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Application of Functional Polynomials to Approximation of Matrix-Valued Functional Integrals

E. A. Ayryan*, **V. B. Malyutin**†

* *Laboratory of Information Technologies*
Joint Institute for Nuclear Research
6, Joliot-Curie str., Dubna, Moscow region, Russia, 141980

† *Institute of Mathematics, The National Academy of Sciences of Belarus,*
11, Surganov str., Minsk, Belarus, 220072

The matrix-valued functional integrals, generated by solutions of the Dirac equation are considered. These integrals are defined on the one-dimensional continuous path $x : |s,t| \to \mathbb{R}$ and take values in the space of complex $d \times d$ matrices. Matrix-valued integrals are widely used in relativistic quantum mechanics for investigation of particle in electromagnetic field. Namely integrals are applied to represent the fundamental solution of the Cauchy problem for the Dirac equation. The method of approximate evaluation of matrix-valued integrals is proposed. This method is based on the expansion of functional in a series. Terms of a series have the form of a product of linear functionals with increasing total power. Taking a finite number of terms in the series and evaluating functional integrals of a product of linear functionals we obtain approximate value of the matrix-valued functional integral. Proposed method can be used for a wide class of integrals because the series converges for a large class of functionals. Application of the suggested method in the case of small and large parameters included in the integral is considered.

Key words and phrases: functional integrals, matrix-valued integrals, functional polynomials, approximation of integrals.

1. Introduction

One approach to approximate evaluation of functional integrals is approximation of the original integrand functional by functional polynomials. Another approach to evaluation of functional integrals is construction of approximate formulas that are exact for the class of functional polynomials given degree [1–3]. There are different types of functional integrals because there are different spaces, measures and ways to define the functional integrals. Functional polynomials and formulas with given degree of accuracy are widely used to approximate evaluation of integrals with respect to Gaussian measure. We propose to use the functional polynomials to approximate evaluation of matrix-valued integrals, generated by solutions of Dirac equation. These integrals are widely used in relativistic quantum mechanics for investigation of particle in electromagnetic field [4, 5]. The method of evaluation of matrix-valued integrals is based on the expansion of functional in a series. Terms of a series have the form of a product of linear functionals with increasing total power. In case of Gaussian integrals the series of integrals of the product of linear functionals converges for a narrow class of functionals. In case of matrix-valued integrals the series converges for a wide class of functionals.

2. Approximation of Matrix-Valued Integrals

Following [4, 5] matrix-valued functional integral is defined on space of functions $x(\tau), s \leqslant \tau \leqslant t$, satisfying a Lipschitz condition with the order of 1, that is for any $s \leqslant a < b \leqslant t$, $|x(b) - x(a)| \leqslant M|b - a|$. Integral is defined by

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$$\int F(x(\cdot))\mathrm{d}\mu(x) =$$

$$= \lim_{\max\limits_j \Delta t_j \to 0} \int\limits_{\mathbf{R}^n} F\left(\sum_{j=1}^{n} x_j \chi_{[t_{j-1},t_j]}(\cdot)\right) \prod_{j=n}^{1} S\left(t_j - t_{j-1}, x_j - x_{j-1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

if this limit exists for any partition of segment $[s,t]$ by dots $s = t_0 < t_1 < \ldots < t_n = t$. Here $x_j = x(t_j)$, $\chi_{]t_{j-1},t_j]}(\tau)$ — characteristic function of segment $[t_{j-1},t_j]$, $S(t_j - t_{j-1}, x_j - x_{j-1})$ — transition function which is the fundamental solution of equation

$$\frac{\partial S(t,x)}{\partial t} = a\alpha \frac{\partial S(t,x)}{\partial x} + b\beta S(t,x),$$

where a, b — real parameters, α, β — anticommuting values (operators or matrices), that is $\alpha\beta + \beta\alpha = 0$. We suppose that $\alpha^2 = \beta^2 = E$, E — identity matrix or operator.

In this work we consider approximation of functional integrals of the cylindrical functionals having the form

$$F(x(\cdot)) = F\left(\int\limits_s^t f_1(\tau)\mathrm{d}x(\tau), \ldots, \int\limits_s^t f_d(\tau)\mathrm{d}x(\tau)\right),$$

where functions $f_j(\tau), 1 \leqslant j \leqslant d$, are Riemann integrable on $[s,t]$.

To approximate functional integral we use the expansion of functional F in a series

$$F(x(\cdot)) = F\left(\int\limits_s^t f_1(\tau)\mathrm{d}x(\tau), \ldots, \int\limits_s^t f_d(\tau)\mathrm{d}x(\tau)\right) =$$

$$= \sum_{a_1,\ldots,a_d=0}^{\infty} F_{a_1,\ldots,a_d} \left(\int\limits_s^t f_1(\tau)\mathrm{d}x(\tau)\right)^{a_1} \cdots \left(\int\limits_s^t f_d(\tau)\mathrm{d}x(\tau)\right)^{a_d}.$$

Then we evaluate functional integrals of a product of linear functionals. In this work we do not give detailed calculations. We present only the final result.

Proposition 1. Let functions $f_j(\tau), 1 \leqslant j \leqslant d$, are Riemann integrable on $[s,t]$. Let function $F(u_1,\ldots,u_d)$ is expanded in a series $\sum\limits_{a_1,\ldots,a_d=0}^{\infty} F_{a_1,\ldots,a_d} u_1^{a_1} \cdots u_d^{a_d}$ and series $\sum\limits_{a_1,\ldots,a_d=0}^{\infty} |F_{a_1,\ldots,a_d}||u_1|^{a_1} \cdots |u_d|^{a_d}$ converges for $u_j = ac_j$, $c_j > \int_s^t |f_j(\tau)|\mathrm{d}\tau, 1 \leqslant j \leqslant d$. Then

$$\int F\left(\int\limits_s^t f_1(\tau)\mathrm{d}x(\tau), \ldots, \int\limits_s^t f_d(\tau)\mathrm{d}x(\tau)\right)\mathrm{d}\mu(x) = \sum_{a_1,\ldots,a_d=0}^{\infty} F_{a_1,\ldots,a_d} \times$$

$$\times (-a)^A \int\limits_s^t (A) \int\limits_s^t \prod_{k=1}^{A} g_k(\tau_k) \prod_{k=0}^{A} \exp\{(\tau_{m_k} - \tau_{m_{k+1}})(-1)^k b\beta\} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_A \alpha^A \quad (1)$$

and series converges in the above equality.

Here $A = a_1 + \ldots + a_d$, $g_i(\tau) = f_k(\tau)$, $\sum\limits_{m=0}^{k-1} a_m \leqslant i < \sum\limits_{m=0}^{k} a_m$, $a_0 = 1$, $1 \leqslant k \leqslant d$, $(m_1, \ldots, m_A)$ — permutation of $(1, \ldots, A)$ such that $\tau_{m_1} \geqslant \tau_{m_2} \geqslant \ldots \geqslant \tau_{m_A}$, $\tau_{m_0} = t$, $\tau_{m_{A+1}} = s$.

In next section we consider the application of formula (1) to calculate some functional integrals.

3. Integrals Containing Small and Large Parameters

Formula (1) can be used to evaluate a wide class of integrals. In some cases the proposed formula is very effective for approximation of functional integrals. Let us suppose that $|f_j(\tau)| \leqslant C, 1 \leqslant j \leqslant d$. Then expression

$$a^A \int\limits_s^t (A) \int\limits_s^t \prod_{k=1}^{A} |g_k(\tau_k)| \prod_{k=0}^{A} \left\| \exp\left\{ (\tau_{m_k} - \tau_{m_{k+1}})(-1)^k b\beta \right\} \right\| \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_A$$

is estimated by

$$a^A C^A A! \| \exp(-b\beta t) \| \exp\left((2t - s) \|b\beta\| \right) H,$$

where $H = \left(\dfrac{t}{2\|b\beta\|} \right)^{\frac{A+2}{2}}$ if A — even and $H = \dfrac{1}{t} \left(\dfrac{t}{2\|b\beta\|} \right)^{\frac{A+3}{2}}$ if A — odd.

From this estimate it follows that the proposed method of approximation of integrals is effective for small a and for large b.

As example we consider approximation of integral $\int \exp \left\{ \int\limits_s^t \lambda(\tau) \mathrm{d}x(\tau) \right\} \mathrm{d}\mu(x)$.

Using the expansion (1) we obtain that

$$\int \exp \left\{ \int\limits_s^t \lambda(\tau) \mathrm{d}x(\tau) \right\} \mathrm{d}\mu(x) \approx$$

$$\approx \sum_{a_1=0}^{n} \frac{1}{a_1!} (-a)^{a_1} \int\limits_s^t (a_1) \int\limits_s^t \prod_{k=1}^{a_1} \lambda(\tau_k) \prod_{k=0}^{a_1} \exp\{ (\tau_{m_k} - \tau_{m_{k+1}})(-1)^k b\beta \} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{a_1} \alpha^{a_1} =$$

$$= \sum_{a_1=0}^{n} (-a)^{a_1} \int\limits_s^t (a_1) \int\limits_{\tau_3}^t \int\limits_{\tau_2}^t \prod_{k=1}^{a_1} \lambda(\tau_k) \times$$

$$\times \exp\{ (t - 2\tau_1 + 2\tau_2 - \ldots + (-1)^{a_1} 2\tau_{a_1} - (-1)^{a_1} s) b\beta \} \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{a_1} \alpha^{a_1}.$$

For $t = 2$, $s = 1$, $\lambda \equiv 1$, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = 1$ and small $a = 0.1$ exact value of the integral is $\begin{pmatrix} 1.431 & 1.177 \\ 1.177 & 1.667 \end{pmatrix}$.

Approximate values for $n = 0$ and $n = 1$, respectively, are $\begin{pmatrix} 1.543 & 1.175 \\ 1.175 & 1.543 \end{pmatrix}$, $\begin{pmatrix} 1.426 & 1.175 \\ 1.175 & 1.661 \end{pmatrix}$.

For $t = 2$, $s = 1$, $\lambda \equiv 1$, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $a = 1$ and large $b = 10$ exact value of the integral is $\begin{pmatrix} 10414 & 11507 \\ 11507 & 12715 \end{pmatrix}$.

Approximate values for $n = 0$ and $n = 1$, respectively, are $\begin{pmatrix} 11014 & 11014 \\ 11014 & 11014 \end{pmatrix}$, $\begin{pmatrix} 9912 & 11014 \\ 11014 & 12115 \end{pmatrix}$.

4. Conclusion

The method of approximate evaluation of matrix-valued integrals is presented. The method relies on approximation of the functional under the integral sign by functional polynomials and evaluation of the integral of polynomials. As a result we get approximation of functional integral by Riemann integrals on a finite interval.

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Применение функциональных полиномов к аппроксимации матрично-значных функциональных интегралов

Э. А. Айрян[*], В. Б. Малютин[†]

[*] *Лаборатория информационных технологий*
Объединённый институт ядерных исследований
ул. Жолио-Кюри, д. 6, Дубна, Московская область, Россия, 141980
[†] *Институт математики НАН Беларуси*
ул. Сурганова, д. 11, Минск, Беларусь, 220072

Рассматриваются матричнозначные функциональные интегралы, порождённые решением уравнения Дирака. Эти интегралы определяются на одномерных непрерывных путях $x : [s,t] \to \mathbb{R}$ и принимают значения в пространстве комплексных $d \times d$ матриц. Матричнозначные интегралы широко используются в релятивистской квантовой механике для изучения частиц в электромагнитном поле. А именно, интегралы применяются для того, чтобы представить фундаментальное решение задачи Коши для уравнения Дирака. Предложен метод приближённого вычисления матричнозначных функциональных интегралов. Этот метод основан на разложении функционала в ряд. Члены ряда имеют вид произведения линейных функционалов с возрастающей суммарной степенью. Взяв конечное число членов ряда и вычислив функциональные интегралы от произведения линейных функционалов, мы получаем приближённое значение для матричнозначного функционального интеграла. Указанный метод может быть использован для широкого класса интегралов, так как ряд сходится для большого класса функционалов. Рассмотрено применение предложенного метода в случае малых и больших параметров, входящих в интеграл.

Ключевые слова: функциональные интегралы, матричнозначные интегралы, функциональные полиномы, аппроксимация интегралов.