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A Brief Description of Higher-Order Accurate Numerical Solution of Burgers' Equation

T. Zhanlav*, **O. Chuluunbaatar**†, **V. Ulziibayar**‡

* *Faculty of Mathematics and Computer Science*
National University of Mongolia, Mongolia
† *Laboratory of Information Technologies*
Joint Institute for Nuclear Research
6, Joliot-Curie str., Dubna, Moscow region, Russia, 141980
‡ *Faculty of Mathematics*
Mongolian University of Science and Technology
P.O.Box 46/520, Ulaanbaatar, Mongolia, 210646

Two new higher-order accurate finite-difference schemes for the numerical solution of boundary-value problem of the Burgers' equation are suggested. Burgers equation is a one-dimensional analogue of the Navier-Stokes equations describing the dynamics of fluids and it possesses all of its mathematical properties. Besides the Burgers' equation, one of the few nonlinear partial differential equations which has the exact solution, and it can be used as a test model to compare the properties of different numerical methods. A first scheme is purposed for the numerical solution of the heat equation. It has a sixth-order approximation in the space variable, and a third-order one in the time variable. A second scheme is used for finding a numerical solution for the Burgers's equation using the relationship between the heat and Burgers' equations. This scheme also has a sixth-order approximation in the space variable. The numerical results of test examples are found in good agreement with exact solutions and confirm the approximation orders of the schemes proposed.

Key words and phrases: Burgers' equation, higher-order accurate numerical solution.

1. Introduction

We consider a one-dimensional quasi-linear parabolic partial differential equation which is known as Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \quad a < x < b, \quad t > 0, \tag{1}$$

with an initial condition

$$u(x,0) = \varphi(x), \quad a < x < b, \tag{2}$$

and boundary conditions

$$u(a,t) = f(t) \quad \text{and} \quad u(b,t) = g(t), \quad t > 0, \tag{3}$$

where $\nu > 0$ is a coefficient of the kinematic viscosity and $\varphi(x)$, $f(t)$ and $g(t)$ are known functions.

The Burgers' equation can be considered as an approach to the Navier-Stokes equations [1, 2]. Since both contain nonlinear terms of the type: unknown functions multiplied by a first derivative and both contain higher-order terms multiplied by a small parameter. On the other hand, the Burgers' equation is one of a few nonlinear equations which can be solved exactly for an arbitrary initial and boundary conditions [3]. However these exact solutions are impractical for the small values of viscosity constant due to a slow convergence of series solutions. Thus many numerical schemes are constructed for a numerical solution of the Burgers' equation for small values of viscosity constant which corresponds to a steep front in the propagation of dynamic

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wave forms [3–8]. The study of the general properties of the Burgers' equation has motivated considerable attention due to its applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. [3]. The extended version of this paper will be published elsewhere.

We consider the Burgers' equation (1) with the initial condition

$$u(x,0) = \sin(\pi x), \quad 0 < x < 1, \tag{4}$$

and the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0. \tag{5}$$

It is well known that, by the Hopf-Cole transformation

$$u(x,t) = -2\nu \frac{\vartheta'(x,t)}{\vartheta(x,t)} \tag{6}$$

the Burgers' equation transforms to the linear heat equation

$$\frac{\partial \vartheta(x,t)}{\partial t} = \nu \frac{\partial^2 \vartheta(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0 \tag{7}$$

with initial condition

$$\vartheta(x,0) = \exp\left(-\frac{1-\cos(\pi x)}{2\pi\nu}\right), \quad 0 < x < 1, \tag{8}$$

and Neumann boundary conditions

$$\vartheta'(0,t) = \vartheta'(1,t) = 0, \quad t > 0. \tag{9}$$

Symbol "$'$" denotes the derivative with respect to variable x. Thus, if $\vartheta(x,t)$ is any solution of the heat equation (7) subject to the conditions (8) and (9), then the function (6) is a solution of the Burgers' equation (1) with the conditions (4) and (5).

We assume that the numerical solution of the heat problem defined by Eqs. (7)–(9) is found by any of known methods with higher accuracy. For example, this problem can be solved by well-known Crank-Nicolson scheme [9] and a more accurate explicit scheme proposed by Zhanlav in [10]

$$\vartheta_i^{n+1} = \frac{\beta-\gamma}{\beta+\gamma}\vartheta_i^{n-1} + \frac{\beta\gamma}{\beta+\gamma}\left(\vartheta_{i-1}^n - 2\vartheta_i^n + \vartheta_{i+1}^n\right) + \frac{2\gamma}{\beta+\gamma}\vartheta_i^n, \tag{10}$$

$$\gamma = \frac{2\tau\nu}{h^2}, \quad i = 1, \dots, N-1, \quad Nh = 1, \quad n = 1, 2, \dots.$$

Here and throughout the work, ϑ_i^n is the approximate solution at the mesh points $(x_i = ih, t_n = n\tau)$, where h is a spatial step, τ is a time step. Easy to show that the scheme (10) is stable and its truncation error is of the order $O(\tau^3 + h^6)$ provided that

$$\beta = 0.2, \quad \frac{\tau\nu}{h^2} = \frac{1}{\sqrt{60}}. \tag{11}$$

When $\beta = 1$, the scheme (10) leads to the well-known DuFort-Frankel's one [9].

It should be mentioned that the scheme (10) is a three-level one in time. Hence, in order to find ϑ_i^n at level two it requires two values ϑ_i^n at level 0 and 1, i.e., ϑ_i^0 and ϑ_i^1.

Using the Taylor expansion of $\vartheta(x,\tau)$ at point $(x,0)$ and Eq. (7), we obtain

$$\vartheta(x,\tau)=\vartheta(x,0)+\nu\frac{\partial^2\vartheta(x,0)}{\partial^2 x}\tau+\frac{\nu^2}{2}\frac{\partial^4\vartheta(x,0)}{\partial^4 x}\tau^2+O(\tau^3). \quad (12)$$

From Eq. (12) values of ϑ_i^1 were obtained.

2. Construction of Higher-Order Accurate Finite-difference Schemes for Spatial Variable

The solution domain $\{(x,t) : x\in[0,1],\, t\in(0,\infty)\}$ is discretized into cells described by the node set (x_i,t_n) in which $x_i=ih$, $t_n=n\tau$, $i=0,1,\ldots,N$, $Nh=1$, $n=0,1\ldots$

We suppose that the solution of Eqs. (7)–(9) is a sufficiently smooth function with respect to x and t. So, from the Taylor expansions of $\vartheta(x_{i+1},t)$ and $\vartheta(x_{i-1},t)$ at point (x_i,t) we have

$$\frac{\vartheta(x_{i+1},t)-\vartheta(x_{i-1},t)}{2h}=\vartheta'(x_i,t)+\frac{\vartheta'''(x_i,t)}{6}h^2+\frac{\vartheta^{(5)}(x_i,t)}{120}h^4+O(h^6), \quad (13)$$

$$-1.1cm\frac{\vartheta'(x_{i+1},t)-2\vartheta'(x_i,t)+\vartheta'(x_{i-1},t)}{h^2}=\vartheta'''(x_i,t)+\frac{\vartheta^{(5)}(x_i,t)}{12}h^2+O(h^4). \quad (14)$$

Eliminating ϑ''' from (13) and (14), we obtain

$$\frac{\vartheta(x_{i+1},t)-\vartheta(x_{i-1},t)}{2h}=\frac{\vartheta'(x_{i+1},t)+4\vartheta'(x_i,t)+\vartheta'(x_{i-1},t)}{6}-\\-\frac{\vartheta^{(5)}(x_i,t)}{180}h^4+O(h^6). \quad (15)$$

Omitting the small term in the right-hand side of the obtained finite-difference scheme:

$$\frac{\vartheta_{i+1}^n-\vartheta_{i-1}^n}{2h}=\frac{\vartheta_{i-1}'^n+4\vartheta_i'^n+\vartheta_{i+1}'^n}{6},\quad i=1,2,\ldots,N-1. \quad (16)$$

The truncation error of this scheme is $O(h^4)$. Finding ϑ' from (6) and substituting it into (16), we obtain a compact finite-difference scheme for approximate solution $y_i^n\equiv y(x_i,t_n)$ of $u(x_i,t_n)$:

$$\vartheta_{i-1}^n y_{i-1}^n+4\vartheta_i^n y_i^n+\vartheta_{i+1}^n y_{i+1}^n=-\frac{6\nu}{h}\left(\vartheta_{i+1}^n-\vartheta_{i-1}^n\right), \quad (17)$$

$$i=1,2,\ldots,N-1,\quad n=1,2,\ldots$$

with boundary conditions

$$y_0^n=y_N^n=0. \quad (18)$$

If we denote $v_i^n=\vartheta_i^n y_i^n$, then the scheme (17), (18) leads to

$$v_{i-1}^n+4v_i^n+v_{i+1}^n=-\frac{6\nu}{h}\left(\vartheta_{i+1}^n-\vartheta_{i-1}^n\right), \quad (19)$$

$$v_0^n=v_N^n=0. \quad (20)$$

The last system has a unique solution set $(v_0^n,v_1^n,\ldots,v_N^n)$ since its the matrix is diagonally dominant. It means that the tridiagonal system (17), (18) has a unique solution set $(y_0^n,y_1^n,\ \ldots,y_N^n)$ for each $n=1,2,\ldots$, and it can be solved by efficient elimination method [11]. Moreover, it is also possible to obtain a higher accurate finite-difference scheme than (17), (18).

Using the Taylor expansions of $\vartheta(x_{i+2},t)$ and $\vartheta(x_{i-2},t)$ at the point (x_i,t) we have

$$\frac{\vartheta(x_{i+2},t)-\vartheta(x_{i-2},t)}{4h}=\vartheta'(x_i,t)+\frac{\vartheta'''(x_i,t)}{6}4h^2+\frac{\vartheta^{(5)}(x_i,t)}{120}16h^4+O(h^6). \tag{21}$$

We can eliminate the term with $\vartheta^{(5)}(x_i,t)$ from (21) and (13). As a result we have

$$16\frac{\vartheta(x_{i+1},t)-\vartheta(x_{i-1},t)}{2h}-\frac{\vartheta(x_{i+2},t)-\vartheta(x_{i-2},t)}{4h}$$
$$=15\vartheta'(x_i,t)+2\vartheta'''(x_i,t)h^2+O(h^6). \tag{22}$$

We also use the well-known five-point approximation formula for $\vartheta'''(x_i,t)$

$$\vartheta_i'''=\frac{1}{12h^2}\left(-\vartheta_{i-2}'+16\vartheta_{i-1}'-30\vartheta_i'+16\vartheta_{i+1}'-\vartheta_{i+2}'\right)+O(h^4), \tag{23}$$

which holds for sufficiently smooth function $\vartheta(x,t)$ with respect to x variable. Substituting (23) into (22) and using the Hopf-Cole transformation given by Eq. (6) we obtain

$$v_{i-2}^n-16v_{i-1}^n-60v_i^n-16v_{i+1}^n+v_{i+2}^n=c_i^n, \tag{24}$$

$$c_i^n=-\frac{3\nu}{h}\left(-\vartheta_{i-2}^n+32\vartheta_{i-1}^n-32\vartheta_{i+1}^n+\vartheta_{i+2}^n\right),\ i=2,3,\dots,N-2.$$

Of course, besides of $v_0^n=v_N^n=0$ we need additionally two end conditions v_1^n and v_{N-1}^n in order to solve the system (24). Differentiating Eq. (7) $(2k-1)$-times with respect to x, and taking into account (9), we obtain

$$\vartheta^{(2k+1)}(x_0,t)=\vartheta^{(2k+1)}(x_N,t)=0,\quad k=0,1,\dots \tag{25}$$

Then from (13) it follows that

$$\vartheta(x_1,t)=\vartheta(x_{-1},t),\quad \vartheta(x_{N+1},t)=\vartheta(x_{N-1},t), \tag{26}$$

where $x_{-1}=x_0-h$ and $x_{N+1}=x_N+h$. Also differentiating Eq. (6) $(2k)$-times with respect to x, and taking into account (25), we have

$$v^{(2k)}(x_0,t)=v^{(2k)}(x_N,t)=0,\quad k=0,1,\dots \tag{27}$$

Then substituting (27) in Taylor expansions of $v(x_{\pm1},t)$ and $v(x_{N\pm1},t)$ at the point x_0 and x_N, respectively, we conclude that

$$v(x_1,t)=-v(x_{-1},t),\quad v(x_{N+1},t)=-v(x_{N-1},t). \tag{28}$$

Hence, taking into account (26) and (28), the finite-difference scheme (24) for $i=1,N-1$ has the forms

$$-61v_1^n-16v_2^n+v_3^n=-\frac{3\nu}{h}\left(32\vartheta_0^n-\vartheta_1^n-32\vartheta_2^n+\vartheta_3^n\right), \tag{29}$$

$$v_{N-3}^n-16v_{N-2}^n-61v_{N-1}^n=-\frac{3\nu}{h}\left(-\vartheta_{N-3}^n+32\vartheta_{N-2}^n+\vartheta_{N-1}^n-32\vartheta_N^n\right).$$

Thus, we have five-point finite-difference schemes (24), (29) with truncation error $O(h^6)$.

3. Numerical Results

In this section we demonstrate the accuracy of the proposed finite-difference schemes (10), (11), (24), (29) by solving exact solvable Burgers' equation (1), (4), (5) for $\nu = 1$ and compare the numerical results with the existing results. The computations are performed using MatLab.

Table 1 displays convergence of the proposed schemes for the numerical solution $y(x_i, T)$ to the exact solution $u(x_i, T)$ at $T = (10\sqrt{15})^{-1}$ versus the number of nodes N. Table 2 presents the maximum absolute error $\|e\|_\infty = \max\limits_{1 \leqslant i \leqslant N-1} |y(x_i, T) - u(x_i, T)|$ versus the number of nodes N.

Table 1

Convergence of the proposed schemes for the numerical solution $y(x_i, T)$ to the exact solution $u(x_i, T)$ versus the number of nodes N. Here $\nu = 1$, $T = (10\sqrt{15})^{-1}$

x	Numerical solution				Exact solution
	$N = 10$	$N = 20$	$N = 40$	$N = 80$	
0.1	0.228649	0.22865030	0.2286503154	0.2286503156451	0.2286503156477
0.2	0.437766	0.43776771	0.4377677345	0.4377677347901	0.4377677347942
0.3	0.608777	0.60877832	0.6087783451	0.6087783454149	0.6087783454190
0.4	0.725196	0.72519674	0.7251967567	0.7251967569572	0.7251967569600
0.5	0.774045	0.77404614	0.7740461512	0.7740461512588	0.7740461512595
0.6	0.747568	0.74756837	0.7475683734	0.7475683733302	0.7475683733289
0.7	0.645016	0.64501619	0.6450161825	0.6450161823893	0.6450161823870
0.8	0.474055	0.47405491	0.4740549069	0.4740549067930	0.4740549067907
0.9	0.251102	0.25110176	0.2511017581	0.2511017580559	0.2511017580546

Table 2

The maximum absolute error $\|e\|_\infty = \max\limits_{1 \leqslant i \leqslant N-1} |y(x_i, T) - u(x_i, T)|$ between numerical and exact solutions versus the number of nodes N, and corresponding Runge coefficients

N	$\|e\|_\infty$	$\|e\|_{\infty h} / \|e\|_{\infty h/2}$
10	1.058410630083717e-006	
20	1.679794564557469e-008	63.008
40	2.635179296994750e-010	63.744
80	4.136968545509490e-012	63.698

From Tables 1, 2 we observed that the numerical results obtained by proposed schemes are reasonably in good agreement with the exact solution. The corresponding Runge coefficients are consistent with the theoretical expectation of $O(h^6)$.

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Краткое описание высокоточного метода численного решения уравнения Бюргерса

Т. Жанлав*, О. Чулуунбаатар†, В. Улзийбаяр‡

* *Факультет математики и компьютерных наук*
Монгольский государственный университет, Монголия
† *Лаборатория информационных технологий*
Объединённый институт ядерных исследований
ул. Жолио-Кюри, д. 6, Дубна, Московская область, Россия, 141980
‡ *Факультет математики*
Монгольский государственный университет науки и технологии
Улан-Батор, Монголия, 210646

Предложены две новые разностные схемы повышенной точности для численного решения начально-краевой задачи уравнения Бюргерса. Уравнение Бюргерса является одномерным аналогом уравнения Навье–Стокса, описывающего динамику жидкости, и обладает всеми его математическими свойствами. Кроме того, уравнение Бюргерса относится к числу немногих нелинейных уравнений в частных производных, для которых известно аналитическое решение, что позволяет использовать его в качестве тестовой модели для сравнения свойств различных численных методов. Первая схема, предназначенная для численного решения уравнения теплопроводности, имеет шестой порядок аппроксимации по пространственной переменной и третий порядок по временной переменной. Вторая схема используется для нахождения численного решения уравнения Бюргерса на основе связи между уравнением теплопроводности с уравнением Бюргерса. Данная схема также имеет шестой порядок аппроксимации по пространственной переменной. Полученные на тестовых примерах численные результаты хорошо согласуются с аналитическими решениями уравнения Бюргерса и подтверждают порядок аппроксимации предложенных схем.

Ключевые слова: уравнение Бюргерса, повышенной точности численного решения.