p}$; $\delta = \dfrac{2}{q}$; $\alpha = 0$.
Then

$$u = -\frac{2}{p+q}, \quad v = -\frac{p + 2p^2 + q + 2q^2 + 2pq}{pq(p+q)}.$$

LC^{-1} :

$$f = -\begin{cases} 2y, & y < x, \\ 2x, & y \geqslant x, \end{cases} \qquad g = \begin{cases} (2+y)x, & y < x, \\ y(2+x), & y \geqslant x. \end{cases}$$

2. Consider the equation of parabolic type $\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial f}{\partial y} = xy$. Initial conditions:

$$f(0,y) = a(y); \quad \left.\frac{\partial f(x,y)}{\partial x}\right|_{x=0} = b(y); \quad f(x,0) = c(x);$$

$$LC: f(x,y) \mapsto u(p,q),\ a(y) \mapsto \alpha(q), \quad b(y) \mapsto \beta(q), \quad c(x) \mapsto \gamma(p),$$

LC: $p^2u - p^2\alpha - p\beta - qu + q\gamma = \dfrac{1}{pq}$.
Then

$$u = \frac{1 + p^3q\alpha + p^2q\beta - pq^2\gamma}{pq(p^2 - q)}.$$

Substituting $p = \sqrt{g}$ into the nominator of u, we obtain the compatibility condition: $\alpha = \gamma; \gamma = -\dfrac{1}{q^2}$. For example $\alpha = \gamma = 1, \quad \gamma = -\dfrac{1}{q^2}$. Correspondingly, we obtain

$$u = \frac{1 + p^3q\alpha + p^2q\beta - pq^2\gamma}{pq(p^2 - q)},$$

and as a result of LC^{-1} $f(x,y) = 1 - \dfrac{xy^2}{2}$ under the initial conditions $a(y) = c(x) = 1$, $b(y) = -\dfrac{y^2}{2}$.

4. Conclusion

Let us adduce advantages of the algorithm presented in the paper.

1. The Laplace–Carson transform is a way for symbolic solving of differential equations as it reduces the solution process to algebraic manipulations.
2. Representation of righthand side functions by sums of exponents with polynomial coefficients (in a case when it is possible) makes the Laplace–Carson transform completely symbolic.
3. The algebraic system obtained after the Laplace transform may be solved by methods most convenient and efficient for each specific case.
4. Representation of the solutions of algebraic system as sums of proper fractions with exponential coefficients provides a symbolic character of the inverse Laplace transform.
5. The application of Laplace–Carson transform permits to obtain compatibility conditions in a symbolic way for many types of PDE equations and systems of PDE equations.

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Символьное решение дифференциальных уравнений в частных производных

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Предлагается алгоритм для символьного решения систем дифференциальных уравнений в частных производных посредством многомерного преобразования Лапласа–Карсона. Рассмотрена система K уравнений с M как наивысшим порядком частных производных и правой частью особого типа, который допускает символьное преобразование Лапласа–Карсона. Начальные условия являются входными. В результате Лаплас–Карсоновского преобразования системы по начальным условиям получаем алгебраическую систему уравнений. Существуют эффективные методы решения систем такого типа. Это дает возможность применять предлагаемый метод для решения больших систем уравнений в частных производных. Обсуждается метод получения условий совместности. Применение преобразования Лапласа–Карсона позволяет выполнить это в символьном виде.

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