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Description of Lepton and Baryon Phases in Skyrme–Faddeev Spinor Model

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8-spinor field is suggested to unify Skyrme and Faddeev models describing baryons and leptons as topological solitons. In the Skyrme model the particles-solitons possess the topological charge of the degree type, which is interpreted as the baryon number. In the Faddeev model the particles are endowed with the topological invariant of the Hopf type, which is interpreted as the Lepton number. The special 8-spinor Brioschi identity is used to include leptons and baryons as two possible phases of the effective spinor field model with Higgs potential depending on the $j^{\mu}j_{\mu}$ being added to the Lagrangian.

In the present paper the generalization of the Mie electrodynamics within the scope of the effective 8-spinor field model is suggested. For this field model the quadratic spinor quantities entering the Brioschi identity are constructed. Also we find the symmetry groups, which generate S^2- and S^3-submanifolds in general S^8 biquadratic spinor manifold.

As a result we have the homotopy groups $\pi_3(S^2)$ and $\pi_3(S^3)$, which describes lepton and baryon phases. To unifying these phases, we find common vacuum state which conserves only one component in two cases. Finally, we obtain the resulting 8-spinor model permitting unified description of baryon and lepton.

Key words and phrases: 8-spinor, topological charge, solitons, homotopy groups, Brioshi identity, Skyrme–Faddeev model, Hopf index.

1. Introduction. Nonlinear 8-spinor field theory

There are known several models, which describe elementary particles as topological solitons. One can mention the magnetic monopole theory, the Skyrme model (1954) and the Faddeev one (1972). The Skyrme's fruitful idea to describe baryons as topological solitons was based on the identification of baryon number B with the topological charge of the degree type $B = \deg(S^3 \to S^3)$. It serves as the generator of the third homotopy group $\pi_3(S^3) = \mathbb{Z}$. The similar idea to describe leptons as topological solitons was announced by Faddeev. He identified the lepton number L with the Hopf invariant Q_H. The unification of these two approaches appears as an attractive goal and an attempt to proceed in this direction was made in [1]. Baryons and leptons should be considered as two possible phases of the effective spinor field model.

For unification of Skyrme's and Faddeev's ideas we suggest the generalization of the Mie electrodynamics within the scope of effective 8-spinor field model with the Lagrangian including Higgs-like potential and higher degrees of the invariant $A_\mu A^\mu$. For the classification of all possible chiral type models, the special 8-spinor Brioschi identity is used [2]:

$$j_\mu j^\mu - \tilde{j}_\mu \tilde{j}^\mu = s^2 + p^2 + v^2 + a^2. \tag{1}$$

In this identity the following quadratic spinor quantities are introduced:

$$S = \overline{\psi}\psi, \quad p = i\overline{\psi}\gamma_5\psi, \quad v = \overline{\psi}\lambda\psi, \quad a = i\overline{\psi}\gamma_5\lambda\psi, \quad j_\mu = \overline{\psi}\gamma_\mu\psi, \quad \tilde{j}_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi,$$

with $\overline{\psi} = \psi^+\gamma_0$ and matrix $\lambda = \sigma_i \otimes I_4$ standing for the Pauli matrices in the isotopic spinor space. Here and below we use the Weyl representation for Dirac matrices γ_μ,

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$\mu = 0,\ 1,\ 2,\ 3$, which may be written in the block form using the Pauli matrices σ_1, σ_2, σ_3:

$$\gamma_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3.$$

Hence the matrix γ_5, which is the product of the four gamma matrices, is written as follows:

$$\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{2}$$

We use the denotation I_n for the unit matrix of size n, $n \in \mathbb{N}$.

By analogy with [3, 4] we consider the 8-spinor space, in which 8-spinor is defined as column:

$$\psi = col(\varphi, \chi, \xi, \theta), \tag{3}$$

with $\varphi = \mathrm{col}(\varphi_1, \varphi_2)$, $\chi = \mathrm{col}(\chi_1, \chi_2)$, $\xi = \mathrm{col}(\xi_1, \xi_2)$, $\theta = \mathrm{col}(\theta_1, \theta_2)$ being 2-spinors. Now we can write down the quadratic spinor quantities s, p, v, a entering the Brioschi identity (1):

$$s = \overline{\psi}\psi = \psi^+\gamma_0\psi = \varphi^+\chi + \chi^+\varphi + \xi^+\theta + \theta^+\xi, \tag{4}$$
$$p = i\overline{\psi}\gamma_5\psi = i\left[-\chi^+\varphi + \varphi^+\chi + \xi^+\theta - \theta^+\xi\right], \tag{5}$$
$$v_1 = \overline{\psi}\lambda_1\psi = \theta^+\varphi + \xi^+\chi + \varphi^+\theta + \chi^+\xi, \tag{6}$$
$$v_2 = \overline{\psi}\lambda_2\psi = i\left[\theta^+\varphi + \xi^+\chi - \varphi^+\theta - \chi^+\xi\right], \tag{7}$$
$$v_3 = \overline{\psi}\lambda_3\psi = \varphi^+\chi + \chi^+\varphi - \xi^+\theta - \theta^+\xi, \tag{8}$$
$$a_1 = i\overline{\psi}\gamma_5\lambda_1\psi = i\left[-\theta^+\varphi + \xi^+\chi + \varphi^+\theta - \chi^+\xi\right], \tag{9}$$
$$a_2 = i\overline{\psi}\gamma_5\lambda_2\psi = \theta^+\varphi - \xi^+\chi + \varphi^+\theta - \chi^+\xi, \tag{10}$$
$$a_3 = i\overline{\psi}\gamma_5\lambda_3\psi = i\left[\varphi^+\chi - \chi^+\varphi - \xi^+\theta + \theta^+\xi\right]. \tag{11}$$

In this 8-spinor space we search for S^2- and S^3-submanifolds to unify Skyrme and Faddeev models.

2. Lepton phase

To describe the lepton phase we should find group $SU(2)$. To this end let us seek the symmetry group as follows:

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = U_L \begin{pmatrix} \xi \\ \theta \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi \\ \theta \end{pmatrix}, \tag{12}$$

where a, b take complex values and U_L is unitary matrix. Here and further, we consider 2-spinor fields denoted by Greek letters.

Let us use the following substitution: $b = -b^* = i\beta$, $\beta \in \mathbb{R}$ and $a = 0$:

$$s = (1+\beta^2)(\xi^+\theta + \theta^+\xi), \quad p = i(\beta^2 - 1)(\xi^+\theta - \theta^+\xi),$$
$$v_1 = 0, \quad v_2 = -2\beta(\theta\theta^+ + \xi\xi^+), \quad v_3 = (\beta^2 - 1)(\xi^+\theta + \theta^+\xi),$$
$$a_1 = 2\beta(\theta\theta^+ - \xi\xi^+), \quad a_2 = 0, \quad a_3 = -i(\beta^2 + 1)(\theta^+\xi - \xi^+\theta).$$

Simple calculations show that with $\beta = 1$ the only surviving combination is $s^2 + a_1^2 + a_3^2 \neq 0$. As a result we have the homotopy group $\pi_3(S^2)$, which generates the topological Hopf-like charge:

$$s = 2(\xi^+\theta + \theta^+\xi), \quad a_1 = 2(\theta\theta^+ - \xi\xi^+), \quad a_3 = -2i(\theta^+\xi - \xi^+\theta),$$
$$s^2 + a_1^2 + a_3^2 = 4\left(\theta^+\theta + \xi^+\xi\right)^2.$$

Let us introduce the normalized vector $\tilde{A}$ with the following components:

$$\tilde{A}_1 = s = \frac{\xi^+\theta + \theta^+\xi}{\theta^+\theta + \xi^+\xi}, \quad \tilde{A}_2 = a_1 = \frac{\theta\theta^+ - \xi\xi^+}{\theta^+\theta + \xi^+\xi}, \quad \tilde{A}_3 = a_3 = -i\frac{\theta^+\xi - \xi^+\theta}{\theta^+\theta + \xi^+\xi}. \tag{13}$$

As was shown by Faddeev, the topological solitons of the twisted closed string type can be realised in this case. The resulting mapping is $\tilde{A} : S^3 \to S^2$. So the $\tilde{A}$-field configurations are classified by elements of the homotopy group $\pi^3(S^2) = \mathbb{Z}$. The Lagrangian of this model has the following structure:

$$\mathcal{L} = \lambda^2(\partial_\mu \tilde{A}^i)^2 - \frac{\varepsilon^2}{4} f_{\mu\nu}^2 - m^2(1 - \tilde{A}_3), \quad \mu, \nu = 0, 1, 2, 3, \tag{14}$$

where ε, λ, m are constant parameters,

$$f_{\mu\nu} = 2\varepsilon_{ijk}\partial_\mu \tilde{A}^i \partial_\nu \tilde{A}^j \tilde{A}^k \equiv \partial_\mu \aleph_\nu - \partial_\nu \aleph_\mu. \tag{15}$$

The mass term $m^2(1 - \tilde{A}_3)$ is added to (14) for providing the required asymptotic behavior at infinity.

The topological invariant in our model is the Hopf index Q_H, which can be calculated by the following formula:

$$Q_H = -\frac{1}{(8\pi)^2}\int \aleph rot(\aleph) \mathrm{d}^3x. \tag{16}$$

The energy has the lower bound:

$$E > \varepsilon\lambda(4\pi)^2\sqrt{2}3^{3/8}|Q_H|^{3/4}, \tag{17}$$

which provides the stability of the regular vortices in the Faddeev model [5].

3. Baryon phase

By analogy with (12) let us seek the symmetry group as follows:

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = U_B \begin{pmatrix} \xi^* \\ \theta^* \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi^* \\ \theta^* \end{pmatrix}, \tag{18}$$

where $a, b \in \mathbb{C}$ and U_B is unitary matrix.

Let us use the following substitution: $a = 0$ and $b = i\beta$, $\beta \in \mathbb{R}$:

$$s = (1 + \beta^2)(\xi^+\theta + \theta^+\xi), \quad p = i(1 + \beta^2)(\xi^+\theta - \theta^+\xi),$$
$$v_1 = \beta\left(\left(\theta^+\right)^2 + \theta^2 + \xi^2 + \left(\xi^+\right)^2\right), \quad v_2 = -\beta\left(\left(\theta^+\right)^2 + \theta^2 + \xi^2 + \left(\xi^+\right)^2\right),$$
$$v_3 = (\beta^2 + 1)(\xi^+\theta + \theta^+\xi),$$
$$a_1 = \beta\left(\left(\theta^+\right)^2 + \theta^2 - \xi^2 - \left(\xi^+\right)^2\right), \quad a_2 = i\beta\left(\left(\theta^+\right)^2 - \theta^2 + \xi^2 - \left(\xi^+\right)^2\right),$$
$$a_3 = i(1 - \beta^2)(\xi^+\theta - \theta^+\xi).$$

Putting $\beta = 1$, we find S^3-manifold generated by the structure $p^2 + s^2 + a_1^2 + a_2^2$:

$$s = 2(\xi^+\theta + \theta^+\xi), \quad p = 2i(\xi^+\theta - \theta^+\xi),$$
$$a_1 = \left(\left(\theta^+\right)^2 + \theta^2 - \xi^2 - \left(\xi^+\right)^2\right), \quad a_2 = i\left(\left(\theta^+\right)^2 - \theta^2 + \xi^2 - \left(\xi^+\right)^2\right),$$
$$s^2 + p^2 + a_1^2 + a_2^2 = 4\left[\left(\theta^+\theta + \xi^+\xi\right)^2 - \left(\xi^+\theta - \theta^+\xi\right)^2\right].$$

4. Unification of two phases

To unify the two phases, we should find common vacuum state, which conserves only one component in the lepton and the baryon phases. The scalar quantity s has the same structure for the two phases:

$$s = 2(\xi^+\theta + \theta^+\xi).$$

With $\theta = \xi = C$, $C \in \mathbb{C}$ only this component survives. As a result, 8-spinor in the vacuum state has the following form:

$$\psi_V = \begin{pmatrix} iC \\ iC \\ C \\ C \end{pmatrix}. \tag{19}$$

In this vacuum state the current is invariant: $j_0 = 4C^*C,\ j_k = 0,\ k = 1, 2, 3.$

The complex value C can be found from the following condition:

$$\mathcal{L}[\psi_V] = 0.$$

5. Conclusion

Finally we give the summary of the results obtained:

– The generalization of the Mie electrodynamics within the scope of the effective 8-spinor field model is suggested. For this field model the quadratic spinor quantities are constructed.
– The lepton and baryon sectors with the common vacuum state are suggested. The topological invariant in the lepton sector is the Hopf index Q_H, which can be calculated by the formula (14).

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Описание лептонного и барионного секторов в спинорной модели Скирма–Фаддеева

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Для объединения моделей Скирма и Фаддеева, описывающих соответственно барионы и лептоны как топологические солитоны, предлагается использовать 8-спинорное поле. В модели Скирма частица-солитон имеет топологический заряд, который интерпретируется как барионное число. В модели Фаддеева, описывающей лептоны, частицы наделены топологическим инвариантом типа Хопфа. Использование специального 8-спинорного тождества Бриоски позволяет рассматривать лептоны и барионы как сектора в общей спинорной модели с потенциалом Хиггса, зависящего от $j^\mu j_\mu$, входящего в лагранжиан.

В настоящей статье рассматривается обобщение электродинамики Ми в рамках эффективной 8-спинорной полевой модели. Для этого полевой модели получен явный вид квадратичных спинорных величин, входящих в тождество Бриоски. Обнаружены группы симметрий, образующие S^2 и S^3 подмногообразия в общем биквадратном спинорном S^8-многообразии.

В результате получаются две гомотопические группы $\pi_3(S^2)$ и $\pi_3(S^3)$, которые могут описывать лептоны и барионы соответственно. Для этих многообразий построено общее вакуумное состояние, сохраняющее лишь одну компоненту в обоих случаях. В результате получается 8-спинорная модель, позволяющая единым образом описать барионы и лептоны.

Ключевые слова: 8-спинор, топологический заряд, солитоны, гомотопические группы, тождество Бриоски, модель Скирма–Фаддеева, индекс Хопфа.

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